Breaking Universality in the Lower Order Terms in the 1-level and 2-level Density of Holomorphic Cusp Newforms

Xiaoyao Huang[1] Say-Yeon Kwon [2] Meiling Laurence[3]

[1]University of Michigan, xyrushac@umich.edu
[2]Princeton University, sk9017@princeton.edu
[3]Yale University, meiling.laurence@yale.edu

Joint work with L. Dillon, V. Muthuvel, S.J. Miller, L. Rowen, P. Saldin, and S. Zanetti.

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Who with?


EMERGENCY DOOR
DO NOT BLOCK
Luke Rowen
Steven J Miller
Pramana Saldin
Lawrence Dillon
Steven Zanetti
Say-Yeon Kwon
Xiaoyao Huang
Vishal Muthuvel
Meiling Laurence

Overview

- We investigate how lower-order terms ($O(\log^{-4} N)$ error) of weighted first- and second-level densities over $H_k^*(N)$ vary depending on how the prime factors of N approach infinity.

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- We investigate how lower-order terms ($O(\log^{-4} N)$ error) of weighted first- and second-level densities over $H_k^*(N)$ vary depending on how the prime factors of N approach infinity.
- By doing this, we break the universality of the main term behavior suggested by the Katz-Sarnak Conjecture.

Hecke Eigenforms

- We care about associating L-functions to certain *modular forms*.
- We define $H_k^*(N)$ to be the space of all holomorphic cusp newforms of weight k and level N.
- For every $f \in H_k^*(N)$, we call its n^{th} Hecke eigenvalue $\lambda_f(n)$

Hecke Eigenforms

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- For every $f \in H_k^*(N)$, we call its n^{th} Hecke eigenvalue $\lambda_f(n)$
- $\lambda_f(p) \in [-2, 2]$ for primes p.
- For Fourier coefficients $a_f(n)$ of f (normalized so that $a_f(1) = 1$), $a_f(n) = \lambda_f(n)n^{(k-1)/2}$.
- Multiplicative property:

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n)\\(d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right).$$

  - In particular, if $(m, n) = 1$, $\lambda_f(mn) = \lambda_f(m)\lambda_f(n)$.

Hecke Eigenforms

- The Satake parameters of f at n, $\alpha_f(n)$ and $\beta_f(n)$, are related to the Hecke-eigenvalues $\lambda_f(n)$. They satisfy the following:
  - $\lambda_f(n) = \alpha_f(n) + \beta_f(n)$.
  - $\alpha_f(n)\beta_f(n) = 1$.
  - $\lambda_f(p^\nu) = \alpha_f(p)^\nu + \beta_f(p)^\nu$

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  - $\alpha_f(n)\beta_f(n) = 1$.
  - $\lambda_f(p^\nu) = \alpha_f(p)^\nu + \beta_f(p)^\nu$
- For $f \in H_k^*(N)$, we define the L-function associated to f as

$$L(s, f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \alpha_f(p)p^{-s}\right)^{-1} \left(1 - \beta_f(p)p^{-s}\right)^{-1},$$

$\mathfrak{Re}(s) > 1$.

n-Level Density

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  $\mathfrak{Re}(s) > 1$.
- GRH: all non-trivial zeros of $L(s,f)$ are of the form $\rho_f = 1/2 + i\gamma_f$, $\gamma_f \in \mathbb{R}$.
- We wish to find a way to study the zeroes near the central point ($s = \frac{1}{2}$). Thus, we associate an **n-level density** to a modular form.

n-Level Density

Definition (n-Level Density)

Let $L(s,f)$ be an L-function associated to a modular form f. Let $\Phi : \mathbf{R}^n \to \mathbf{R}$ be a vector of even Schwartz function whose Fourier transforms have compact support. Then, its **n-level density** is given by:

$$D_n(f;\Phi) \ := \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi_1\left(\gamma_f^{(j_1)} \frac{\log(R)}{2\pi}\right) \cdots \phi_n\left(\gamma_f^{(j_n)} \frac{\log(R)}{2\pi}\right),$$

where we enumerate the imaginary part $0 \leq \gamma_f^{(1)} \leq \gamma_f^{(2)} \leq \dots$ of the zeros of $L(s,f)$ with $\gamma_f^{(-j)} = -\gamma_f^{(j)}$ by symmetry and R is the analytic conductor of f (it measures the size of the gaps between zeroes near the central point).

Weighted densities averaged over families

Definition

For non-negative weights $w_R(f)$, we can define the weighted n-level density of the space over holomorphic cusp newforms $H_k^*(N)$ by

$$D_n(H_k^*(N)) := \lim_{N\to\infty} \frac{1}{\sum_{f\in H_k^*(N)} w_R(f)} \sum_{f\in H_k^*(N)} w_R(f) D_n(f,\Phi).$$

- We use harmonic weights.

Katz-Sarnak Conjecture

Conjecture (Katz-Sarnak)

As we average over $\mathcal{F} = \cup \mathcal{F}_N$, a family of L-functions ordered by conductors N, and take $N \to \infty$, n-level density converges to a scaled distribution of eigenvalues near 1 of a classical compact group, i.e.

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} D_n(f,\phi) = \int \Phi(\overrightarrow{x})W_{n,G(\mathcal{F})}(\overrightarrow{x})d\overrightarrow{x},$$

where $W_G(\mathcal{F})$ represents the limiting distribution of a similar statistic for the eigenvalues of random matrices in some classical compact group as their rank goes to infinity.

- While the main terms of the n-level density with the family $H_k^*(N)$ have been shown to agree with random matrix theory, lower order terms break this universality.

Katz-Sarnak Conjecture

- In 2009, S.J. Miller studied 1-parameter families of elliptic curves and compared the lower order terms with newforms of prime level[Mil09].

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- We investigate how lower-order terms ($O(\log^{-4} N)$ error) of weighted first- and second-level densities over $H_k^*(N)$ vary depending on how the prime factors of N approach infinity.

