# Generalized More-Sum-Than Difference Sets

Geoffrey Iyer[1], Liyang Zhang[2], Oleg Lazarev[3]

1. University of Michigan, 2. Williams College, 3. Princeton University

Advisor Steven J Miller (sjm1@williams.edu)

http://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

**Maine-Quebec Number Theory Conference**
**October 2, 2011**

## Motivation

We often care about the sum/difference of a set $A \subseteq \mathbb{Z}$.

## Motivation

We often care about the sum/difference of a set $A \subseteq \mathbb{Z}$.

- Goldbach's Conjecture: $\{\text{Evens}\} \setminus \{2 \subseteq P + P$
- Fermat's Last Theorem: If $A_n$ is the set of positive $n^{\text{th}}$ powers, then $A_n + A_n \cap A_n = \emptyset$ for all $n \geq 3$

## Motivation

We often care about the sum/difference of a set $A \subseteq \mathbb{Z}$.

- Goldbach's Conjecture: $\{\text{Evens}\} \setminus \{2 \subseteq P + P$
- Fermat's Last Theorem: If $A_n$ is the set of positive $n^{\text{th}}$ powers, then $A_n + A_n \cap A_n = \emptyset$ for all $n \geq 3$

Natural question: What are the sizes of the sum/difference sets?

## Definitions

$A$ finite set of integers, $|A|$ its size. Form

- Sumset: $A + A = \{a_i + a_j : a_i, a_j \in A\}$.
- Difference set: $A - A = \{a_i - a_j : a_i, a_j \in A\}$.

## Definitions

$A$ finite set of integers, $|A|$ its size. Form

- Sumset: $A + A = \{a_i + a_j : a_i, a_j \in A\}$.
- Difference set: $A - A = \{a_i - a_j : a_i, a_j \in A\}$.

### Definition

Difference dominated: $|A - A| > |A + A|$
Balanced: $|A - A| = |A + A|$
Sum dominated (or MSTD): $|A + A| > |A - A|$.

## History

What could cause a set to be sum-dominated? Difference-dominated?

- $x + x = 2x$, but $x - x = 0$.
- $x + y = y + x$, but $x - y \neq y - x$.

## History

What could cause a set to be sum-dominated? Difference-dominated?

- $x + x = 2x$, but $x - x = 0$.
- $x + y = y + x$, but $x - y \neq y - x$.

Nathanson, *Problems in Additive Number Theory*. "With the right way of counting the vast majority of sets satisfy $|A - A| > |A + A|$."

## History

Theorem (Martin-O'Bryant): If each set $A \subseteq [0, n-1]$ is equally likely, then a positive percentage of sets are sum-dominant in the limit. More precisely:

$$\lim_{n\to\infty} \frac{\#\{A \subseteq [0, n-1];\ A \text{ is sum-dominant}\}}{2^n} \approx 0.00045.$$

## History

How is it possible for a positive percent of sets to be sum-dominant?

## History

How is it possible for a positive percent of sets to be sum-dominant?

Martin-O'Bryant: We have the expected values

- $|A + A| \sim 2n - 11$,
- $|A - A| \sim 2n - 7$.

## History

Say $A \subseteq [0, n-1]$, $x \in A+A$ if we can find $a_1, a_2 \in A$ such that $a_1 + a_2 = x$.

## History

Say $A \subseteq [0, n-1]$, $x \in A + A$ if we can find $a_1, a_2 \in A$ such that $a_1 + a_2 = x$.

If $x$ is near $n$ there are many possibilities for $a_1, a_2$.

## History

Say $A \subseteq [0, n-1]$, $x \in A + A$ if we can find $a_1, a_2 \in A$ such that $a_1 + a_2 = x$.

If $x$ is near $n$ there are many possibilities for $a_1, a_2$.

With high probability, the middle will be full.

## History

Say $A \subseteq [0, n-1]$, $x \in A + A$ if we can find $a_1, a_2 \in A$ such that $a_1 + a_2 = x$.

If $x$ is near $n$ there are many possibilities for $a_1, a_2$.

