Sum and Difference Sets in Generalized Dihedral Groups

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SMALL REU 2022 at Williams College
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CANT 2023

May 25, 2023

Definitions

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Given a set of integers A, we define the sumset and difference set of A as follows:

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We want to compare the sizes of these two sets:

- $|A + A| > |A - A|$: A has more sums than differences (MSTD).
- $|A + A| = |A - A|$: A is sum-difference balanced.
- $|A + A| < |A - A|$: A has more differences than sums (MDTS).

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Fermat's last theorem says that $(A_n + A_n) \cap A_n = \emptyset$, where A_n is the set of positive n^{th} powers for $n \geq 3$.

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Theorem (Martin-O'Bryant, 2006)

Let P be any arithmetic progression with length n. On average, the difference set of a subset of P has 4 more elements than its sumset:

$$\frac{1}{2^n}\sum_{A\subseteq P}|A-A| \sim 2n-7,$$

$$\frac{1}{2^n}\sum_{A\subseteq P}|A+A| \sim 2n-11.$$

MSTD sets of integers

Theorem (Martin-O'Bryant, 2006)

For $n \geq 15$, *the number of MSTD subsets of* $\{0, 1, 2, \ldots, n-1\}$ *is at least* $(2 \cdot 10^{-7})2^n$.

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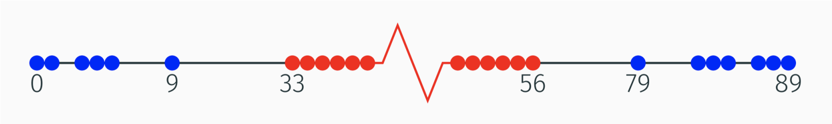

Example

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Let $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$.

$$A + A = \{0, 1, \ldots, 28\} \backslash \{1, 20, 27\}, \ |A + A| = 26,$$

$$A - A = \{-14, -13, \ldots, 14\} \backslash \{-13, -6, 6, 13\}, \ |A - A| = 25.$$

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Theorem (Miller-Vissuet 2014)

Let G_n be a family of finite groups such that $|G_n| \to \infty$. If $A_n \subseteq G_n$ is chosen uniformly at random, then

$$\mathbb{P}(A_n + A_n = A_n - A_n = G_n) \to 1 \text{ as } n \to \infty.$$

Dihedral groups

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$R+R$ and $-R+F$ contribute to $A+A$ and not $A-A$.
Only $R-R$ contributes to $A-A$ and not $A+A$.

Partitioning by size

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We further extended this piecemeal approach:

Lemma (A. et al. 2022+)

$\mathcal{S}_3$ has strictly more MSTD subsets than MDTS subsets.

Large subsets

Haviland et al. (2020) also showed that sufficiently large subsets must be sum-difference balanced:

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Given $A \subseteq D_{2n}$, if $|A| > n$, then $A + A = A - A = D_{2n}$.

It remains to show that $\mathcal{S}_m$ does not have more MDTS sets than MSTD sets for $4 \leq m \leq n$.

Composition of A

Our main new idea: further partition $\mathcal{S}_m$ by the number of rotation elements versus flip elements.

Writing each $A \subseteq D_{2n}$ as $R \cup F$ (rotations and flips), we have:

Lemma (A. et al. 2022+)

If $|R| > \frac{n}{2}$ or $|F| > \frac{n}{2}$, then A cannot be MDTS.

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Proof. Recall this table:

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We can only have $|A-A| > |A+A|$ if $R-R$ contains rotation elements that $A+A$ does not have. But if $|R| > \frac{n}{2}$ (resp. $|F| > \frac{n}{2}$), then $R+R$ (resp. $F+F$) contributes all of the possible rotations in D_{2n}.

Counting collisions

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For any n, more of the subsets in $\mathcal{S}_m$ are MSTD than MDTS for $6 \leq m \leq c \cdot \sqrt{n}$ where c is a global constant.

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Even more can be said if we further restrict m:

Theorem (A. et al. 2022+)

For any $\epsilon > 0$, there exist m_ϵ and c_ϵ such that for all $n \gg 0$, if $m_\epsilon \leq m \leq c_\epsilon \sqrt{n}$, the proportion of MSTD sets in $\mathcal{S}_m$ is at least $1 - \epsilon$.

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Let $|A| = m$, $|F| = k$, and $|R| = m - k$. Assuming no overlaps, and not counting $F + F$:

Type	A+A	A-A
Rotations	$\binom{m-k}{2} + (m - k)$	$2\binom{m-k}{2}$
Flips	$2(m - k)k$	$(m - k)k$

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This implies that, with no overlaps, A is MSTD if

$$\binom{m-k}{2} + (m-k) + 2(m-k)k \ > \ 2\binom{m-k}{2} + (m-k)k.$$

Collisions

Definition

Let $A \in \mathcal{S}_m$, and let $i = (a, b, c, d) \in A^4$. We call the event that $ab = cd$ (or equivalently, $d = c^{-1}ab$) a *collision*.