Katz-Sarnak Conjecture

- In 2009, S.J. Miller studied 1-parameter families of elliptic curves and compared the lower order terms with newforms of prime level[Mil09].
- We investigate how lower-order terms ($O(\log^{-4} N)$ error) of weighted first- and second-level densities over $H_k^*(N)$ vary depending on how the prime factors of N approach infinity.
- We consider four different scenarios in which the level approaches infinity.
  - $N = q$ prime.
  - $N = q_1 q_2$, $q_1 \neq q_2$ primes, q_1 fixed, $q_2 \to \infty$.
  - $N = q_1 q_2$, $q_1 \neq q_2$ primes with $q_1 \sim N^{\delta}$, $q_2 \sim N^{1-\delta}$ where $\delta \in (0, 1/2]$.
  - $N = p^2$.

Explicit Formula for 1-level

- Our main tool is the following explicit formula for the 1-level density.

Theorem (Iwaniec, Luo, and Sarnak [ILS00])

Given the same conditions,

$$D_1(f;\phi) = \frac{A}{\log R} - 2\sum_p \sum_{m=1}^{\infty} \left(\frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}} \right) \hat{\phi}\left(m\frac{\log p}{\log R} \right) \frac{\log p}{\log R}$$

where A represents a sum of digamma $(\Gamma'(s)/\Gamma(s))$ *factors.*

- Higher level densities can be obtained from this using inclusion-exclusion.

Trace Formulae

- Harmonic weights lead to sums

$$\Delta^*_{k,N}(m,n) = \sum_{f \in H^*_k(M)} \frac{Z_N(1,f)}{Z(1,f)} \lambda_f(m)\lambda_f(n).$$

- For N square-free, we use:

Theorem (Iwaniec, Luo, Sarnak 2000)

Let N be squarefree, $(m,N)=1$, and $(n,N^2)|N$. Then

$$\Delta^*_{k,N}(m,n) \;=\; \frac{k-1}{12} \sum_{ML=N} \frac{\mu(L)M}{\nu((n,L))} \sum_{\ell | L^\infty} \ell^{-1} \Delta_{k,M}(m\ell^2, n). \quad (1)$$

Trace Formulae

- For N not square-free, we use:

Theorem (Barrett et al. 2016)

Let $(m, N) = 1$ *and* $(n, N) = 1$. *Then*

$$\Delta^*_{k,N}(m,n) \;=\; \frac{k-1}{12} \sum_{ML=N} \prod_{p^2|M} \left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{\substack{\ell | L^\infty \\ (\ell,M)=1}} \ell^{-1} \Delta_{k,M}(m\ell^2, n). \tag{2}$$

Our Work

Our Work

Theorem (DHKLMMRSZ 25+)

Consider the Weighted First and Second Level Density, $D_1(H_k^*(N))$ and $D_2(H_k^*(N))$, computed with the level N going to infinity through

- $N = q$ *prime.*
- $N = q_1 q_2$, $q_1 \neq q_2$ *primes with* $q_1 \sim N^{\delta}$, $q_2 \sim N^{1-\delta}$ *where* $\delta \in (0, 1/2]$.
- $N = p^2$.
- $N = q_1 q_2$, $q_1 \neq q_2$ *primes,* q_1 *fixed,* $q_2 \to \infty$.

The first three cases are the same up to error of $O\left(\frac{1}{\log^4(R)}\right)$ *while, when* $N = q_1 q_2$ *for* q_1 *fixed, new terms dependent on* q_1 *come out.*

Coming up with Formulas - Weighted 1st Level Density

- We can use the Explicit Formula of [ILS00] to write $D_1(H_k^*(N))$, the weighted 1-level density, in terms of $S_1(H_k^*(N), \phi)$ where

$$\mathcal{S}_1(H_k^*(N), \phi)$$
$$:= -2 \sum_p \sum_{m=1}^{\infty} \frac{1}{W_R(H_k^*(N))} \sum_{f \in H_k^*(N)} w_R(f) \frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}} \frac{\log(p)}{\log(R)} \hat{\phi}\left(m \frac{\log(p)}{\log(R)}\right).$$

Coming up with Formulas - Weighted 1st Level Density

Lemma (DHKLMMRSZ 25+)

Let $\mathcal{S}_1(H_k^*(N), \phi) :=$

$$\sum_p \sum_{m=1}^{\infty} \frac{-2}{W_R(H_k^*(N))} \sum_{f \in H_k^*(N)} w_R(f) \frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}} \frac{\log(p)}{\log(R)} \hat{\phi}\left(m \frac{\log(p)}{\log(R)}\right),$$