With high probability, the middle will be full.

The trick is to control the fringes.

## Notation

- As adding sets and not multiplying, set

$$kA \;=\; \underbrace{A + \cdots + A}_{\text{k times}}.$$

- $[a, b] \;=\; \{a, a+1, \dots, b\}$.

## Questions

- Can we find a set $A$ such that $|kA + kA| > |kA - kA|$?
- Can we find a set $A$ such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$?
- Can we find a set $A$ such that $|kA + kA| > |kA - kA|$ for all $k$?

## Questions

- Can we find a set $A$ such that $|kA + kA| > |kA - kA|$? Yes.
- Can we find a set $A$ such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$? Yes.
- Can we find a set $A$ such that $|kA + kA| > |kA - kA|$ for all $k$? No. (No such set exists.)

## Initial Observations

Question: Can we find $A$ with $|kA + kA| > |kA - kA|$?

## Initial Observations

Question: Can we find $A$ with $|kA + kA| > |kA - kA|$?

- One set is enough to show a positive percentage.

## Initial Observations

Question: Can we find $A$ with $|kA + kA| > |kA - kA|$?

- One set is enough to show a positive percentage.
- How do we find one set?

## Initial Observations

Question: Can we find $A$ with $|kA + kA| > |kA - kA|$?

- One set is enough to show a positive percentage.
- How do we find one set?
- Computer simulations? We couldn't find a set for $k = 2$; the probability of finding some of these sets is less than $10^{-8}$.

## Initial Observations

Question: Can we find $A$ with $|kA + kA| > |kA - kA|$?

- One set is enough to show a positive percentage.
- How do we find one set?
- Computer simulations? We couldn't find a set for $k = 2$; the probability of finding some of these sets is less than $10^{-8}$.

If $A$ is symmetric ($A = c - A$ for some $c$) then

$$|A + A| = |A + (c - A)| = |A - A|.$$

## $|2A+2A| > |2A-2A|$

Example: $|2A+2A| > |2A-2A|$

## $|2A+2A| > |2A-2A|$

Example: $|2A+2A| > |2A-2A|$

$$A = \{0,1,3,4,5,9\} \cup [33,56] \cup \{79,83,84,85,87,88,89\}$$

# $|2A + 2A| > |2A - 2A|$

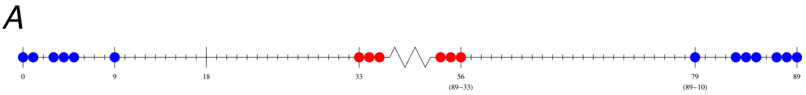
Example: $|2A + 2A| > |2A - 2A|$

$$A = \{0, 1, 3, 4, 5, 9\} \cup [33, 56] \cup \{79, 83, 84, 85, 87, 88, 89\}$$

# $|2A + 2A| > |2A - 2A|$

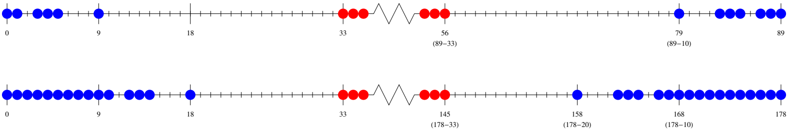
$$A + A = [0, 9] \cup \{10, 12, 13, 14, 18\} \cup [33, 145] \cup \{158, 162, 163, 164, 166, 167\} \cup [168, 178]$$

# $|2A+2A| > |2A-2A|$



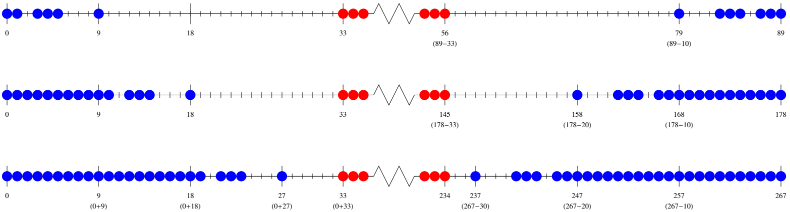
# $|2A + 2A| > |2A - 2A|$

$A + A$


0
9
18
33
56
(89−33)
79
(89−10)
89
0
9
18
33
145
(178−33)
158
(178−20)
168
(178−10)
178

