When Almost All Sets Are Difference Dominated

Steven J Miller
Williams College

Steven.J.Miller@williams.edu
http://www.williams.edu/go/math/sjmiller/

Workshop on Combinatorial and Additive Number Theory
(CANT 2009)
CUNY Graduate Center, New York, May 2009
Summary

- History of the problem.
- Examples.
- Main results and proofs.
- Describe open problems.

This is joint work with Peter Hegarty, Brooke Orosz and Dan Scheinerman.
Introduction
A finite set of integers, \(|A|\) its size. Form

- **Sumset**: \(A + A = \{a_i + a_j : a_j, a_j \in A\}\).
- **Difference set**: \(A - A = \{a_i - a_j : a_i, a_j \in A\}\).
A finite set of integers, $|A|$ its size. Form

- **Sumset**: $A + A = \{a_i + a_j : a_i, a_j \in A\}$.
- **Difference set**: $A - A = \{a_i - a_j : a_i, a_j \in A\}$.

**Definition**

We say $A$ is **difference dominated** if $|A - A| > |A + A|$, **balanced** if $|A - A| = |A + A|$ and **sum dominated (or an MSTD set)** if $|A + A| > |A - A|$. 
Expect generic set to be difference dominated:

- addition is commutative, subtraction isn’t:
- Generic pair \((x, y)\) gives 1 sum, 2 differences.
Questions

Expect **generic** set to be difference dominated:
- addition is commutative, subtraction isn’t:
- Generic pair \((x, y)\) gives 1 sum, 2 differences.

**Questions**

- Do there exist sum-dominated sets?
- If yes, how many?
Examples
Examples

- Conway: \( \{0, 2, 3, 4, 7, 11, 12, 14\} \).

- Marica (1969): \( \{0, 1, 2, 4, 7, 8, 12, 14, 15\} \).

- Freiman and Pigarev (1973): \( \{0, 1, 2, 4, 5, 9, 12, 13, 14, 16, 17, 21, 24, 25, 26, 28, 29\} \).

- Computer search of random subsets of \( \{1, \ldots, 100\} \):
  \[ \{2, 6, 7, 9, 13, 14, 16, 18, 19, 22, 23, 25, 30, 31, 33, 37, 39, 41, 42, 45, 46, 47, 48, 49, 51, 52, 54, 57, 58, 59, 61, 64, 65, 66, 67, 68, 72, 73, 74, 75, 81, 83, 84, 87, 88, 91, 93, 94, 95, 98, 100\} \).

- Recently infinite families (Hegarty, Nathanson).
Key observation

If $A$ is an arithmetic progression, $|A + A| = |A - A|$. 
Infinite Families

Key observation

If $A$ is an arithmetic progression, $|A + A| = |A - A|$.

Proof:

- WLOG, $A = \{0, 1, \ldots, n\}$ as $A \rightarrow \alpha A + \beta$ doesn’t change $|A + A|, |A - A|$.
## Key observation

If $A$ is an arithmetic progression, $|A + A| = |A - A|$.

Proof:

- WLOG, $A = \{0, 1, \ldots, n\}$ as $A \rightarrow \alpha A + \beta$ doesn't change $|A + A|, |A - A|$.

- $A + A = \{0, \ldots, 2n\}, A - A = \{-n, \ldots, n\}$, both of size $2n + 1$. □
Previous Constructions

Most constructions perturb an arithmetic progression.

Example:

- MSTD set $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$.

- $A = \{0, 2\} \cup \{3, 7, 11\} \cup (14 - \{0, 2\}) \cup \{4\}$.
Example (Nathanson)

Theorem

$m, d, k \in \mathbb{N}$ with $m \geq 4$, $1 \leq d \leq m - 1$, $d \neq m/2$, $k \geq 3$ if $d < m/2$ else $k \geq 4$. Let

- $B = [0, m - 1] \setminus \{d\}$.
- $L = \{m - d, 2m - d, \ldots, km - d\}$.
- $a^* = (k + 1)m - 2d$.
- $A^* = B \cup L \cup (a^* - B)$.
- $A = A^* \cup \{m\}$.