then

$$\mathcal{S}_1(H_k^*(N), \phi) \;=\; S_{A'}(H_k^*(N)) + S_A(H_k^*(N))$$

where

Coming up with Formulas - Weighted 1st Level Density

$$
\begin{aligned}
S_{A'}(H_k^*(N)) \; := \; & -2\sum_p \sum_{m=1}^{\infty} \frac{A'_{m,H_k^*(N)}(p)}{p^{m/2}} \frac{\log(p)}{\log(R)} \hat{\phi}\left(m\frac{\log(p)}{\log(R)}\right) \\
S_A(H_k^*(N)) \; := \; & -2\hat{\phi}(0)\sum_p \frac{2A_{0,H_k^*(N)}(p)\log(p)}{p(p+1)\log(R)} + 2\sum_p \frac{2A_{0,H_k^*(N)}(p)\log(p)}{p\log(R)} \hat{\phi}\left(2\frac{\log(p)}{\log(R)}\right) \\
& -2\sum_p \frac{A_{1,H_k^*(N)}(p)\log(p)}{p^{1/2}\log(R)} \hat{\phi}\left(\frac{\log(p)}{\log(R)}\right) + 2\hat{\phi}(0)\sum_p \frac{A_{1,H_k^*(N)}(p)(3p+1)\log(p)}{p^{1/2}(p+1)^2\log(R)} \\
& -2\sum_p \frac{A_{2,H_k^*(N)}(p)\log(p)}{p\log(R)} \hat{\phi}\left(2\frac{\log(p)}{\log(R)}\right) + 2\hat{\phi}(0)\sum_p \frac{A_{2,H_k^*(N)}(p)(p^2+3p+1)\log(p)}{p(p+1)^3\log(R)} \\
& +\hat{\phi}''(0)\sum_p \frac{A_{0,H_k^*(N)}(p)(32p^2+24p+8)\log^3(p)}{p(p+1)^3\log(R)} - \hat{\phi}''(0)\sum_p \frac{A_{1,H_k^*(N)}(p)(27p^3-17p^2+5p+1)\log^3(p)}{p^{1/2}(p+1)^4} \\
& -\hat{\phi}''(0)\sum_p \frac{A_{2,H_k^*(N)}(p)(64p^4-4p^3+44p^2+20p+4)\log^3(p)}{p(p+1)^5\log^3(R)} \\
& -2\hat{\phi}(0)\sum_p \sum_{r=3}^{\infty} \frac{A_{r,H_k^*(N)}(p)p^{r/2}(p-1)\log(p)}{(p+1)^{r+1}\log(R)} \\
& +\hat{\phi}''(0)\sum_p \sum_{r=3}^{\infty} \frac{A_{r,H_k^*(N)}(p-1)(r^2(p-1)^2-12rp-8p)p^{r/2}\log^3(p)}{(p+1)^{r+3}\log(R)} + O\left(\frac{1}{\log^4(R)}\right).
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&\quad -2\sum_p \frac{A_{1,H_k^*(N)}(p)\log(p)}{p^{1/2}\log(R)} \hat{\phi}\left(\frac{\log(p)}{\log(R)}\right) + 2\hat{\phi}(0)\sum_p \frac{A_{1,H_k^*(N)}(p)(3p+1)\log(p)}{p^{1/2}(p+1)^2\log(R)} \\
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&\quad +\hat{\phi}''(0)\sum_p\sum_{r=3}^{\infty} \frac{A_{r,H_k^*(N)}(p-1)(r^2(p-1)^2-12rp-8p)p^{r/2}\log^3(p)}{(p+1)^{r+3}\log(R)} + O\left(\frac{1}{\log^4(R)}\right).
\end{aligned}
$$

Coming up with Formulas - Weighted 1st Level Density

Above, we define

$$A'_{r,H_k^*(N)}(p) := \frac{1}{W_R(H_k^*(N))} \sum_{\substack{f \in H_k^*(N) \\ p \mid N}} w_R(f)\lambda_f(p)^r$$

$$A_{r,H_k^*(N)}(p) := \frac{1}{W_R(H_k^*(N))} \sum_{\substack{f \in H_k^*(N) \\ p \nmid N}} w_R(f)\lambda_f(p)^r.$$

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- We can compute A' and A to see behaviors dependent on factorization of N on $D_1(H_k^*(N))$.

Coming up with Formulas - Weighted 2nd Level Density

- We used inclusion-exclusion to write $D_2(H_k^*(N))$ followed by the Explicit Formula, the weighted 2-level density, in terms of $S_2(H_k^*(N), \phi)$

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- We used inclusion-exclusion to write $D_2(H_k^*(N))$ followed by the Explicit Formula, the weighted 2-level density, in terms of $S_2(H_k^*(N), \phi)$ where $S_2(H_k^*(N), \phi)$ is defined as

Coming up with Formulas - Weighted 2nd Level Density

$$
\begin{aligned}
S_2(H_k^*(N), \phi_1, \phi_2) \ =& \\
& \frac{1}{W_R(H_k^*(N))} \sum_{p_1, p_2} \sum_{\substack{m_1 \in N \\ m_2 \in N}} \sum_{\substack{f \in H_k^*(N) \\ p_1 | N_f \\ p_2 | N_f}} w_R(f) \frac{\lambda_f(p_1)^{m_1}}{p_1^{m_1/2}} \frac{\lambda_f(p_2)^{m_2}}{p_2^{m_2/2}} \frac{\log(p_1)\log(p_2)}{\log^2(R)} \hat{\phi}_1\left(m_1 \frac{\log(p_1)}{\log(R)}\right) \hat{\phi}_2\left(m_2 \frac{\log(p_2)}{\log(R)}\right) \\
+& \frac{1}{W_R(H_k^*(N))} \sum_{p_1, p_2} \sum_{\substack{m_1 \in N \\ m_2 \in N}} \sum_{\substack{f \in H_k^*(N) \\ p_1 | N_f \\ p_2 \nmid N_f}} w_R(f) \frac{\lambda_f(p_1)^{m_1}}{p_1^{m_1/2}} \frac{\alpha_f(p_2)^{m_2} + \beta_f(p_2)^{m_2}}{p_2^{m_2/2}} \frac{\log(p_1)\log(p_2)}{\log^2(R)} \hat{\phi}_1\left(m_1 \frac{\log(p_1)}{\log(R)}\right) \hat{\phi}_2\left(m_2 \frac{\log(p_2)}{\log(R)}\right) \\
+& \frac{1}{W_R(H_k^*(N))} \sum_{p_1, p_2} \sum_{\substack{m_1 \in N \\ m_2 \in N}} \sum_{\substack{f \in H_k^*(N) \\ p_1 \nmid N_f \\ p_2 | N_f}} w_R(f) \frac{\alpha_f(p_1)^{m_1} + \beta_f(p_1)^{m_1}}{p_1^{m_1/2}} \frac{\lambda_f(p_2)^{m_2}}{p_2^{m_2/2}} \frac{\log(p_1)\log(p_2)}{\log^2(R)} \hat{\phi}_1\left(m_1 \frac{\log(p_1)}{\log(R)}\right) \hat{\phi}_2\left(m_2 \frac{\log(p_2))}{\log(R)}\right) \\
+& \frac{1}{W_R(H_k^*(N))} \sum_{p_1, p_2} \sum_{\substack{m_1 \in N \\ m_2 \in N}} \sum_{\substack{f \in H_k^*(N) \\ p_1 \nmid N_f \\ p_2 \nmid N_f}} w_R(f) \frac{\alpha_f(p_1)^{m_1} + \beta_f(p_1)^{m_1}}{p_1^{m_1/2}} \frac{\alpha_f(p_2)^{m_2} + \beta_f(p_2)^{m_2}}{p_2^{m_2/2}} \\
& \qquad \cdot \frac{\log(p_1)\log(p_2)}{\log^2(R)} \hat{\phi}_1\left(m_1 \frac{\log(p_1)}{\log(R)}\right) \hat{\phi}_2\left(m_2 \frac{\log(p_2)}{\log(R)}\right).
\end{aligned}
$$

