Using Graph Theory to Investigate the Size of a Random Product or Quotient Set

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Product and Quotient Sets

Let G be a finitely presented multiplicative group equipped with a word metric. Let $A = \{a_1, \ldots, a_n\} \subseteq G$ and let $A^{-1} := \{a_1^{-1}, \ldots, a_n^{-1}\}$. The ***product set*** and the ***right*** and ***left difference sets*** of A are given, respectively, by

$$AA := \{a_i \cdot a_j : a_i, a_j \in A\},$$
$$AA^{-1} := \{a_i \cdot a_j^{-1} : a_i, a_j \in A\},$$
$$A^{-1}A := \{a_i^{-1} \cdot a_j : a_i, a_j \in A\}.$$

Following Lazarev-Miller-O'Bryant [1], we use graph theory as a framework to compute the probability that AA or AA^{-1} contain a word of a specified length, where A is random.

Condition Graphs

Let $R \geq 0$ and B_R be the set of words in G of length $\leq R$. **What can we say about the size of a uniform random subset A of B_R?**

Definition

The ***condition graph*** $C(w_1, \ldots, w_k \notin AA^{-1})$ is a graph with vertex set G and edges (u, v) whenever $uv^{-1} = w_i$ or $u^{-1}v = w_i$ for some $i \in \{1, \ldots, k\}$.

Similarly, the condition graph $C(w_1, \ldots, w_k \notin AA)$ is a graph with vertex set G and edges (u, v) whenever $uv = w_i$ or $vu = w_i$ for some $i \in \{1, \ldots, k\}$.

The condition graph $C(w_1, \ldots, w_k \notin S)$ represents all the ways that the words $w_1, \ldots, w_k$ could appear as an element of S through a specified random process. In order to consider questions of probability, we restrict our attention to the subgraphs $C_R(w_1, \ldots, w_k \notin S)$ induced by B_R, the set of all words in G of length $\leq R$.

The connected components of the condition graph $C(w \notin AA^{-1})$ are paths and cycles.

Lemma. Structure of $C(w \notin AA^{-1})$.

Let X be a connected component of of $C(w \notin AA^{-1})$. Then, X is isomorphic to one of the following.

(1) A cycle;
(2) A path of infinite length;
(3) A path of length 2;
(4) A singleton.

Case 3 occurs only when w is a square. Case 4 only occurs when $w = e$, in which case every connected component is a singleton.

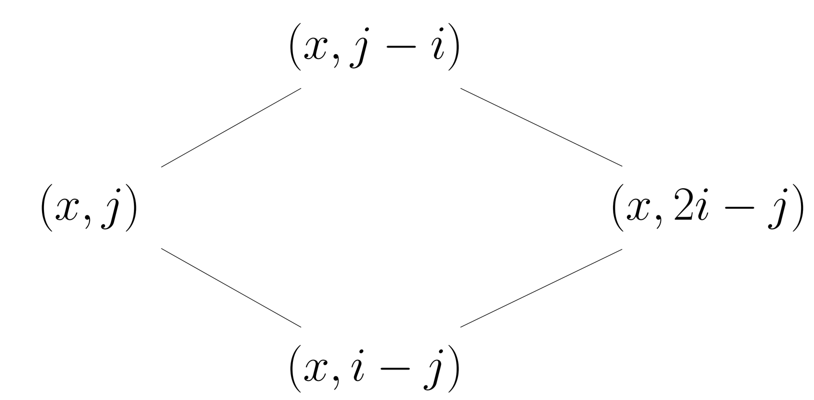
Counting the numbers of path and cycle connected components allows us to determine the probability that $w \notin AA^{-1}$. Let $p(k)$ be the number of length k path components in $C_R(w \notin S)$ and $c(k)$ be the number of length k cycle components in $C_R(w \notin S)$. Then, we have the following formula in terms of the Fibonacci numbers $F_1 = 1$, $F_2 = 2$, and $F_k = F_{k-1} + F_{k-2}$.

