Maass Forms and Random Matrix Theory

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1. Maass

1.1 Introduction

Modular forms are cool GL_2 automorphic forms. But Iwaniec, Luo, and Sarnak pounced on that family from the get-go. GL_1 is in safe hands with Rubinstein, Gao, Ozluk-Snyder, Fouvry-Iwaniec, Fiorilli-Miller, ... So why not complete GL_2?

Let's gather the results of our preliminary messing around (note: here everything was made "messing around" by the very strong work of Kuznetsov). Let $K \in 4\mathbb{Z}^+$ be a large positive multiple of 4, which we'll specify later. Let $\tilde{h} \in C^\infty\left(\left(-\frac{1}{2K}, \frac{1}{2K}\right)\right)$ be odd. Let $h := \hat{\tilde{h}}^K$, a (large) power of the Fourier transform of $\tilde{h}$. We assume $K \geq 8$. We use the weight function

$$h_T := r \mapsto \frac{h\left(\frac{ir}{T}\right)}{\cosh\left(\frac{\pi r}{T}\right)},$$

and we take T to be a (large) odd integer. We study the behavior of the one-level densities of Maass form L-functions with spectral parameter near T (weighted by h_T), as $T \to \infty$:

$$\frac{1}{\sum_j \frac{h(t_j)}{\|u_j\|^2}} \sum_j \frac{h(t_j)}{\|u_j\|^2} \sum_j \phi\left(\frac{\gamma}{2\pi} \log R\right).$$

All of our constants may depend on h (and hence K). Furthermore, let η be such that

$$\operatorname{Supp} \hat{\phi} \subseteq [-\eta, \eta]\,.$$

All of our constants may also depend on η. We prove the analytic conductor of our family is T^2, and

$$\sum_u \frac{h_T(t_u)}{||u||^2} \asymp T^2.$$

The Eisenstein and diagonal terms of Kuznetsov present us no problem. Determining the 1-level density reduces to proving the following.

Theorem:

$$\sum_{p \leq T^{2\eta}} \frac{\log p}{\sqrt{p}\log T} \hat{\phi}\left(\frac{\log p}{2\log T}\right) \sum_{c \geq 1} \frac{S(p,1;c)}{c} \int_{\mathbb{R}} J_{2ir}\left(\frac{4\pi\sqrt{p}}{c}\right) \frac{rh(ir)}{\cosh(\pi r)\cosh\left(\frac{\pi r}{T}\right)} dr \ll T^{2-\epsilon}$$

for some $\epsilon > 0$.

Using the commutativity of addition, after some work we see that the Kuznetsov trace formula is equivalent to the following: for $h : \{x + iy \,||y| < \frac{1}{2} + \epsilon\} \to \mathbb{C}$ holomorphic, even, and such that $h(x+iy) \ll (1+|x|)^{-2-\delta}$,

$$\begin{aligned}\sum_u \frac{h(t_u)}{||u||^2} \lambda_m \bar{\lambda}_n = {} & \frac{\delta_{m,n}}{\pi} \int_{\mathbb{R}} rh(r) \tanh(r) dr \\ & - \frac{1}{\pi} \int_{\mathbb{R}} m^{ir} \sigma_{ir}(m) n^{-ir} \sigma_{-ir}(n) \frac{h(r)}{|\zeta(1+2ir)|^2} dr \\ & + \frac{2i}{\pi} \sum_{c \geq 1} \frac{S(m,n;c)}{c} \int_{\mathbb{R}} J_{2ir}\left(\frac{4\pi\sqrt{mn}}{c}\right) \frac{rh(r)}{\cosh(\pi r)} dr,\end{aligned}$$

the sum taken over the Hecke-Maass eigenforms of level 1. Here λ_m are the Hecke eigenvalues, $||u||$ is the L^2 norm of u (on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$), $\sigma_k(n) := \sum_{d|n} n^k$, $J_\alpha(x)$ is the usual Bessel function, and $S(m,n;c)$ is the usual Kloosterman sum. Using Poisson summation, the method of stationary phase, and some other nonsense, we are able to determine the one-level density for test functions whose Fourier transform is supported in $[-1-\epsilon, 1+\epsilon]$ for an $\epsilon \geq 0$. This allows us to uniquely determine the corresponding classical compact group.