Coming up with Formulas - Weighted 2nd Level Density

Lemma (DHKLMMRSZ 25+)

Let $S_2(H_k^*(N), \phi_1, \phi_2)$ be defined as above. Then, we have

$$S_2(H_k^*(N), \phi_1, \phi_2) = S_{B''}(H_k^*(N)) + S_{B'}(H_k^*(N)) + S_{B_f}(H_k^*(N)) + S_{B_\infty}(H_k^*(N)) + O\left(\frac{1}{\log^4(R)}\right).$$

Coming up with Formulas - Weighted 2nd Level Density

$$S_{B''}(H_k^*(N)) = \sum_{p_1,p_2} \sum_{m_1,m_2=1}^{\infty} B''_{m_1,m_2}(p_1,p_2) \frac{\log(p_1)\log(p_2)}{p_1^{m_1/2} p_2^{m_2/2} \log^2(R)} \hat{\phi}_1\left(m_1 \frac{\log(p_1)}{\log(R)}\right) \hat{\phi}_2\left(m_2 \frac{\log(p_2)}{\log(R)}\right),$$

$$\begin{aligned} S_{B'}(H_k^*(N)) = & \sum_{p_1,p_2} \sum_{m_1=1}^{\infty} B'_{m_1,1}(p_1,p_2) \frac{\log(p_1)\log(p_2)}{p_1^{m_1/2}\sqrt{p_2}\log^2(R)} \left(\sum_{(i,j)\in A} \hat{\phi}_i\left(\frac{m_1\log(p_1)}{\log(R)}\right)\hat{\phi}_j\left(\frac{\log(p_2)}{\log(R)}\right)\right) \\ & + \sum_{p_1,p_2} \sum_{m_1=1}^{\infty} (B'_{m_1,2}(p_1,p_2) - 2B'_{m_1,0}(p_1,p_2)) \frac{\log(p_1)\log(p_2)}{p_1^{m_1/2} p_2 \log^2(R)} \left(\sum_{(i,j)\in A} \hat{\phi}_i\left(m_1\frac{\log(p_1)}{\log(R)}\right)\hat{\phi}_j\left(2\frac{\log(p_2)}{\log(R)}\right)\right) \\ & + \sum_{p_1,p_2} \sum_{m_1=1,m_2=0}^{\infty} B'_{m_1,m_2}(p_1,p_2) \frac{P_{m_2}(p_2)}{p_1^{m_1/2}} \frac{\log(p_1)\log(p_2)}{\log^2(R)} \left(\sum_{(i,j)\in A} \hat{\phi}_i\left(\frac{m_1\log(p_1)}{\log(R)}\right)\hat{\phi}_j(0)\right), \end{aligned}$$

Coming up with Formulas - Weighted 2nd Level Density

$$
\begin{aligned}
S_{B_f}(H_k^*(N)) &= \sum_{p_1,p_2} B_{1,1}(p_1,p_2)\frac{\log(p_1)\log(p_2)}{\sqrt{p_1}\sqrt{p_2}\log^2(R)}\hat{\phi}_1\left(\frac{\log(p_1)}{\log(R)}\right)\hat{\phi}_2\left(\frac{\log(p_2)}{\log(R)}\right)\\
&+\sum_{p_1,p_2}(B_{1,2}(p_1,p_2)-2B_{1,0}(p_1))\frac{\log(p_1)\log(p_2)}{\sqrt{p_1}p_2\log^2(R)}\hat{\phi}_1\left(\frac{\log(p_1)}{\log(R)}\right)\hat{\phi}_2\left(\frac{2\log(p_2)}{\log(R)}\right)\\
&+\sum_{p_1,p_2}(B_{2,1}(p_1,p_2)-2B_{0,1}(p_2))\frac{\log(p_1)\log(p_2)}{p_1\sqrt{p_2}\log^2(R)}\hat{\phi}_1\left(\frac{2\log(p_1)}{\log(R)}\right)\hat{\phi}_2\left(\frac{\log(p_2)}{\log(R)}\right)\\
&+\sum_{p_1,p_2}(B_{2,2}(p_1,p_2)-2B_{2,0}(p_1)-2B_{0,2}(p_2)+4)\frac{\log(p_1)\log(p_2)}{p_1p_2\log^2(R)}\hat{\phi}_1\left(\frac{2\log(p_1)}{\log(R)}\right)\hat{\phi}_2\left(\frac{2\log(p_2)}{\log(R)}\right),\\
S_{B_\infty}(H_k^*(N)) &= \hat{\phi}_1(0)\sum_{p_1,p_2}\sum_{m_1=0}^{\infty}\frac{\log(p_1)\log(p_2)}{\log^2(R)}\hat{\phi}_2\left(\frac{2\log(p_2)}{\log(R)}\right)(B_{m_1,2}(p_1,p_2)-2B_{m_1,0}(p_1,p_2))\frac{P_{m_1}(p_1)}{p_2}\\
&+\hat{\phi}_1(0)\sum_{p_1,p_1}\sum_{m_1=0}^{\infty}\hat{\phi}_2\left(\frac{\log(p_2)}{\log(R)}\right)\frac{\log(p_1)\log(p_2)}{\log^2(R)}B_{m_1,1}(p_1,p_2)\frac{P_{m_1}(p_1)}{\sqrt{p_2}}\\
&+\hat{\phi}_2(0)\sum_{p_1,p_1}\sum_{m_1=0}^{\infty}\hat{\phi}_1\left(\frac{\log(p_1)}{\log(R)}\right)\frac{\log(p_1)\log(p_2)}{\log^2(R)}B_{1,m_1}(p_1,p_2)\frac{P_{m_1}(p_2)}{\sqrt{p_1}}\\
&+\hat{\phi}_2(0)\sum_{p_1,p_2}\sum_{m_1=0}^{\infty}\frac{\log(p_1)\log(p_2)}{\log^2(R)}\hat{\phi}_1\left(\frac{2\log(p_1)}{\log(R)}\right)(B_{2,m_1}(p_1,p_2)-2B_{0,m_1}(p_1,p_2))\frac{P_{m_1}(p_2)}{p_1}\\
&+\hat{\phi}_1(0)\hat{\phi}_2(0)\sum_{p_1,p_2}\sum_{m_1,m_2=0}^{\infty}\frac{\log(p_1)\log(p_2)}{\log^2(R)}B_{m_1,m_2}(p_1,p_2)P_{m_1}(p_1)P_{m_2}(p_2)
\end{aligned}
$$