$$\mathbb{P}(w \notin AA^{-1}) = \prod_{k=1}^{\infty} \left(\frac{F_{k+1}}{2^k}\right)^{p(k)} \prod_{k=1}^{\infty} \left(\frac{F_{k-2} + F_k}{2^k}\right)^{c(k)}.$$

Condition Graphs for $\mathbb{Z}_2 * \mathbb{Z}_2$

Words in $\mathbb{Z}_2 * \mathbb{Z}_2$ are alternating strings of x, y. Any word can be encoded as a pair (c, i) where c is the starting character (either x or y) and i is the length of the string. In $\mathbb{Z}_2 * \mathbb{Z}_2$, the condition graphs $C_R(w \notin AA)$ have paths and cycles.

Example. If $w = (x, i)$ and $j \geq 1$ is odd, we have the cycle



Theorem. $C_R(w \notin AA)$ in $\mathbb{Z}_2 * \mathbb{Z}_2$

Let $w \in \mathbb{Z}_2 * \mathbb{Z}_2$. Then for AA,

(1) $C_R(e)$ consists of $\begin{cases} 2\lceil \frac{R}{2} \rceil + 1 \text{ self-loops,} \\ \lfloor \frac{R}{2} \rfloor \text{ paths of length 1.} \end{cases}$

(2) **if w is of even length $i \geq 2$,** write $R = k(\frac{i}{2}) + j$ for $j, k \in \mathbb{Z}_{\geq 0}$ and $j < \frac{i}{2}$. Then,

(i) **if $4 \mid i$,** $C_R(w)$ consists of $\begin{cases} 2\lceil \frac{j}{2} \rceil \text{ paths of length } k \text{ if } k \geq 1, \\ \frac{i}{2} - 2\lceil \frac{j}{2} \rceil \text{ paths of length } k-1 \text{ if } k \geq 2, \\ \lfloor \frac{2R-i-2}{4} \rfloor \text{ paths of length 1 if } R \geq \frac{i}{2}, \\ 1 \text{ self-loop if } R \geq \frac{i}{2}. \end{cases}$

(ii) **if $4 \nmid i$,** $C_R(w)$ consists of
$\begin{cases} j+1 \text{ (resp. } j) \text{ paths of length } k \text{ if } k \geq 1, R \text{ odd (resp. even),} \\ \frac{i}{2} - j - 1 \text{ (r. } \frac{i}{2} - j) \text{ paths of length } k-1 \text{ if } k \geq 2, R \text{ odd (r. even),} \\ \lfloor \frac{2R-i}{4} \rfloor \text{ paths of length 1 if } R \geq \frac{i}{2}. \end{cases}$

(3) **if w is of odd length i,** then $C_R(w)$ consists of
$\begin{cases} \lfloor \frac{R-i}{2} \rfloor \text{ 4-cycles if } R \geq i, \\ R - \lfloor \frac{i}{2} \rfloor \text{ paths of length 2 if } \lfloor \frac{i}{2} \rfloor < R < i, \\ \lfloor \frac{R}{2} \rfloor - \lfloor \frac{R-i}{2} \rfloor \text{ paths of length 2 if } R \geq i, \\ 1 \text{ path of length 1 if } R \geq i. \end{cases}$

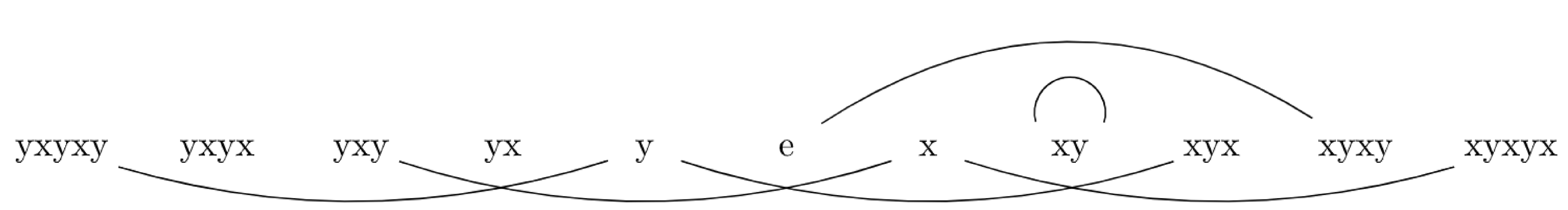


Figure 1: $C_5(xyxy)$ for AA.