2. Random Matrix Theory

2.1 Introduction

An important problem in random matrix theory involves investigating the distribution of eigenvalues of random matrix ensembles. Such a study has applications from nuclear physics to number theory. Previous work has given the eigenvalue distribution of real symmetric matrices and Toeplitz matrices. We provide a way to investigate behavior between these two previously studied ensembles by looking at the Signed Toeplitz matrix ensemble, which are constant along the diagonal up to a randomly chosen sign for each entry:

$$\begin{pmatrix} \epsilon_{11}b_0 & \epsilon_{12}b_1 & \epsilon_{13}b_2 & \epsilon_{14}b_3 \\ \epsilon_{21}b_1 & \epsilon_{22}b_0 & \epsilon_{23}b_1 & \epsilon_{24}b_2 \\ \epsilon_{31}b_2 & \epsilon_{32}b_1 & \epsilon_{33}b_0 & \epsilon_{34}b_1 \\ \epsilon_{41}b_3 & \epsilon_{42}b_2 & \epsilon_{43}b_1 & \epsilon_{44}b_0 \end{pmatrix}$$

where $\epsilon_{ij} = \epsilon_{ji} \in \{1, -1\}$ and $p = \mathbb{P}(\epsilon_{ij}) = 1$.

2.2 Methods

Markov's Method of Moments We attempt to show a typical eigenvalue measure $\mu_{A,N}(x)$ converges to a probability distribution P by controlling convergence of average moments of the measures as $N \to \infty$ to the moments of P.

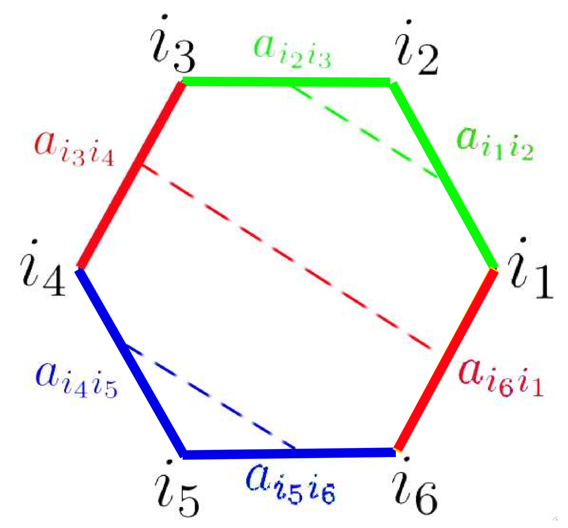
Eigenvalue Trace Lemma For any non-negative integer k, if A is an $N \times N$ matrix with eigenvalues $\lambda_i(A)$, then

$$\operatorname{Trace}\left(A^k\right) = \sum_{i=1}^N \lambda_i(A)^k\,.$$

We get the following formula for the average k^{th} moment, $M_k(N) = \mathbb{E}[M_k(A_N)]$, is:

$$\frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \leq i_1, \ldots, i_k \leq N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

Circle Configurations We represent the different terms in the sums as ways of pairing vertices on a circle. For example, a configuration of the 6^{th} moment:



2.3 Results: LMT '12

2.3.1 Theorem 1: For palindromic Toeplitz matrices, the depression of the contribution depends only on the crossing number.

- Dependency does not hold for doubly palindromic Toeplitz, consider 6th moment.

2.3.2 Theorem 2: Given any matrix ensemble, when $p = 1/2$, the limiting spectral measure is the semicircle distribution (special dependencies allowed between matrix elements).

- In noncrossing, contrib. at most $(x(c)-1)(2p-1)^4 + 1$ and at least $(x(c)-1)(2p-1)^{2k} + 1$. In crossing, contrib. at most $x(c)(2p-1)^{e(c)}$ and at least $x(c)(2p-1)^{2k}$.

2.3.3 Theorem 3: Any distribution that had unbounded or bounded support before weighting still has unbounded or bounded support after weighting (for general real symmetric matrix ensemble).

- In noncrossing, contribution to the $2k$th moment reduced from $x(c)$ to at most $(2p-1)^2(x(c)-1)+1$. In crossing, contribution to $2k$th moment reduced from $x(c)$ to at most $(2p-1)^2 x(c)$.
- When $p = \frac{1}{2}$, obtain semicircle distribution.