Coming up with Formulas - Weighted 2nd Level Density

Above, we define

$$B''_{r_1,r_2,H_k^*(N)}(p_1,p_2) := \frac{1}{W_R(H_k^*(N))} \sum_{f\in H_k^*(N),\ p_1|N_f,\ p_2|N_f} w_R(f)\lambda_f(p_1)^{r_1}\lambda_f(p_2)^{r_2}$$

$$B'_{r_1,r_2,H_k^*(N)}(p_1,p_2) := \frac{1}{W_R(H_k^*(N))} \sum_{f\in H_k^*(N),\ p_1|N_f,\ p_2\nmid N_f} w_R(f)\lambda_f(p_1)^{r_1}\lambda_f(p_2)^{r_2}$$

$$B_{r_1,r_2,H_k^*(N)}(p_1,p_2) := \frac{1}{W_R(H_k^*(N))} \sum_{f\in H_k^*(N),\ p_1\nmid N_f,\ p_2\nmid N_f} w_R(f)\lambda_f(p_1)^{r_1}\lambda_f(p_2)^{r_2}.$$

Sum of the weights

- Goal: $A'_r(p), A_r(p), B''_{r_1,r_2}(p_1,p_2), B'_{r_1,r_2}(p_1,p_2)$, and $B_{r_1,r_2}(p_1,p_2)$, which determine the values of $S_1(H_k^*(N),\phi_1,\phi_2)$ and $S_2(H_k^*(N),\phi_1,\phi_2)$.
- The definition of all above terms involve the sum of the weights $W_R(H_k^*(N)) := \sum_{f\in H_k^*(N)} w_R(f)$.

Harmonic weights

$$A'_{r,H_k^*(N)}(p) := \frac{1}{W_R(H_k^*(N))} \sum_{\substack{f\in H_k^*(N)\\ p|N}} w_R(f)\lambda_f(p)^r$$

$$A_{r,H_k^*(N)}(p) := \frac{1}{W_R(H_k^*(N))} \sum_{\substack{f\in H_k^*(N)\\ p\nmid N}} w_R(f)\lambda_f(p)^r.$$

$$B''_{r_1,r_2,H_k^*(N)}(p_1,p_2) := \frac{1}{W_R(H_k^*(N))} \sum_{f\in H_k^*(N),\ p_1|N_f,\ p_2|N_f} w_R(f)\lambda_f(p_1)^{r_1}\lambda_f(p_2)^{r_2}$$

$$B'_{r_1,r_2,H_k^*(N)}(p_1,p_2) := \frac{1}{W_R(H_k^*(N))} \sum_{f\in H_k^*(N),\ p_1|N_f,\ p_2\nmid N_f} w_R(f)\lambda_f(p_1)^{r_1}\lambda_f(p_2)^{r_2}$$

$$B_{r_1,r_2,H_k^*(N)}(p_1,p_2) := \frac{1}{W_R(H_k^*(N))} \sum_{f\in H_k^*(N),\ p_1\nmid N_f,\ p_2\nmid N_f} w_R(f)\lambda_f(p_1)^{r_1}\lambda_f(p_2)^{r_2}.$$

Harmonic weights

We use the harmonic weights to compute the asymptotic explicitly

$$w_R(f) = \frac{Z_N(1,f)}{Z(1,f)}.$$

$\mathcal{F} = H_k^*(N)$: the family of holomorphic cusp newforms with weight k and level N.
Scenarios ($N \to \infty$):

- N is prime.
- $N = q_1 q_2$ for two primes q_1, q_2, with $q_1 \neq q_2$ and q_1 fixed.
- $N = q_1 q_2$ for two primes with $q_1 \sim N^\delta$ and $q_2 \sim N^{1-\delta}$ where $\delta \in (0, 1/2]$.
- $N = p^2$ where p is prime.

Two cases for sum of weights

Square-free cases: use Petersson trace formula as in [ILS00]

- N is prime.
- $N = q_1 q_2$ for two primes q_1, q_2, with $q_1 \neq q_2$ and q_1 fixed.
- $N = q_1 q_2$ for two primes with $q_1 \sim N^{\delta}$ and $q_2 \sim N^{1-\delta}$ where $\delta \in (0, 1/2]$.

Non-square-free case: use [BBD$^+$16]

- $N = p^2$ where p is prime.

N square-free

Lemma (DHKLMMRSZ 25+)

Let $N = q_1 q_2$ be square-free with q_1, q_2 being distinct primes with

$$q_1 \sim C_1 N^{\delta}, q_2 \sim C_2 N^{1-\delta}.$$

Here, $\delta \in [0, 1/2]$ while C_1, C_2 can be both prime constants if $\delta \neq 0$, or $C_1 = 1$ or prime and C_2 prime if $\delta = 0$. Then

$$W_R(H_k^*(N)) := \sum_{f \in H_k^*(N)} w_r(f) = \frac{k-1}{12}\varphi(N) + O(E(N,\delta)),$$

where

$$E(N,\delta) = \begin{cases} \frac{\log(N)}{N^{\frac{1-\delta}{2}}} & \text{if } 0 < \delta \leq \frac{1}{2} \text{ or } \delta = 0, C_1 = 1 \\ \frac{\log^2(N)}{N^{1/2}} & \text{if } \delta = 0 \text{ and } C_1 \text{ prime.} \end{cases}$$

N square-free

Using [ILS00]

$$W_R(H_k^*(N)) = \Delta_{k,N}^*(1,1) = \frac{k-1}{12}\sum_{LM=N}\mu(L)M\sum_{\ell \mid L^\infty}\ell^{-1}\Delta_{k,M}(\ell^2,1).$$