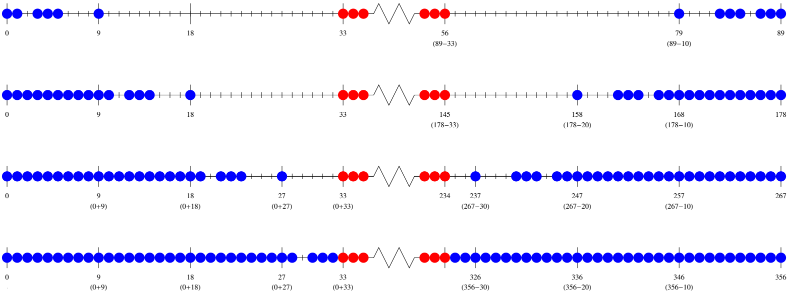
# $|2A + 2A| > |2A - 2A|$

$A + A + A$



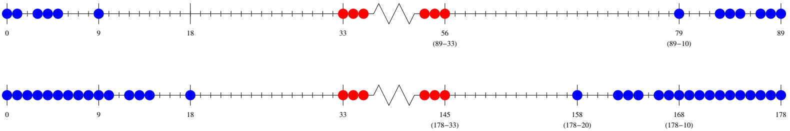
# $|2A + 2A| > |2A - 2A|$

$A + A + A + A$



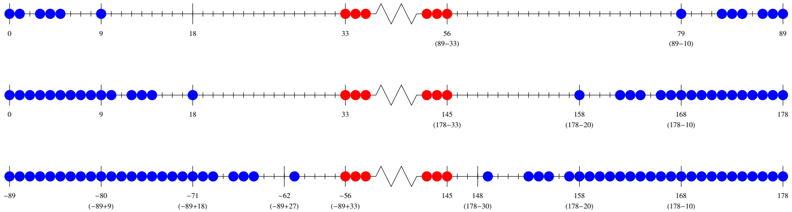
# $|2A + 2A| > |2A - 2A|$

$A + A$


0
9
18
33
56
(89−33)
79
(89−10)
89
0
9
18
33
145
(178−33)
158
(178−20)
168
(178−10)
178

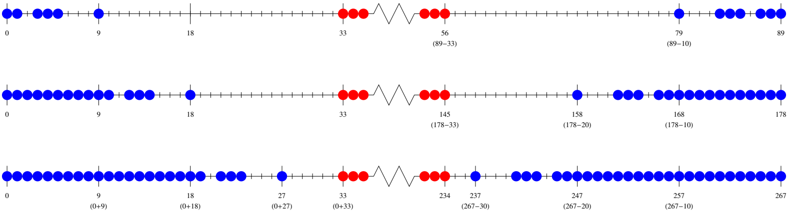
# $|2A + 2A| > |2A - 2A|$

$A + A - A$



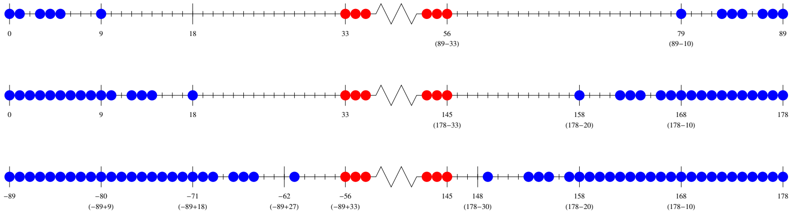
# $|2A + 2A| > |2A - 2A|$

$A + A + A$



# $|2A + 2A| > |2A - 2A|$

## $A + A - A$



## $|2A+2A| > |2A-2A|$

Say that $L$ is the left fringe of $A$, $R$ the right fringe.

## $|2A+2A| > |2A-2A|$

Say that $L$ is the left fringe of $A$, $R$ the right fringe.