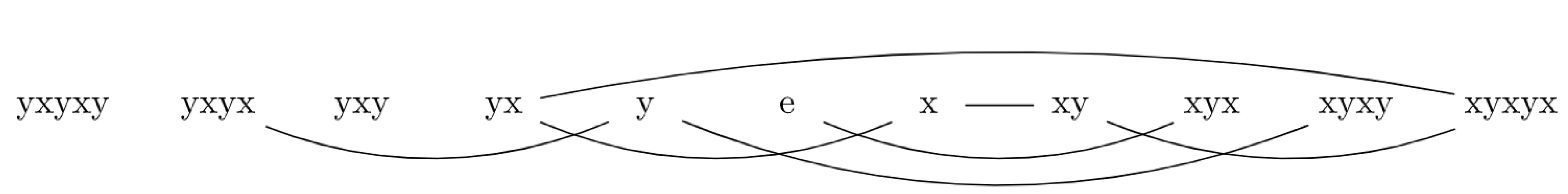


Figure 2: $C_5(xyx)$ for AA.

$\mathbb{Z}_2 * \mathbb{Z}_2$ Continued

Theorem. $C_R(w \notin AA^{-1})$ in $\mathbb{Z}_2 * \mathbb{Z}_2$)

Let $w \in \mathbb{Z}_2 * \mathbb{Z}_2$. Then,

(1) $C_R(e \notin AA^{-1})$ consists of $2R + 1$ self-loops.

(2) **if w is of even length $i \geq 2$,** write $R = k(\frac{i}{2}) + j$ for $j, k \in \mathbb{Z}_{\geq 0}$ and $j < \frac{i}{2}$. Then, $C_R(w \notin AA^{-1})$ consists of
$\begin{cases} 2j + 1 \text{ paths of length } k \text{ if } k \geq 1, \\ i - 2j - 1 \text{ paths of length } k - 1 \text{ if } k \geq 2. \end{cases}$

(3) **if w is of odd length i,** then $C_R(w \notin AA^{-1})$ consists of $R - \lfloor \frac{i}{2} \rfloor$ paths of length 1 if $R \geq \frac{i}{2}$.

Using the Lemma, we can find $\mathbb{P}(w \notin AA^{-1})$ in each of these cases where $A \subseteq B_R$ and R is sufficiently large.

Future Work

We hope to extend the condition graphs framework to

(1) Condition graphs $C(w_1, \ldots, w_k \notin S)$ on $\mathbb{Z}_2 * \mathbb{Z}_2$ involving multiple words;
(2) The free group F_2 (partial progress);
(3) Random subsets A where each element is included indpendently with probability p (not just $p = 1/2$).

On the free group on 2-generators F_2, we have made progress when all elements of A are of uniform length. In this case,

$$\mathbb{P}(w \notin AA^{-1}) = \begin{cases} 0 & |w| \text{ odd} \\ (\frac{3}{4})^{3^{R-|w|/2}} & |w| \text{ even and middle characters the same} \\ (\frac{3}{4})^{2*3^{R-|w|/2-1}} & |w| \text{ even and middle characters different.} \end{cases}$$

This work also suggests ways to compute the expected sizes $\mathbb{E}|AA|$ and $\mathbb{E}|AA^{-1}|$ as well as the variances $\text{Var}|AA|$ and $\text{Var}|AA^{-1}|$.

References

[1] Oleg Lazarev, Steven J. Miller, and Kevin O'Bryant.
Distribution of missing sums in sumsets.
Exp. Math., 22(2):132–156, 2013.

[2] G Martin and K O'Bryant.
Many sets have more sums than differences, Additive combinatorics, 287–305.
In *CRM Proc. Lecture Notes*, volume 43.

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