Low-lying zeros of cuspidal Maass forms

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Introduction
**L-functions**

$L$-functions generalizes the Riemann zeta-function:

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s, f)^{-1}, \quad \text{Re}(s) > 1.
\]

**Explicit Formula:** Relates sums over zeros to sums over primes.

**Functional Equation:**

\[
\Lambda(s, f) = \Lambda_{\infty}(s, f)L(s, f) = \Lambda(1 - s, f).
\]

**Generalized Riemann Hypothesis (RH):**

All non-trivial zeros have \(\text{Re}(s) = \frac{1}{2}\); can write zeros as \(\frac{1}{2} + i\gamma\).
Measures of Spacings: $n$-Level Density

$n$-level density for one function

$$D_{n,f}(\phi) = \sum_{j_1, \ldots, j_n \text{ distinct}} \phi_1\left(\gamma_{f}^{(j_1)}\right) \cdots \phi_n\left(\gamma_{f}^{(j_n)}\right)$$

- Test function $\phi(x) := \prod_i \phi_i(x_i)$, $\phi_i$ is even Schwartz function.
- Fourier Transforms $\hat{\phi}$ has compact support: $(-\sigma, \sigma)$.
- Zeros scaled by $L_f$.
- Most of contribution is from low zeros.
Katz-Sarnak Conjecture

Conjecture (Katz-Sarnak)
(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Need to average $n$-level density over a family and take the limit of this parameter; as $|N| \rightarrow \infty$,

$$
\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) \rightarrow \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx.
$$
Cuspidal Maass Forms
Maass Forms

Definition: Maass Forms

A Maass form on a group $\Gamma \subset \text{PSL}(2, \mathbb{R})$ is a function $f : \mathcal{H} \to \mathbb{R}$ which satisfies:

1. $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
2. $f$ vanishes at the cusps of $\Gamma$, and
3. $\Delta f = \lambda f$ for some $\lambda = s(1 - s) > 0$, where

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace-Beltrami operator on $\mathcal{H}$.

- Coefficients contain information about partitions.
- For full modular group, $s = 1/2 + it_j$ with $t_j \in \mathbb{R}$.
- Test Katz-Sarnak conjecture.
Write Fourier expansion of Maass form $u_j$ as

$$u_j(z) = \cosh(t_j) \sum_{n \neq 0} \sqrt{y} \lambda_j(n) K_{it_j}(2\pi |n| y) e^{2\pi i nx}.$$
Write Fourier expansion of Maass form $u_j$ as

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Define $L$-function attached to $u_j$ as

$$L(s, u_j) = \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1}$$

where $\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1, \quad \lambda_j(1) = 1$. 

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where $\alpha_j(p) + \beta_j(p) = \lambda_j(p)$, $\alpha_j(p)\beta_j(p) = 1$, $\lambda_j(1) = 1$.

Also,

$$\lambda_j(p) \ll p^{7/64}.$$
Recall for Katz-Sarnak Conjecture,

\[ \frac{1}{|F_N|} \sum_{f \in F_N} D_{n,f}(\phi) = \frac{1}{|F_N|} \sum_{f \in F_N} \sum_{j_1, \ldots, j_n \neq \pm j_k} \prod_{i} \phi_i \left( L_{f, \gamma_E}^{(j_i)} \right) \]

\[ \rightarrow \int \cdots \int \phi(x) W_{n,G}(x) dx. \]
Recall for Katz-Sarnak Conjecture,

\[
\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{j_1, \ldots, j_n \neq \pm j_k} \prod_{i} \phi_i \left( L_f \gamma_E^{(j_i)} \right) \\
\to \int \cdots \int \phi(x) W_{n,\gamma(\mathcal{F})}(x) dx.
\]

For Dirichlet/cuspidal newform $L$-functions, there are many with a given conductor.

**Problem:** For Maass forms, expect at most one with a given conductor.
Solution: Average over Laplace eigenvalues $\lambda_f = 1/4 + t_j^2$. 
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two choices for the weight function $h_T$:

$$h_{1,T}(t_j) = \exp(-t_j^2/T^2),$$

which picks out eigenvalues near the origin,
Solution: Average over Laplace eigenvalues $\lambda_f = 1/4 + t_j^2$.

Two choices for the weight function $h_T$:

$$h_{1,T}(t_j) = \exp\left(-\frac{t_j^2}{T^2}\right),$$

which picks out eigenvalues near the origin, or

$$h_{2,T}(t_j) = \exp\left(-\frac{(t_j - T)^2}{L^2}\right) + \exp\left(-\frac{(t_j + T)^2}{L^2}\right),$$

which picks out eigenvalues centered at $\pm T$. 
Solution: Average over Laplace eigenvalues $\lambda_f = 1/4 + t_j^2$.

- two choices for the weight function $h_T$:

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\]

which picks out eigenvalues centered at $\pm T$.