Then $A$ is an MSTD set.
New Construction: Notation

- \([a, b] = \{k \in \mathbb{Z} : a \leq k \leq b\}\).

- \(A\) is a \(P_n\)-set if its sumset and its difference set contain all but the first and last \(n\) possible elements (and of course it may or may not contain some of these fringe elements).
Theorem (Miller-Scheinerman ’09)

- \( A = L \cup R \) be a \( P_n \), MSTD set where \( L \subset [1, n] \), \( R \subset [n + 1, 2n] \), and \( 1, 2n \in A \).
- Fix a \( k \geq n \) and let \( m \) be arbitrary.
- \( M \) any subset of \([n + k + 1, n + k + m]\) st no run of more than \( k \) missing elements. Assume \( n + k + 1 \not\in M \).
- Set \( A(M) = L \cup O_1 \cup M \cup O_2 \cup R' \), where \( O_1 = [n + 1, n + k] \), \( O_2 = [n + k + m + 1, n + 2k + m] \), and \( R' = R + 2k + m \).

Then \( A(M) \) is an MSTD set, and \( \exists C > 0 \) st the percentage of subsets of \( \{0, \ldots, r\} \) that are in this family (and thus are MSTD sets) is at least \( C / r^4 \).
Generalization: Miller-Orosz-Scheinerman

Can we find $A$ so that:

$$|\epsilon_1 A + \cdots + \epsilon_n A| > |\tilde{\epsilon}_1 A + \cdots + \tilde{\epsilon}_n A|,$$

where $\epsilon_i, \tilde{\epsilon}_i \in \{-1, 1\}$.

Consider the generalized sumset

$$f_{j_1, j_2}(A) = A + A + \cdots + A - A - A - \cdots - A,$$

where there are $j_1$ pluses and $j_2$ minuses, and set $j = j_1 + j_2$.

$P^j_n$-set

Let $A \subset [1, k]$ with $1, k, \in A$. We say $A$ is a $P^j_n$-set if any $f_{j_1, j_2}(A)$ contains all but the first $n$ and last $n$ possible elements. (Note that a $P^2_n$-set is the same as what we called a $P_n$-set earlier.)
Generalization: Miller-Orosz-Scheinerman

**Conjecture (MOS)**

For any $f_{j_1, j_2}$ and $f_{j_1', j_2'}$, there exists a finite set of integers $A$ which is (1) a $P_n^j$-set; (2) $A \subset [1, 2n]$ and $1, 2n \in A$; and (3) $|f_{j_1, j_2}(A)| > |f_{j_1', j_2'}(A)|$.

- Problem is finding an $A$ with $|f_{j_1, j_2}(A)| > |f_{j_1', j_2'}(A)|$; once we find such a set, we can mirror previous construction and construct infinitely many.
- Theorem: The conjecture is true for $j \in \{2, 3\}$. 
Proof of Generalization

• Needed input set for $j = 3$: $A = \{1, 2, 5, 6, 16, 19, 22, 26, 32, 34, 35, 39, 43, 48, 49, 50\}$. Found by taking elements in $\{2, \ldots, 49\}$ to be in $A$ with probability $1/3$; it took about $300000$ sets to find the first one satisfying our conditions. To be a $P^3_{25}$-set we need to have $A + A + A \supset [n + 3, 6n - n] = [28, 125]$ and $A + A - A \supset [-n + 2, 3n - 1] = [-23, 74]$. Have $A + A + A = [3, 150]$ (all possible elements), while $A + A - A = [-48, 99]\{-34\}$ (i.e., all but -34). Thus $A$ is a $P^3_{25}$-set satisfying $|A + A + A| > |A + A - A|$, and have the needed example.