The left fringe of $A+A$ is $L+L$, the right fringe is $R+R$.

## $|2A + 2A| > |2A - 2A|$

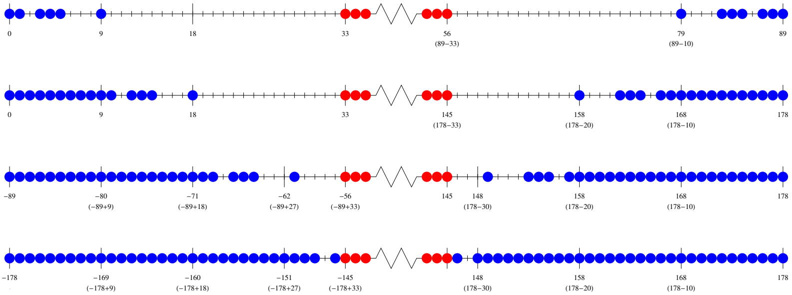
Say that $L$ is the left fringe of $A$, $R$ the right fringe.

The left fringe of $A + A$ is $L + L$, the right fringe is $R + R$.

The left fringe of $A - A$ is $L - R$, the right fringe is $R - L$.

# $|2A + 2A| > |2A - 2A|$

$A + A - A - A$



## Generalization

After dealing with some technical obstructions, we can generalize:

## Generalization

After dealing with some technical obstructions, we can generalize:

For all nontrivial choices of $s_1, d_1, s_2, d_2$, $\exists\, A \subseteq \mathbb{Z}$ such that $|s_1 A - d_1 A| > |s_2 A - d_2 A|$.

## Generalization

After dealing with some technical obstructions, we can generalize:

For all nontrivial choices of $s_1, d_1, s_2, d_2$, $\exists\, A \subseteq \mathbb{Z}$ such that $|s_1 A - d_1 A| > |s_2 A - d_2 A|$.

Example: We can have $|A+A+A+A| > |A+A+A-A|$:

$$A = \{0, 1, 3, 4, 5, 9, 33, 34, 35, 50, 54, 55, 56, 58, 59, 60\}$$

## *k*-Generational Sets

Question: Does a set $A$ exist such that $|A+A| > |A-A|$ and $|A+A+A+A| > |A+A-A-A|$?

## $k$-Generational Sets

Question: Does a set $A$ exist such that $|A + A| > |A - A|$ and $|A + A + A + A| > |A + A - A - A|$?

Say $A$ is $k$-generational if $A, 2A, \ldots, kA$ all sum-dominant.

## *k*-Generational Sets

Question: Does a set $A$ exist such that $|A+A| > |A-A|$ and $|A+A+A+A| > |A+A-A-A|$?

## *k*-Generational Sets

Question: Does a set $A$ exist such that $|A+A| > |A-A|$ and $|A+A+A+A| > |A+A-A-A|$?

Yes!

$$A = \{0, 1, 3, 4, 7, 26, 27, 29, 30, 33, 37, 38, 40, 41, 42, 43, 46, 49, 50, 52, 53, 54, 72, 75, 76, 78, 79, 80\}$$

In fact, we can find a $k$-generational set for all $k$.

## *k*-Generational Sets

Idea of proof: We can find $A_j$ such that $|jA_j + jA_j| > |jA_j - jA_j|$ for a specific $1 \leq j \leq k$.

## $k$-Generational Sets

Idea of proof: We can find $A_j$ such that $|jA_j + jA_j| > |jA_j - jA_j|$ for a specific $1 \leq j \leq k$.

Combine the $A_j$ using the method of base expansion.

## Base Expansion

Base Expansion: For sets $A_1, A_2$ and $m \in \mathbb{N}$ sufficiently large (relative to $A_1, A_2$) the set

$$A = m \cdot A_1 + A_2$$

behaves like the direct product $A_1 \times A_2 \subseteq \mathbb{Z} \times \mathbb{Z}$.