If $C_1 = 1$, then N is a prime and

$$W_R(H_k^*(N)) = \frac{k-1}{12}\left(N\Delta_{k,N}(1,1) - \sum_{\ell \mid N^\infty}\ell^{-1}\Delta_{k,q_1}(\ell^2,1)\right).$$

If $C_1 \neq 1$, N is product of 2 distinct primes and

$$W_R(H_k^*()) = \frac{k-1}{12}\left[N\Delta_{k,N}(1,1) - q_1\sum_{\ell \mid q_2^\infty}\ell^{-1}\Delta_{k,q_1}(\ell^2,1) - q_2\sum_{\ell \mid q_1^\infty}\ell^{-1}\Delta_{k,q_2}(\ell^2,1) + \sum_{\ell \mid N^\infty}\ell^{-1}\Delta_{k,1}(\ell^2,1)\right].$$

$$N\Delta_{k,N}(1,1) = N + O_k\left(\frac{1}{N^{1/2}}\right), \quad \sum_{\ell \mid N^\infty}\ell^{-1}\Delta_{k,1}(\ell^2,1) = 1 + O_k\left(\frac{\log(N)}{N^{1/2}}\right),$$

$$q_1\sum_{\ell \mid q_2^\infty}\ell^{-1}\Delta_{k,q_1}(\ell^2,1) = q_1 + O\left(\frac{\log(N)}{N^{(1-\delta)/2}}\right), \quad q_2\sum_{\ell \mid q_1^\infty}\ell^{-1}\Delta_{k,q_2}(\ell^2,1) = q_2 + O_k\left(\sum_{m=0}^{\infty}\frac{\log(2q_1^{2m})}{(q_1^m + kq_2)^{1/2}}\right).$$

N prime square

Lemma (DHKLMMRSZ 25+)

Let $N = p^2$ for some prime p, then

$$W_R(H_k^*(N)) = \frac{k-1}{12}(\varphi(N)-1) + O\left(\frac{1}{N^{1/4}}\right).$$

We appeal [BBD$^+$16].

N is prime

Lemma (DHKLMMRSZ 25+)

For $H_k^*(= H_k^*(N))$ where N is a prime, the following holds:

$$
\begin{aligned}
A'_r(p) &= O(N^{-r/2}) \\
A_r(p) &= \begin{cases} C_{r/2} + O\left(\frac{r2^r p^{r/4}\log(p^r N)}{N}\right) & \text{if } r \text{ even,} \\ O\left(\frac{r2^r p^{r/4}\log(p^r N)}{N}\right) & \text{if } r \text{ odd.} \end{cases} \\
B''_{r_1,r_2}(p_1,p_2) &= O\left(N^{-(r_1+r_2)/2}\right) \\
B'_{r_1,r_2}(p_1,p_2) &= O\left(2^{r_2} N^{-r_1/2}\right)
\end{aligned}
$$

* C_r is the r^{th} Catalan number.

N is prime

Lemma (DHKLMMRSZ 25+)

[cont.]

$$B_{r_1,r_2}(p_1,p_2) \;=\; \begin{cases} C_{(r_1+r_2)/2} + O\left(2^{r_1+r_2}(p_1^{r_1}p_2^{r_2})^{1/4}\frac{\log(2p_1^{r_1}p_2^{r_2}N)\log(N)}{k^{5/6}N}\right) \\ \text{if } p_1 = p_2 \text{ and } (r_1 - r_2 \bmod 2) = 0, \\ C_{r_1/2}C_{r_2/2} + O\left(2^{r_1+r_2}(p_1^{r_1}p_2^{r_2})^{1/4}\frac{\log(2p_1^{r_1}p_2^{r_2}N)\log(N)}{k^{5/6}N}\right) \\ \text{if } p_1 \neq p_2 \text{ and } r_1, r_2 \text{ even}, \\ O\left(2^{r_1+r_2}(p_1^{r_1}p_2^{r_2})^{1/4}\frac{\log(2p_1^{r_1}p_2^{r_2}N)\log(N)}{k^{5/6}N}\right) \\ \text{otherwise}. \end{cases}$$

Proof

For $p = N$, $\lambda_f(N)^2 \; = \; \frac{1}{N}$

$$A_r'(N) \; = \; O(N^{-r/2}).$$

For $p \neq N$

$$\begin{aligned}
A_r(p) \; &= \; \frac{1}{W_R(H_k^*()} \sum_{f_N^*} w_R(f)\lambda_f(p)^r \\
&= \; \frac{1}{W_R(H_N^*)} \sum_{f_N^*} w_R(f) \sum_{k=0}^{\lfloor r/2 \rfloor} b_{r,r-2k}\lambda_f(p^{r-2k}) \\
&= \; \sum_{\ell=0}^{\lfloor r/2 \rfloor} b_{r,r-2\ell} \left(\delta\left(p^{r-2\ell}, 1\right) + O\left(p^{\frac{r-2\ell}{4}} \frac{\log(2p^{r-2\ell}N)}{N} \right) \right) \\
&= \; \begin{cases} C_{r/2} + O\Big(\frac{r2^r p^{r/4}\log(p^r N)}{N}\Big) & \text{if } r \text{ even,} \\ O\Big(\frac{r2^r p^{r/4}\log(p^r N)}{N}\Big) & \text{if } r \text{ odd.} \end{cases}
\end{aligned}$$

Proof (cont.)

N is prime. If a prime p divides N, then $p = N$.

For all primes $p \mid N$, we have $\lambda_f(p)^2 = \frac{1}{p}$.

$$B''_{r_1,r_2,{}^*_k(N)}(N,N) \lesssim \frac{1}{N^{(r_1+r_2)/2}}.$$

For $p_2 \nmid N$, we have

$$\begin{aligned} B'_{r_1,r_2,{}^*_k(N)}(N,p_2) &= \frac{1}{W_R(H_N^*)} \sum_{f \in H_N^*} w_R(f)\lambda_f(N)^{r_1}\lambda_f(p_2)^{r_2} \\ &\lesssim \frac{1}{N^{r_1/2}} \left(\frac{1}{W_R(H_k^*(N))} \sum_{f \in H_k^*(N)} w_R(f)\lambda_f(p_2)^{r_2} \right) \lesssim \frac{2^{r_2}}{N^{r_1/2}}. \end{aligned}$$

Proof (cont.)