• Could also take $A = \{1, 2, 3, 4, 8, 12, 18, 22, 23, 25, 26, 29, 30, 31, 32, 34, 45, 46, 49, 50\}$. 
Results
Probability Review

$X$ random variable with density $f(x)$ means

- $f(x) \geq 0$;
- $\int_{-\infty}^{\infty} f(x) = 1$;
- $\text{Prob}(X \in [a, b]) = \int_{a}^{b} f(x) \, dx$.

Key quantities:

- Expected (Average) Value: $\mathbb{E}[X] = \int xf(x) \, dx$.
- Variance: $\sigma^2 = \int (x - \mathbb{E}[X])^2 f(x) \, dx$. 
Binomial model

Binomial model, parameter $p(n)$

Each $k \in \{0, \ldots, n\}$ is in $A$ with probability $p(n)$.

Consider uniform model ($p(n) = 1/2$):

- Let $A \in \{0, \ldots, n\}$. Most elements in $\{0, \ldots, 2n\}$ in $A + A$ and in $\{-n, \ldots, n\}$ in $A - A$.

- $\mathbb{E}[|A + A|] = 2n - 11$, $\mathbb{E}[|A - A|] = 2n - 7$. 

Theorem

Let $A$ be chosen from $\{0, \ldots, N\}$ according to the binomial model with constant parameter $p$ (thus $k \in A$ with probability $p$). At least $k_{SD; p} 2^{N+1}$ subsets are sum dominated.
Theorem

Let $A$ be chosen from $\{0, \ldots, N\}$ according to the binomial model with constant parameter $p$ (thus $k \in A$ with probability $p$). At least $k_{SD;p}2^{N+1}$ subsets are sum dominated.

- $k_{SD;1/2} \geq 10^{-7}$, expect about $10^{-3}$.  

Theorem

Let $A$ be chosen from $\{0, \ldots, N\}$ according to the binomial model with constant parameter $p$ (thus $k \in A$ with probability $p$). At least $k_{SD;p} 2^{N+1}$ subsets are sum dominated.

- $k_{SD;1/2} \geq 10^{-7}$, expect about $10^{-3}$.

Proof ($p = 1/2$): Generically $|A| = \frac{N}{2} + O(\sqrt{N})$.

- about $\frac{N}{4} - \frac{|N-k|}{4}$ ways write $k \in A + A$.
- about $\frac{N}{4} - \frac{|k|}{4}$ ways write $k \in A - A$.
- Almost all numbers that can be in $A \pm A$ are.
- Win by controlling fringes.
Notation

- $X \sim f(N)$ means $\forall \epsilon_1, \epsilon_2 > 0, \exists N_{\epsilon_1, \epsilon_2}$ st $\forall N \geq N_{\epsilon_1, \epsilon_2}$

  $$\text{Prob}(X \notin [(1 - \epsilon_1)f(N), (1 + \epsilon_1)f(N)]) < \epsilon_2.$$
Notation

- \( X \sim f(N) \) means \( \forall \epsilon_1, \epsilon_2 > 0, \exists N_{\epsilon_1, \epsilon_2} \text{ st } \forall N \geq N_{\epsilon_1, \epsilon_2} \)
  \[ \text{Prob} \left( X \notin [(1 - \epsilon_1)f(N), (1 + \epsilon_1)f(N)] \right) < \epsilon_2. \]

- \( S = |A + A|, \ D = |A - A|, \)
  \( S^c = 2N + 1 - S, \ D^c = 2N + 1 - D. \)
Notation

• $X \sim f(N)$ means $\forall \epsilon_1, \epsilon_2 > 0, \exists N_{\epsilon_1, \epsilon_2}$ st $\forall N \geq N_{\epsilon_1, \epsilon_2}$

$$\text{Prob}(X \notin [(1 - \epsilon_1)f(N), (1 + \epsilon_1)f(N)]) < \epsilon_2.$$ 

• $S = |A + A|, D = |A - A|,$

$\quad S^c = 2N + 1 - S, D^c = 2N + 1 - D.$

New model: Binomial with parameter $p(N)$:

• $1/N = o(p(N))$ and $p(N) = o(1);$ 

• $\text{Prob}(k \in A) = p(N).$

Conjecture (Martin-O’Bryant)

As $N \to \infty$, $A$ is a.s. difference dominated.
Main Result

**Theorem (Hegarty-Miller)**

\[ p(N) \text{ as above, } g(x) = 2^{e^{-x}-(1-x)/x}. \]

- \[ p(N) = o(N^{-1/2}): D \sim 2S \sim (Np(N))^2; \]
- \[ p(N) = cN^{-1/2}: D \sim g(c^2)N, S \sim g\left(\frac{c^2}{2}\right)N \]
  \( (c \to 0, D/S \to 2; c \to \infty, D/S \to 1); \)
- \[ N^{-1/2} = o(p(N)): S^c \sim 2D^c \sim 4/p(N)^2. \]

Can generalize to binary linear forms, still have critical threshold.
Inputs

Key input: recent strong concentration results of Kim and Vu (Applications: combinatorial number theory, random graphs, ...).
Key input: recent strong concentration results of Kim and Vu (Applications: combinatorial number theory, random graphs, ...).

Example (Chernoff): $t_i$ iid binary random variables, $Y = \sum_{i=1}^{n} t_i$, then

$$\forall \lambda > 0 : \text{Prob} \left( |Y - \mathbb{E}[Y]| \geq \sqrt{\lambda n} \right) \leq 2e^{-\lambda/2}.$$
Key input: recent strong concentration results of Kim and Vu (Applications: combinatorial number theory, random graphs, ...).

Example (Chernoff): $t_i$ iid binary random variables, $Y = \sum_{i=1}^{n} t_i$, then

$$\forall \lambda > 0 : \text{Prob} \left( |Y - \mathbb{E}[Y]| \geq \sqrt{\lambda n} \right) \leq 2e^{-\lambda/2}.$$  

Need to allow dependent random variables.
Inputs

Key input: recent strong concentration results of Kim and Vu (Applications: combinatorial number theory, random graphs, ...).

Example (Chernoff): \( t_i \) iid binary random variables, \( Y = \sum_{i=1}^{n} t_i \), then

\[
\forall \lambda > 0 : \quad \text{Prob} \left( |Y - \mathbb{E}[Y]| \geq \sqrt{\lambda n} \right) \leq 2e^{-\lambda/2}.
\]

Need to allow dependent random variables.
Sketch of proofs: \( \mathcal{X} \in \{S, D, S^c, D^c\} \).

1. Prove \( \mathbb{E}[\mathcal{X}] \) behaves asymptotically as claimed;
2. Prove \( \mathcal{X} \) is strongly concentrated about mean.
Proofs
Note: only need strong concentration for $N^{-1/2} = o(p(N))$. 
Note: only need strong concentration for $N^{-1/2} = o(p(N))$.

Will assume $p(N) = o(N^{-1/2})$ as proofs are elementary (i.e., Chebyshev: $\text{Prob}(|Y - \mathbb{E}[Y]| \geq k\sigma_Y) \leq 1/k^2$).
Note: only need strong concentration for $N^{-1/2} = o(p(N))$.

Will assume $p(N) = o(N^{-1/2})$ as proofs are elementary (i.e., Chebyshev: $\text{Prob}(|Y - \mathbb{E}[Y]| \geq k\sigma_Y) \leq 1/k^2$).

For convenience let $p(N) = N^{-\delta}$, $\delta \in (1/2, 1)$.

