From the Manhattan Project to Elliptic Curves

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Goals

- Determine correct scale and statistics to study zeros of $L$-functions.
- See similar behavior in different systems.
- Discuss the tools and techniques needed to prove the results.
Introduction
Fundamental Problem: Spacing Between Events

**General Formulation:** Studying system, observe values at $t_1, t_2, t_3, \ldots$.

**Question:** What rules govern the spacings between the $t_i$?
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**Examples:** Spacings between
- Energy Levels of Nuclei.
- Eigenvalues of Matrices.
- Zeros of $L$-functions.
- Summands in Zeckendorf Decompositions.
- Primes.
- $n^k \alpha \mod 1$. 
Fundamental Problem: Spacing Between Events

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Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.

2. Develop an explicit formula relating what we want to study to something we understand.

3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!
Classical Random Matrix Theory

With Olivia Beckwith, Leo Goldmakher, Chris Hammond, Steven Jackson, Cap Khoury, Murat Koloğlu, Gene Kopp, Victor Luo, Adam Massey, Eve Ninsuwan, Vincent Pham, Karen Shen, Jon Sinsheimer, Fred Strauch, Nicholas Triantafillou, Wentao Xiong
Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.
Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.

Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

\[ H \psi_n = E_n \psi_n \]

- \( H \): matrix, entries depend on system
- \( E_n \): energy levels
- \( \psi_n \): energy eigenfunctions
Origins of Random Matrix Theory

- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\bar{A}^T = A$).
Random Matrix Ensembles

\[ A = \begin{pmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\
  a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN}
\end{pmatrix} = A^T, \quad a_{ij} = a_{ji} \]

Fix \( p \), define

\[ \text{Prob}(A) = \prod_{1 \leq i \leq j \leq N} p(a_{ij}). \]

This means

\[ \text{Prob}(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]) = \prod_{1 \leq i \leq j \leq N} \int_{x_{ij} = \alpha_{ij}}^{\beta_{ij}} p(x_{ij}) \, dx_{ij}. \]

Want to understand eigenvalues of \( A \).
Eigenvalue Distribution

\[ \delta(x - x_0) \] is a unit point mass at \( x_0 \):
\[
\int f(x) \delta(x - x_0) \, dx = f(x_0).
\]
Eigenvalue Distribution

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\]

To each \( A \), attach a probability measure:

\[
\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)
\]
Eigenvalue Distribution

\[ \delta(x - x_0) \] is a unit point mass at \( x_0 \):
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\]

\[
\int_{a}^{b} \mu_{A,N}(x)\,dx = \frac{\# \left\{ \lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a, b] \right\}}{N}
\]
Eigenvalue Distribution

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\]

\[ k^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N} \lambda_i(A)^k}{2^k N^{k/2 + 1}} = \frac{\text{Trace}(A^k)}{2^k N^{k/2 + 1}}. \]
Wigner’s Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all $A$, as $N \to \infty$

$$\mu_{A,N}(x) \to \begin{cases} \frac{2}{\pi} \sqrt{1 - x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$
SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of $A$, but choose the matrix elements randomly and independently.

**Eigenvalue Trace Lemma**

Let $A$ be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) = \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$
SKETCH OF PROOF: Correct Scale

\[
\text{Trace}(A^2) = \sum_{i=1}^{N} \lambda_i(A)^2.
\]

By the Central Limit Theorem:

\[
\text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} a_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \sim N^2
\]

\[
\sum_{i=1}^{N} \lambda_i(A)^2 \sim N^2
\]

Gives \( N \text{Ave}(\lambda_i(A)^2) \sim N^2 \) or \( \text{Ave}(\lambda_i(A)) \sim \sqrt{N} \).
SKETCH OF PROOF: Averaging Formula

Recall $k$-th moment of $\mu_{A,N}(x)$ is $\text{Trace}(A^k)/2^kN^{k/2+1}$.