For $p_1 \nmid N$ and $p_2 \nmid N$

$$
\begin{aligned}
& B_{r_1,r_2,H_k^*(N)}(p_1,p_2) \\
&= \sum_{k_1=0}^{\lfloor r_1/2 \rfloor} \sum_{k_2=0}^{\lfloor r_2/2 \rfloor} b_{r_1,r_1-2k_1} b_{r_2,r_2-2k_2} \delta(p_1^{r_1-2k_1}, p_2^{r_2-2k_2}) + O\left((p_1^{r_1} p_2^{r_2})^{1/4} \frac{\log(2p_1^{r_1} p_2^{r_2} N)}{k^{5/6} N} \right) \\
&= \begin{cases}
O\left(2^{r_1+r_2} (p_1^{r_1} p_2^{r_2})^{1/4} \frac{\log(2p_1^{r_1} p_2^{r_2} N)}{k^{5/6} N} \right) & \text{if } p_1 = p_2 \text{ and } (r_1 - r_2 \bmod 2) = 1 \\
C_{(r_1+r_2)/2} + O\left(2^{r_1+r_2} (p_1^{r_1} p_2^{r_2})^{1/4} \frac{\log(2p_1^{r_1} p_2^{r_2} N)}{k^{5/6} N} \right) & \text{if } p_1 = p_2 \text{ and } (r_1 - r_2 \bmod 2) = 0 \\
O\left(2^{r_1+r_2} (p_1^{r_1} p_2^{r_2})^{1/4} \frac{\log(2p_1^{r_1} p_2^{r_2} N)}{k^{5/6} N} \right) & \text{if } p_1 \neq p_2 \text{ and } (r_1 - r_2 \bmod 2) = 1 \\
& \text{or } p_1 \neq p_2 \text{ and } r_1, r_2 \text{ odd} \\
C_{r_1/2} C_{r_2/2} + O\left(2^{r_1+r_2} (p_1^{r_1} p_2^{r_2})^{1/4} \frac{\log(2p_1^{r_1} p_2^{r_2} N)}{k^{5/6} N} \right) & \text{if } p_1 \neq p_2 \text{ and } r_1, r_2 \text{ both even.}
\end{cases}
\end{aligned}
$$

$N = q_1 \cdot q_2$, q_1 fixed, $q_2 \to \infty$.

Lemma (DHKLMMRSZ 25+)

Suppose $H_k^*(N) = H_k^*(N)$ where $N = q_1 \cdot q_2$ for fixed prime q_1 and $q_2 \to \infty$. Then

$$A'_{r,H_k^*(N)}(p) = \begin{cases} q_1^{-r/2} & \text{if } p = q_1 \text{ and } r \text{ even}, \\ O\left(\frac{\log(N)}{N}\right) & \text{if } p = q_1 \text{ and } r \text{ odd}, \\ O\left(N^{-r/2}\right) & \text{if } p = q_2. \end{cases}$$

$$A_{r,H_k^*(N)}(p) = \begin{cases} C_{r/2} + O_k\left(\frac{2^r p^{r/4} \log(p^r N) \log(N)}{N}\right) & \text{if } r \text{ even}, \\ O_k\left(\frac{2^r p^{r/4} \log(p^r N) \log(N)}{N}\right) & \text{if } r \text{ odd}. \end{cases}$$

$N = q_1 \cdot q_2$, q_1 fixed, $q_2 \to \infty$.

$$B''_{r_1,r_2,H_k^*(N)}(p_1,p_2) = \begin{cases} q_1^{-(r_1+r_2)/2} & \text{if } p_1 = p_2 = q_1,\ r_1 \equiv r_2 \pmod 2, \\ O\left(\frac{\log^2(N)}{N}\right) & \text{if } p_1 = p_2 = q_1,\ r_1 \not\equiv r_2 \pmod 2, \\ O\left(N^{-\lfloor \min(r_1,r_2)/2 \rfloor}\right) & \text{if } p_1 = q_2 \text{ or } p_2 = q_2. \end{cases}$$

$$B'_{r_1,r_2,H_k^*(N)}(p_1,p_2) = \begin{cases} \frac{C_{r_2/2}}{q_1^{r_1/2}} + O_{k,q_1}\left(\frac{2^{r_2} p_2^{r_2/4} \log(p_2^{r_2} N)}{N}\right) & \text{if } p_1 = q_2,\ r_1 \text{ even},\ r_2 \text{ even}, \\ O_{k,q_1}\left(\frac{2^{r_2} p_2^{r_2/4} \log(p_2^{r_2} N)}{N}\right) & \text{if } p_1 = q_2,\ r_1 \text{ even},\ r_2 \text{ odd}, \\ O_{q_1,k}\left(\frac{2^{r_2} p_2^{r_2/4} \log(p_2^{r_2})}{N}\right) & \text{if } p_1 = q_1 \text{ and } r_1 \text{ odd}, \\ O\left(\frac{2^{r_2} p_2^{r_2/4} \log(p_2^{r_2} N)}{N^{\lfloor r_1/2 \rfloor + 1}}\right) & \text{if } p_1 = q_2. \end{cases}$$

$N = q_1 \cdot q_2$, q_1 **fixed**, $q_2 \to \infty$.