- Weighted 1-level density becomes

\[
\frac{1}{\sum_j h_T(t_j) \|u_j\|^2} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D_{n,u_j}(\phi)
\]

\[
= \frac{1}{\sum_j h_T(t_j) \|u_j\|^2} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \sum \prod_{i} \phi_i \left( \frac{\gamma}{2\pi} \log R \right)
\]
Results
1-Level Density

1-level density for one function

\[ D(u_j; \phi) = \sum_{\gamma} \phi \left( \frac{\gamma}{2\pi} \log R \right) \]
1-Level Density

1-level density for one function

\[ D(u_j; \phi) = \text{Terms involving } \Gamma + \frac{2}{\log R} \sum_p \frac{\log p}{p} \hat{\phi} \left( \frac{2 \log p}{\log R} \right) \]

\[ - \sum_p \frac{2 \lambda_j(p) \log p}{p^{1/2} \log R} \hat{\phi} \left( \frac{\log p}{\log R} \right) - \sum_p \frac{2 \lambda_j(p^2) \log p}{p \log R} \hat{\phi} \left( \frac{2 \log p}{\log R} \right) \]

\[ + O \left( \frac{1}{\log R} \right) \]

Explicit formula.
1-Level Density

1-level density for one function

\[
D(u_j; \phi) = \hat{\phi}(0) \frac{\log(1 + t_j^2)}{\log R} + \frac{2}{\log R} \sum_p \frac{\log p}{p} \hat{\phi} \left( \frac{2 \log p}{\log R} \right) \\
- \sum_p \frac{2\lambda_j(p) \log p}{p^{\frac{1}{2}} \log R} \hat{\phi} \left( \frac{\log p}{\log R} \right) - \sum_p \frac{2\lambda_j(p^2) \log p}{p \log R} \hat{\phi} \left( \frac{2 \log p}{\log R} \right) \\
+ O \left( \frac{1}{\log R} \right)
\]

1. Explicit formula.
2. Gamma function identities
1-Level Density

1-level density for one function

\[ D(u_j; \phi) = \hat{\phi}(0) \frac{\log(1 + t_j^2)}{\log R} + \frac{\phi(0)}{2} + O \left( \frac{\log \log R}{\log R} \right) \]

\[ - \sum_p \frac{2\lambda_j(p) \log p}{p^{1/2} \log R} \hat{\phi} \left( \frac{\log p}{\log R} \right) - \sum_p \frac{2\lambda_j(p^2) \log p}{p \log R} \hat{\phi} \left( \frac{2 \log p}{\log R} \right) \]

1. Explicit formula.
2. Gamma function identities
3. Prime Number Theorem
Average 1-level density

The weighted 1-level density becomes:

\[
\frac{1}{\sum_j h_T(t_j)} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} D(u_j; \phi)
\]

\[
= \frac{\phi(0)}{2} + O \left( \frac{\log \log R}{\log R} \right) + \frac{1}{\sum_j h_t(t_j)} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} \tilde{\phi}(0) \frac{\log(1 + t_j^2)}{\log R}
\]

\[
- \frac{1}{\sum_j h_T(t_j)} \sum_p \frac{2 \log p}{p^{\frac{1}{2}} \log R} \tilde{\phi} \left( \frac{\log p}{\log R} \right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p)
\]

\[
- \frac{1}{\sum_j h_T(t_j)} \sum_p \frac{2 \log p}{p \log R} \tilde{\phi} \left( \frac{2 \log p}{\log R} \right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p^2)
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- \frac{1}{\sum_j h_T(t_j) \frac{1}{\|u_j\|^2}} \sum_p \frac{2 \log p}{p^{1/2} \log R} \hat{\phi} \left( \frac{\log p}{\log R} \right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p) \\
- \frac{1}{\sum_j h_T(t_j) \frac{1}{\|u_j\|^2}} \sum_p \frac{2 \log p}{p \log R} \hat{\phi} \left( \frac{2 \log p}{\log R} \right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p^2)
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$$\sum_j \frac{h(t_j)}{\|u_j\|^2} \lambda_j(m) \overline{\lambda_j(n)}$$

$$= \text{some function that depends just on } h, m, \text{ and } n$$
Kuznetsov Trace Formula

\[
\sum_{j} \frac{h(t_j)}{\|u_j\|^2} \lambda_j(m) \overline{\lambda_j(n)} + \frac{1}{4\pi} \int_{\mathbb{R}} \tau(m, r) \tau(n, r) \frac{h(r)}{\cosh(\pi r)} dr = \\
\frac{\delta_{n,m}}{\pi^2} \int_{\mathbb{R}} r \tanh(r) h(r) dr + \frac{2i}{\pi} \sum_{c \geq 1} \frac{S(n, m; c)}{c} \int_{\mathbb{R}} J_{ir} \left( \frac{4\pi \sqrt{mn}}{c} \right) \frac{h(r)r}{\cosh(\pi r)} dr
\]