## Base Expansion

Base Expansion: For sets $A_1, A_2$ and $m \in \mathbb{N}$ sufficiently large (relative to $A_1, A_2$) the set

$$A = m \cdot A_1 + A_2$$

behaves like the direct product $A_1 \times A_2 \subseteq \mathbb{Z} \times \mathbb{Z}$.
(here multiplication is the usual scalar multiplication)

## Base Expansion

Base Expansion: For sets $A_1, A_2$ and $m \in \mathbb{N}$ sufficiently large (relative to $A_1, A_2$) the set

$$A = m \cdot A_1 + A_2$$

behaves like the direct product $A_1 \times A_2 \subseteq \mathbb{Z} \times \mathbb{Z}$.
(here multiplication is the usual scalar multiplication)

In particular:

$$|xA - yA| = |xA_1 - yA_1| \cdot |xA_2 - yA_2|$$

whenever $x + y$ is small relative to $m$.

## Generalization

For nontrivial $x_j, y_j, w_j, z_j$ $(2 \leq j \leq k)$, we can find an $A$ such that $|x_jA - y_jA| > |w_jA - z_jA|$ for all $j$.

## Generalization

For nontrivial $x_j, y_j, w_j, z_j$ $(2 \leq j \leq k)$, we can find an $A$ such that $|x_jA - y_jA| > |w_jA - z_jA|$ for all $j$.

Example: We can find an $A$ such that

$$|A + A| > |A - A|$$
$$|A + A - A| > |A + A + A|$$
$$|5A - 2A| > |A - 6A|$$
$$\vdots$$
$$|1870A - 141A| > |1817A - 194A|$$

# Limiting behavior of kA

Question: Can we find $A$ with $|kA + kA| > |kA - kA|$ for all $k$?

## Limiting behavior of kA

Question: Can we find $A$ with $|kA + kA| > |kA - kA|$ for all $k$?

No. No such set exists.

# Limiting behavior of kA

Question: Can we find $A$ with $|kA + kA| > |kA - kA|$ for all $k$?

No. No such set exists.

It turns out that all sets have a sort of limiting behavior.

## Stabilizing Fringes

Example: $A = \{0, 3, 5, 6, 8, 9, 10, 11, 12, 15, 16, 20\}$

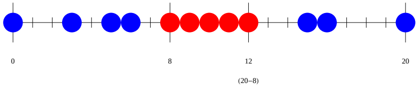



**Figure:** $A$

# Stabilizing Fringes

Example: $A = \{0, 3, 5, 6, 8, 9, 10, 11, 12, 15, 16, 20\}$

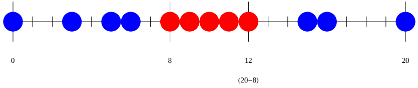



**Figure:** $A$

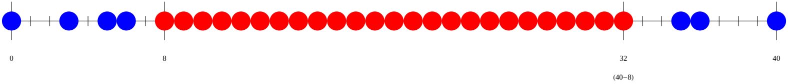



**Figure:** $A + A$

## $|kA - kA|$ vs. $|kA + kA|$

Nathanson: For any set $A$, $kA$ becomes stabilized before $k$ reaches $\max(A)^2 \cdot |A|$.

We improve this bound to $\max(A)$.

## $|kA - kA|$ vs. $|kA + kA|$

**Theorem**

For any set $A$, $kA$ will become difference-dominated or balanced before $k$ reaches $2 \cdot \max(A)$.

Proof Idea:

- The middle will quickly become full, and the remaining fringes are finite.
- $kA \subseteq kA - kA$. Any sum can eventually be written as a difference.

Because the form stabilizes, this means $kA - kA \supseteq kA + kA$ when $k$ large.

## Other Results

Arbitrary Differences: We can create sets $A$ where

$$|kA + kA| - |kA - kA| = m.$$

More generally, $|s_1 A - d_1 A| - |s_2 A - d_2 A| = m$.

Simultaneous Comparison: We can create a set $A$ where

$$|4A| > |3A - A| > |2A - 2A|.$$

More generally, any order and number of (nontrivial) comparisons.

## Thanks

Thanks to:

- Williams College,
- NSF Grant DMS0850577,
- Conference Coordinators.