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For our purposes, we will disregard three types of collisions:

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These *redundant collisions* have already been accounted for in the previous analysis.

MSTD, counting overlaps

Let $|A| = m$, $|F| = k$, and $|R| = m - k$. Let X_A denote the number of nonredundant collisions in A. Then, A is MSTD if

$$\binom{m-k}{2} + (m-k) + 2(m-k)k - X_A \; > \; 2\binom{m-k}{2} + (m-k)k,$$

or, solving for k,

$$\frac{2m - \sqrt{m^2 - 6X_A}}{3} \; \leq \; k \; \leq \; \frac{2m + \sqrt{m^2 - 6X_A}}{3}$$

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$$\frac{2m - \sqrt{m^2 - 6X_A}}{3} \ \leq \ k \ \leq \ \frac{2m + \sqrt{m^2 - 6X_A}}{3}$$

Takeaway: If X_A is at most a small constant times m^2, then for most values of k, A is MSTD.

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When $m \leq 0.12\sqrt{n}$, this bound suffices to show that most subsets in $\mathcal{S}_m$ are MSTD. And, if we further restrict m, we can prove that a very high proportion of subsets in $\mathcal{S}_m$ are MSTD!

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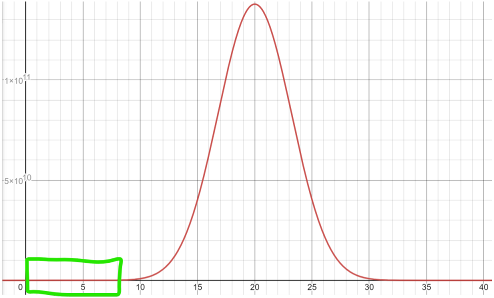

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However, recall that almost all subsets of D_{2n} are balanced. Computer-assisted methods suggest that a phase transition occurs around $m = O(\sqrt{n})$ where $\mathcal{S}_m$ goes from having mostly MSTD subsets to having mostly balanced subsets.

Generalizations

Generalized dihedral groups

Recall that for an abelian group G, the *generalized dihedral group* of G is

$$\mathrm{Dih}(G) = \mathbb{Z}/2 \ltimes G$$

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Our main theorems and methods for D_{2n} translate directly to $\mathrm{Dih}(G)$, as long as G doesn't have too many elements of order 2.

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We can also take $G = \mathbb{Z}^r$ if we restrict the $\mathbb{Z}^r$-components in $\mathrm{Dih}(\mathbb{Z}^r)$ to $[0, n-1]^r$:

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Proof idea: Construct a bijection $\mathbb{Z}/2 \ltimes [0, n-1]^r \to D_{2n^r}$ that preserves collisions.

Future work

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- $c\sqrt{n} < m \leq n$.
- Carefully count collisions.
- Analyze missed elements for m close to n.
- Construct injections from MDTS sets to MSTD sets in $\mathcal{S}_m$.

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Any results we prove for D_{2n} will hopefully translate to generalized dihedral groups.

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Theorem (A. et al. 2022+)

For prime n and a random set $A \subseteq D_{2n}$ with $|A| = m$, we have that

$$\mathbb{E}(|A-A|) = 2n - n\frac{\binom{n}{m}}{\binom{2n}{m}}2^m - n^2(n-1)\sum_{k=1}^{m-1}\frac{\binom{n+k-m-1}{m-k-1}\binom{n-k-1}{k-1}}{\binom{2n}{m}k(m-k)} - \frac{(n-1)(2n)\binom{n-m-1}{m-1}}{(m)\binom{2n}{m}}.$$

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Would also require understanding of variance.

Expected size for difference sets

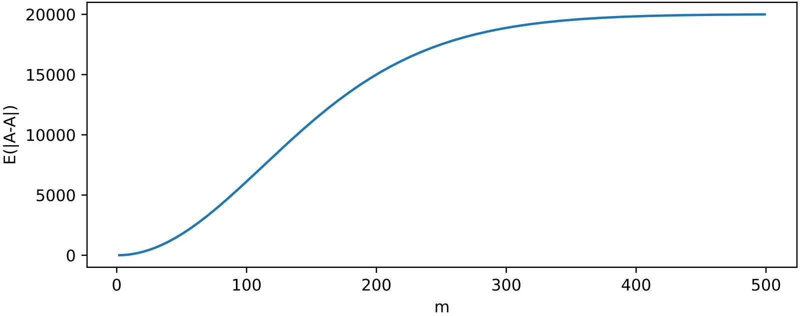

Figure: $\mathbb{E}(|A-A|)$ versus m for $n = 10007$.

Acknowledgments

Thank you!

This research was conducted as part of the **2022 SMALL REU program** at Williams College, and was supported by **NSF Grant DMS1947438**, **Harvey Mudd College**, and **Williams College** funds. We thank Steven J. Miller and our colleagues from the 2022 SMALL REU program for many helpful conversations.

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