Average $k$-th moment is

$$\int \cdots \int \frac{\text{Trace}(A^k)}{2^kN^{k/2+1}} \prod_{i \leq j} p(a_{ij}) \, da_{ij}.$$  

Proof by method of moments: Two steps

- Show average of $k$-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).
SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2 N^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \cdot p(a_{11}) da_{11} \cdots p(a_{NN}) da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij}) da_{ij} \cdot \prod_{\substack{(k,l) \neq (i,j) \\ k<l}} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl}) da_{kl} = 1.$$ 

Higher moments involve more advanced combinatorics (Catalan numbers).
SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

\[
\frac{1}{2^k N^{k/2 + 1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1i_2} \cdots a_{i_ki_1} \cdot \prod_{i \leq j} p(a_{ij}) da_{ij}.
\]

Main contribution when the \(a_{i_\ell i_{\ell+1}}\)'s matched in pairs, not all matchings contribute equally (if did would get a Gaussian and not a semi-circle; this is seen in Real Symmetric Palindromic Toeplitz matrices).


http://arxiv.org/abs/math/0512146
Numerical examples

500 Matrices: Gaussian $400 \times 400$

$$p(x) = \frac{1}{\sqrt{2 \pi}} e^{-x^2/2}$$
Numerical examples

The eigenvalues of the Cauchy distribution are NOT semicircular.

Cauchy Distribution: \( p(x) = \frac{1}{\pi(1+x^2)} \)


Block Circulant Ensemble  
(with Murat Koloğlu, Gene Kopp, Fred Strauch and Wentao Xiong)
The Ensemble of $m$-Block Circulant Matrices

Symmetric matrices periodic with period $m$ on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

\[
\begin{pmatrix}
  c_0 & c_1 & c_2 & c_3 \\
  c_1 & d_0 & d_1 & d_2 \\
  c_2 & d_1 & c_0 & c_1 \\
  c_3 & d_2 & c_1 & d_0 \\
  c_4 & d_3 & c_2 & d_0 \\
  c_2 & c_3 & c_4 & d_3 \\
  c_3 & d_2 & c_1 & d_0 \\
  c_0 & c_1 & c_2 & c_3 \\
\end{pmatrix}
\]

Choose distinct entries i.i.d.r.v.
Oriented Matchings and Dualization

Compute moments of eigenvalue distribution (as $m$ stays fixed and $N \to \infty$) using the combinatorics of pairings. Rewrite:

$$M_n(N) = \frac{1}{N^{2n+1}} \sum_{1 \leq i_1, \ldots, i_n \leq N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1})$$

$$= \frac{1}{N^{2n+1}} \sum_{\sim} \eta(\sim) m_{d_1}(\sim) \cdots m_{d_l}(\sim).$$

where the sum is over oriented matchings on the edges $\{(1, 2), (2, 3), \ldots, (n, 1)\}$ of a regular $n$-gon.
Oriented Matchings and Dualization

Figure: An oriented matching in the expansion for $M_n(N) = M_6(8)$. 
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.
Only Topology Matters

Think of pairings as topological identifications; the contributing ones give rise to orientable surfaces.

Contribution from such a pairing is $m^{-2g}$, where $g$ is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.
Computing the Even Moments

**Theorem: Even Moment Formula**

\[
M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k \left( \frac{1}{N} \right),
\]

with \( \varepsilon_g(k) \) the number of pairings of the edges of a \((2k)\)-gon giving rise to a genus \( g \) surface.

J. Harer and D. Zagier (1986) gave generating functions for the \( \varepsilon_g(k) \).
Harer and Zagier

\[
\begin{align*}
\lfloor k/2 \rfloor \sum_{g=0}^{[k/2]} \varepsilon_g(k) r^{k+1-2g} &= (2k - 1)!! c(k, r) \\
\text{where} \\
1 + 2 \sum_{k=0}^{\infty} c(k, r) x^{k+1} &= \left( \frac{1 + x}{1 - x} \right)^r.
\end{align*}
\]

Thus, we write

\[M_{2k} = m^{-(k+1)}(2k - 1)!! c(k, m).\]
A multiplicative convolution and Cauchy’s residue formula yield the characteristic function of the distribution.