where

\[
\tau(m, r) = \pi^{1/2 + ir} \Gamma(1/2 + ir)^{-1} \zeta(1 + 2ir)^{-1} n^{-1/2} \sum_{ab=\lvert m\rvert} \left( \frac{a}{b} \right)^{ir}.
\]

\[
S(n, m; c) = \sum_{0 \leq x \leq c-1, \gcd(x, c)=1} e^{2\pi i (nx + mx^*) / c}
\]

\[
J_{ir}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + ir + 1)} \left( \frac{1}{2} x \right)^{2m+ir}.
\]
Kuznetsov Formula

\[
\sum_j \frac{h(t_j)}{\|u_j\|^2} \lambda_j(m) \overline{\lambda_j(n)} + \frac{1}{4\pi} \int_{\mathbb{R}} \frac{h(r)}{\tau(m, r) \tau(n, r) \cosh(\pi r)} dr = \\
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\]
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Kuznetsov Formula

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\]

- The only $\lambda(m) \overline{\lambda(n)}$ term that contributes is when $m = n = 1$.
- The $m = 1, n = p$ and $m = 1, n = p^2$ terms do not contribute because of the $\delta_{m,n}$ function.
Result: 1-level density

**Theorem (AILMZ, 2011)**

If $h_T = h_{1,T}$ or $h_{2,T}$, $T \to \infty$ and $L \ll T / \log T$, and $\sigma < 1/6$ then 1-level density is

$$\sum_j \frac{1}{h_T(t_j)} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D(u_j; \phi) = \frac{\phi(0)}{2} + \hat{\phi}(0) + O \left( \frac{\log \log R}{\log R} \right)$$

$$+ O(T^{3\sigma/2-1/4+\epsilon} + T^{\sigma/2-1/4+\epsilon}).$$
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+ O(T^{3\sigma/2 - 1/4 + \epsilon} + T^{\sigma/2 - 1/4 + \epsilon}).
\]

This matches with the orthogonal family density as predicted by Katz-Sarnak.
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Can distinguish unitary and symplectic from the 3 orthogonal groups, but 1-level density cannot distinguish the orthogonal groups from each other if support in $(-1, 1)$. 
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Can distinguish unitary and symplectic from the 3 orthogonal groups, but 1-level density cannot distinguish the orthogonal groups from each other if support in $(-1, 1)$.

2-level density can distinguish orthogonal groups with arbitrarily small support; additional term depending on distribution of signs of functional equations.
To differentiate between even and odd in orthogonal family, we calculated the 2-level density:

\[
D_2^*(\phi) := \frac{1}{\sum_j h_T(t_j) \|u_j\|^2} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \sum_{j_1,j_2} \phi_1(\gamma(j_1))\phi_2(\gamma(j_2))
\]

\[
= \frac{1}{\sum_j h_T(t_j) \|u_j\|^2} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \prod_{i=1}^2 \left( \frac{\phi_i(0)}{2} + \hat{\phi}_i(0) \frac{\log(1 + t_j^2)}{\log R} \right) + O \left( \frac{\log \log R}{\log R} \right)
\]

\[- \sum_p \frac{2\lambda_j(p) \log p}{p^{\frac{1}{2}} \log R} \hat{\phi}_i \left( \frac{\log p}{\log R} \right) - \sum_p \frac{2\lambda_j(p^2) \log p}{p \log R} \hat{\phi}_i \left( \frac{2 \log p}{\log R} \right).\]
2- Level Density

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\]

\[
= \frac{1}{\sum_j h_T(t_j) \|u_j\|^2 \sum_j \phi_i(0) \frac{\log(1 + t_j^2)}{\log R}} + O \left( \frac{\log \log R}{\log R} \right) 
\]

\[
- \sum_p \frac{2\lambda_j(p) \log p}{p^{1\over 2} \log R} \phi_i \left( \frac{\log p}{\log R} \right) - \sum_p \frac{2\lambda_j(p^2) \log p}{p \log R} \phi_i \left( \frac{2 \log p}{\log R} \right) . 
\]

25 terms, handled by Cauchy-Schwarz or Kuznetsov.
Result: 2-level density

Theorem (AILMZ, 2011)

Same conditions as before, for $\sigma < 1/12$ have

$$D^*_{2,F} = \prod_{i=1}^{2} \left[ \frac{\phi_i(0)}{2} + \hat{\phi}_i(0) \right] + 2 \int_{-\infty}^{\infty} |z| \hat{\phi}_1(z) \hat{\phi}_2(z) dz$$

$$-\phi_1(0)\phi_1(0) - 2\phi_1\phi_2(0) + (\phi_1\phi_2)(0) N(-1)$$

$$+ O \left( \frac{\log \log R}{\log R} \right).$$

Note that $N(-1)$ is the weighted percent that have odd sign in functional equation.
Conclusion
Recap

- We calculated 1-level for $\sigma < 1/6$.

- Calculated 2-level densities for $\sigma < 1/12$ in order to distinguish the orthogonal families.

- We showed agreement with Katz-Sarnak conjecture.

Thank you!