IID binary indicator variables:

$$X_{n;N} = \begin{cases} 1 & \text{with probability } N^{-\delta} \\ 0 & \text{with probability } 1 - N^{-\delta}. \end{cases}$$

$$X = \sum_{i=1}^{N} X_{n;N}, \quad \mathbb{E}[X] = N^{1-\delta}.$$
Lemma

\[ P_1(N) = 4N^{-(1-\delta)}, \]
\[ \mathcal{O} = \# \{(m, n) : m < n \in \{1, \ldots, N\} \cap A\}. \]

With probability at least \(1 - P_1(N)\) have

1. \( X \in \left[ \frac{1}{2} N^{1-\delta}, \frac{3}{2} N^{1-\delta} \right]. \)

2. \[ \frac{1}{2} N^{1-\delta} \left( \frac{1}{2} N^{1-\delta} - 1 \right) \leq \mathcal{O} \leq \frac{3}{2} N^{1-\delta} \left( \frac{3}{2} N^{1-\delta} - 1 \right). \]
Proof

Lemma

\[ P_1(N) = 4N^{-(1-\delta)}, \]
\[ \mathcal{O} = \# \{(m, n) : m < n \in \{1, \ldots, N\} \cap A\}. \]

With probability at least \(1 - P_1(N)\) have

1. \( X \in \left[ \frac{1}{2}N^{1-\delta}, \frac{3}{2}N^{1-\delta} \right]. \)
2. \( \frac{1}{2}N^{1-\delta}(\frac{1}{2}N^{1-\delta}-1) \leq \mathcal{O} \leq \frac{3}{2}N^{1-\delta}(\frac{3}{2}N^{1-\delta}-1). \)

Proof:

1. (1) is Chebyshev: \( \text{Var}(X) = N\text{Var}(X_{n;N}) \leq N^{1-\delta}. \)
2. (2) follows from (1) and \( \binom{r}{2} \) ways to choose 2 from \( r \).
Concentration

Lemma

- \( f(\delta) = \min\left(\frac{1}{2}, \frac{3\delta - 1}{2}\right) \), \( g(\delta) \) any function st 0 < \( g(\delta) < f(\delta) \).
- \( p(N) = N^{-\delta} \), \( \delta \in (1/2, 1) \), \( P_1(N) = 4N^{-(1-\delta)} \),
  \( P_2(N) = CN^{-(f(\delta) - g(\delta))} \).

With probability at least 1 − \( P_1(N) − P_2(N) \) have
\( D/S = 2 + O(N^{-g(\delta)}) \).
Concentration

Lemma

- \( f(\delta) = \min \left( \frac{1}{2}, \frac{3\delta - 1}{2} \right) \), \( g(\delta) \) any function st \( 0 < g(\delta) < f(\delta) \).
- \( p(N) = N^{-\delta}, \delta \in (1/2, 1), \ P_1(N) = 4N^{-(1-\delta)}, \ P_2(N) = CN^{-(f(\delta)-g(\delta))} \).

With probability at least \( 1 - P_1(N) - P_2(N) \) have \( \frac{D}{S} = 2 + O(N^{-g(\delta)}) \).

Proof: Show \( D \sim 2O + O(N^{3-4\delta}), \ S \sim O + O(N^{3-4\delta}) \).

As \( O \) is of size \( N^{2-2\delta} \) with high probability, need \( 2 - 2\delta > 3 - 4\delta \) or \( \delta > 1/2 \).
Analysis of $\mathcal{D}$

Contribution from ‘diagonal’ terms lower order, ignore.
Analysis of $D$

Contribution from ‘diagonal’ terms lower order, ignore.

Difficulty: $(m, n)$ and $(m', n')$ could yield same differences.
Contribution from ‘diagonal’ terms lower order, ignore.

Difficulty: \((m, n)\) and \((m', n')\) could yield same differences.

Notation: \(m < n, \, m' < n', \, m \leq m'\),

\[
Y_{m,n,m',n'} = \begin{cases} 
1 & \text{if } n - m = n' - m' \\
0 & \text{otherwise.} 
\end{cases}
\]
Analysis of $\mathcal{D}$

Contribution from ‘diagonal’ terms lower order, ignore.

Difficulty: $(m, n)$ and $(m', n')$ could yield same differences.