Lemma (DHKLMMRSZ 25+)

[cont.]

$$B_{r_1,r_2}(p_1,p_2) = \begin{cases} C_{(r_1+r_2)/2} + O\left(2^{r_1+r_2}(p_1^{r_1}p_2^{r_2})^{1/4}\frac{\log(2p_1^{r_1}p_2^{r_2}N)\log(N)}{k^{5/6}N}\right) \\ \textit{if } p_1 = p_2 \textit{ and } (r_1 - r_2 \bmod 2) = 0, \\ C_{r_1/2}C_{r_2/2} + O\left(2^{r_1+r_2}(p_1^{r_1}p_2^{r_2})^{1/4}\frac{\log(2p_1^{r_1}p_2^{r_2}N)\log(N)}{k^{5/6}N}\right) \\ \textit{if } p_1 \neq p_2 \textit{ and } r_1, r_2 \textit{ even}, \\ O\left(2^{r_1+r_2}(p_1^{r_1}p_2^{r_2})^{1/4}\frac{\log(2p_1^{r_1}p_2^{r_2}N)\log(N)}{k^{5/6}N}\right) \\ \textit{otherwise}. \end{cases}$$

$N = q_1 q_2$, $q_1 \sim N^{\delta}$, $q_2 \sim N^{1-\delta}$

Lemma (DHKLMMRSZ 25+)

Let $H_k^*(N) = H_k^*(N)$ and $N = q_1 q_2$ with q_1, q_2 prime and $q_1 \sim N^{\delta}, q_2 \sim N^{1-\delta}$, where $\delta \in (0, 1/2]$. Then

$$A_r'(p) = O\left(\frac{1}{N^{r\delta/2}}\right)$$

$$A_{r,H_k^*(N)}(p) = \begin{cases} C_{r/2} + O_k\left(\frac{2^r p^{r/4}\log(p^r N)}{N}\right) & \text{if } r \text{ even,} \\ O_k\left(\frac{2^r p^{r/4}\log(p^r N)}{N}\right) & \text{if } r \text{ odd.} \end{cases}$$

$$B''_{r_1,r_2}(p_1,p_2) = O\left(\frac{1}{N^{(r_1+r_2)\delta/2}}\right)$$

$$B'_{r_1,r_2}(p_1,p_2) = O\left(\frac{2^{r_2}}{N^{r_1\delta/2}}\right)$$

$N = q_1 q_2$, $q_1 \sim N^{\delta}$, $q_2 \sim N^{1-\delta}$

Lemma (DHKLMMRSZ 25+)

[cont.]

$$B_{r_1,r_2}(p_1,p_2) = \begin{cases} C_{r_1/2}C_{r_2/2} + O\Big(2^{r_1+r_2}p_1^{r_1/4}p_2^{r_2/4}N^{-1}\log\big(2p_1^{r_1/4}p_2^{r_2/4}N\big)\Big) \\ \text{if } r_1, r_2 \text{ even and } p_1 \neq p_2, \\ C_{(r_1+r_2)/2} + O\Big(2^{r_1+r_2}p_1^{r_1/4}p_2^{r_2/4}N^{-1}\log\big(2p_1^{r_1/4}p_2^{r_2/4}N\big)\Big) \\ \text{if } r_1 - r_2 \equiv 0 \pmod 2 \text{ and } p_1 = p_2, \\ O\Big(2^{r_1+r_2}p_1^{r_1/4}p_2^{r_2/4}N^{-1}\log\big(2p_1^{r_1/4}p_2^{r_2/4}N\big)\Big) \\ \text{otherwise.} \end{cases}$$

Lemma (DHKLMMRSZ 25+)

Let $H_k^*(N) = H_k^*(N)$ and $N = p^2$ with $p \to \infty$. Then

$$\begin{aligned} A'_{r,H_k^*(N)}(q) &= 0 \\ A_{r,H_k^*(N)}(q) &= \begin{cases} C_{r/2} + O_k\left(2^r q^{\frac{r}{4}} \frac{\log(2q^r)}{N}\right) & r \text{ even} \\ O_k\left(2^r q^{\frac{r}{4}} \frac{\log(2q^r)}{N}\right) & r \text{ odd} \end{cases} \end{aligned}$$

$N = p^2$

Lemma (DHKLMMRSZ 25+)

[cont.]

$$B'_{r_1,r_2,H_k^*(N)}(p_1,p_2) = 0$$

$$B_{r_1,r_2,H_k^*(N)}(p_1,p_2) = \begin{cases} C_{r_1/2}C_{r_2/2} + O_k\left(\frac{1}{N}2^{r_1+r_2}p_1^{r_1/4}p_2^{r_2/4}\log\left(2p_1^{r_1}p_2^{r_2}\right)\right) \\ \text{if } p_1 \neq p_2 \text{ and } r_1,\ r_2 \text{ even} \\ O_k\left(\frac{1}{N}2^{r_1+r_2}p_1^{r_1/4}p_2^{r_2/4}\log\left(2p_1^{r_1}p_2^{r_2}\right)\right) \\ \text{if } p_1 \neq p_2 \text{ and } r_1,\ r_2 \text{ odd} \\ C_{\frac{r_1+r_2}{2}} + O_k\left(\frac{1}{N}2^{r_1+r_2}p_1^{r_1/4}p_2^{r_2/4}\log\left(2p_1^{r_1}p_2^{r_2}\right)\right) \\ \text{if } p_1 = p_2 \text{ and } r_1 - r_2 \equiv 0 \mod 2 \\ O_k\left(\frac{1}{N}2^{r_1+r_2}p_1^{r_1/4}p_2^{r_2/4}\log\left(2p_1^{r_1}p_2^{r_2}\right)\right) \\ \text{if } p_1 = p_2 \text{ and } r_1 - r_2 \equiv 1 \mod 2. \end{cases}$$

Important remark

For the three cases where N prime, N product of two primes with both factors going to infinity, and N square, the terms A', B'', and B' do not admit a main term.

On the other hand, for $N = q_1 q_2$ where q_1 is fixed, A', B'', and B' have main terms.

Moreover, in all four cases depending on N, even up to the error terms, the A and B terms are equal.

Lower order terms

We then substitute the results into the formula of $S_1(H_k^*(N), \phi)$ and $S_2(H_k^*(N), \phi_1, \phi_2)$ in order to evaluate the lower order terms.

Closing

Future work

Conjecture (DHKLMMRSZ 25+)

Given N, as long as the prime factors of N go to infinity fast enough relative to the error we are computing up to, the lower order terms of the n level density are the same as the N prime case.

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Conjecture (DHKLMMRSZ 25+)

Given N, as long as the prime factors of N go to infinity fast enough relative to the error we are computing up to, the lower order terms of the n level density are the same as the N prime case.
Interesting new factors emerge when one of the factors goes to infinity slower than the reciprocal of the error term, say at the rate of $\log(N)$, $\log^2(N)$, or $\log^3(N)$.

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