Notation: $m < n$, $m' < n'$, $m \leq m'$,

$$Y_{m,n,m',n'} = \begin{cases} 
1 & \text{if } n - m = n' - m' \\
0 & \text{otherwise}.
\end{cases}$$

$$\mathbb{E}[Y] \leq N^3 \cdot N^{-4\delta} + N^2 \cdot N^{-3\delta} \leq 2N^{3-4\delta}. \text{ As } \delta > 1/2, \text{ expected number bad pairs } \ll |\mathcal{O}|.$$
Analysis of $\mathcal{D}$

Contribution from ‘diagonal’ terms lower order, ignore.

Difficulty: $(m, n)$ and $(m', n')$ could yield same differences.

Notation: $m < n, m' < n', m \leq m'$,

$$Y_{m,n,m',n'} = \begin{cases} 1 & \text{if } n - m = n' - m' \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathbb{E}[Y] \leq N^3 \cdot N^{-4\delta} + N^2 \cdot N^{-3\delta} \leq 2N^{3-4\delta}.$$

As $\delta > 1/2$, expected number bad pairs $\ll |\mathcal{O}|$.

Claim: $\sigma_Y \leq N^{r(\delta)}$ with $r(\delta) = \frac{1}{2} \max(3 - 4\delta, 5 - 7\delta)$. This and Chebyshev conclude proof of theorem.
Proof of claim

Cannot use CLT as $Y_{m,n,m',n'}$ are not independent.
Proof of claim

Cannot use CLT as \( Y_{m,n,m',n'} \) are not independent.

Use \( \text{Var}(U + V) \leq 2\text{Var}(U) + 2\text{Var}(V) \).
Proof of claim

Cannot use CLT as \( Y_{m,n,m',n'} \) are not independent.

Use \( \text{Var}(U + V) \leq 2\text{Var}(U) + 2\text{Var}(V) \).

Write

\[
\sum Y_{m,n,m',n'} = \sum U_{m,n,m',n'} + \sum V_{m,n,n'}
\]

with all indices distinct (at most one in common, if so must be \( n = m' \)).

\[
\text{Var}(U) = \sum \text{Var}(U_{m,n,m',n'}) + 2 \sum \text{CoVar}(U_{m,n,m',n'}, U_{\tilde{m},\tilde{n},\tilde{m}',\tilde{n}'})
\]

\[
\text{Var}(V) = \sum \text{Var}(V_{m,n,m',n'}) + 2 \sum \text{CoVar}(V_{m,n,m',n'}, V_{\tilde{m},\tilde{n},\tilde{m}',\tilde{n}'})
\]
Analyzing $\text{Var}(U_{m,n,m',n'})$

At most $N^3$ tuples.

Each has variance $N^{-4\delta} - N^{-8\delta} \leq N^{-4\delta}$.

Thus $\sum \text{Var}(U_{m,n,m',n'}) \leq N^{3-4\delta}$. 
Analyzing $\text{CoVar}(U_{m,n,m',n'}, U_{\tilde{m},\tilde{n},\tilde{m}',\tilde{n}'}$)

- All 8 indices distinct: independent, covariance of 0.
- 7 indices distinct: At most $N^3$ choices for first tuple, at most $N^2$ for second, get

$$\mathbb{E}[U_{(1)} U_{(2)}] - \mathbb{E}[U_{(1)}] \mathbb{E}[U_{(2)}] = N^{-7\delta} - N^{-4\delta} N^{-4\delta} \leq N^{-7\delta}.$$

- Argue similarly for rest, get $\ll N^{5-7\delta} + N^{3-4\delta}$. 
Open Problems
Probability $k$ in an MSTD set (uniform model)

$$
\gamma(k, n) := \text{Prob}(k \in A : A \subset [1, n] \text{ is an MSTD set})
$$

Figure: Observed $\gamma(k, 100)$, random sample 4458 MSTD sets.

Conjecture

Fix a constant $0 < \alpha < 1$. Then $\lim_{n \to \infty} \gamma(k, n) = 1/2$ for $\lfloor \alpha n \rfloor \leq k \leq n - \lfloor \alpha n \rfloor$. 
Generalization of main result

Theorem (Hegarty-M): Binomial model with parameter $p(N)$ as before, $u, v$ be non-zero integers with $u \geq |v|$, $\gcd(u, v) = 1$ and $(u, v) \neq (1, 1)$. Put $f(x, y) := ux + vy$ and let $D_f$ denote the random variable $|f(A)|$. Then the following three situations arise:

1. $p(N) = o(N^{-1/2})$: Then
   $$D_f \sim (N \cdot p(N))^2.$$  

2. $p(N) = c \cdot N^{-1/2}$ for some $c \in (0, \infty)$: Define the function $g_{u,v} : (0, \infty) \to (0, u + |v|)$ by
   $$g_{u,v}(x) := (u + |v|) - 2|v| \left(\frac{1 - e^{-x}}{x}\right) - (u - |v|)e^{-x}.$$  
   Then
   $$D_f \sim g_{u,v} \left(\frac{c^2}{u}\right) N.$$  

3. $N^{-1/2} = o(p(N))$: Let $D_f^c := (u + |v|)N - D_f$. Then $D_f^c \sim \frac{2u|v|}{p(N)^2}$.  

Generalization of main results (cont)

Let $f, g$ be two binary linear forms. Say $f$ dominates $g$ for the parameter $p(N)$ if, as $N \to \infty$, $|f(A)| > |g(A)|$ almost surely when $A$ is a random subset (binomial model with parameter $p(N)$).

Theorem (Hegarty-M): $f(x, y) = u_1x + u_2y$ and $g(x, y) = u_2x + g_2y$, where $u_i \geq |v_i| > 0$, $\gcd(u_i, v_i) = 1$ and $(u_2, v_2) \neq (u_1, \pm v_1)$. Let

$$\alpha(u, v) := \frac{1}{u^2} \left( \frac{|v|}{3} + \frac{u - |v|}{2} \right) = \frac{3u - |v|}{6u^2}.$$ 

The following two situations can be distinguished:

- $u_1 + |v_1| \geq u_2 + |v_2|$ and $\alpha(u_1, v_1) < \alpha(u_2, v_2)$. Then $f$ dominates $g$ for all $p$ such that $N^{-3/5} = o(p(N))$ and $p(N) = o(1)$. In particular, every other difference form dominates the form $x - y$ in this range.

- $u_1 + |v_1| > u_2 + |v_2|$ and $\alpha(u_1, v_1) > \alpha(u_2, v_2)$. Then there exists $c_{f,g} > 0$ such that one form dominates for $p(N) < cN^{-1/2}$ ($c < c_{f,g}$) and other dominates for $p(N) > cN^{-1/2}$ ($c > c_{f,g}$).
One unresolved matter is the comparison of arbitrary difference forms in the range where $N^{-3/4} = O(p)$ and $p = O(N^{-3/5})$. Note that the property of one binary form dominating another is not monotone, or even convex.

A very tantalizing problem is to investigate what happens while crossing a sharp threshold.

One can ask if the various concentration estimates can be improved (i.e., made explicit).
Programs
Mathematica Code: Computing Sum/Difference Set

```mathematica
setA = {1, 2, 5, 7, 11, 13, 17, 19};
sumset = {};
diffset = {};
n = Length[setA];
For[i = 1, i <= n, i++,
    For[j = 1, j <= n, j++,
        {sum = setA[[i]] + setA[[j]];
         diff = setA[[i]] - setA[[j]];
         If[MemberQ[sumset, sum] == False, sumset = AppendTo[sumset, sum]];
         If[MemberQ[diffset, diff] == False, diffset = AppendTo[diffset, diff]];
    ];
    sumset = Sort[sumset];
diffset = Sort[diffset];
Print[sumset];
Print[diffset];
Print["Size of sumset = ", Length[sumset], ", size of difference set = ", Length[diffset], "."];
```

Bibliography