\[
\phi(t) = \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} = \frac{1}{m} \sum_{k=0}^{\infty} \frac{(-t^2/2m)^k}{k!} c(k, m)
\]

\[
= \frac{1}{2\pi im} \oint_{|z|=2} \frac{1}{2z^{-1}} \left( \left( \frac{1+z^{-1}}{1-z^{-1}} \right)^m - 1 \right) e^{-t^2z/2m} \frac{dz}{z}
\]

\[
= \frac{1}{m} e^{-t^2/2m} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{1}{(\ell-1)!} \left( \frac{-t^2}{m} \right)^{\ell-1}
\]
Fourier transform and algebra yields

**Theorem: Koloğlu, Kopp and Miller**

The limiting spectral density function $f_m(x)$ of the real symmetric $m$-block circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \left(\frac{m}{r+s+1}\right) \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$ 

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.
Results (continued)

Figure: Plot for $f_1$ and histogram of eigenvalues of 100 circulant matrices of size $400 \times 400$. 
Results (continued)

**Figure:** Plot for $f_2$ and histogram of eigenvalues of 100 2-block circulant matrices of size $400 \times 400$. 
Results (continued)

**Figure**: Plot for $f_3$ and histogram of eigenvalues of 100 3-block circulant matrices of size $402 \times 402$. 
Results (continued)

**Figure:** Plot for $f_4$ and histogram of eigenvalues of 100 4-block circulant matrices of size $400 \times 400$. 
Figure: Plot for $f_8$ and histogram of eigenvalues of 100 8-block circulant matrices of size $400 \times 400$. 
Results (continued)

Figure: Plot for $f_{20}$ and histogram of eigenvalues of 100 20-block circulant matrices of size $400 \times 400$. 

...
Results (continued)

Figure: Plot of convergence to the semi-circle.

Introduction to $L$-Functions
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]
Riemann Zeta Function

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Unique Factorization: \( n = p_1^{r_1} \cdots p_m^{r_m}. \)
Riemann Zeta Function

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Unique Factorization: \( n = p_1^{r_1} \cdots p_m^{r_m}. \)

\[
\prod_{p} \left( 1 - \frac{1}{p^s} \right)^{-1} = \left[ 1 + \frac{1}{2^s} + \left( \frac{1}{2^s} \right)^2 + \cdots \right] \left[ 1 + \frac{1}{3^s} + \left( \frac{1}{3^s} \right)^2 + \cdots \right] \cdots \\
= \sum_{n} \frac{1}{n^s}.
\]
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \#\{p : p \text{ is prime, } p \leq x\} \]

Properties of \( \zeta(s) \) and Primes:
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \# \{ p : p \text{ is prime, } p \leq x \} \]

Properties of \( \zeta(s) \) and Primes:

- \( \lim_{s \to 1^+} \zeta(s) = \infty, \pi(x) \to \infty. \)
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \# \{ p : p \text{ is prime, } p \leq x \} \]

Properties of \( \zeta(s) \) and Primes:

- \( \lim_{s \to 1+} \zeta(s) = \infty, \pi(x) \to \infty. \)
- \( \zeta(2) = \frac{\pi^2}{6}, \pi(x) \to \infty. \)
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]

**Functional Equation:**

\[ \xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} \zeta(s) = \xi(1 - s). \]

**Riemann Hypothesis (RH):**

All non-trivial zeros have \( \text{Re}(s) = \frac{1}{2} \); can write zeros as \( \frac{1}{2} + i\gamma \).

**Observation:** Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices \( \overline{A^T} = A \).
General $L$-functions

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{\text{p prime}} L_p(s, f)^{-1}, \quad \text{Re}(s) > 1.$$  

**Functional Equation:**

$$\Lambda(s, f) = \Lambda_{\infty}(s, f)L(s, f) = \Lambda(1 - s, f).$$

**Generalized Riemann Hypothesis (RH):**

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

**Observation:** Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$. 

```
Elliptic Curves: Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.

Addition of distinct points $P$ and $Q$

Adding a point $P$ to itself

$E(\mathbb{Q}) \approx E(\mathbb{Q})_{tors} \oplus \mathbb{Z}^r$
Elliptic curve $L$-function

$E : y^2 = x^3 + ax + b$, associate $L$-function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \text{ mod } p\}.$$
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$E : y^2 = x^3 + ax + b$, associate $L$-function

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where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \text{ mod } p\}.$$
Properties of zeros of $L$-functions

- infinitude of primes, primes in arithmetic progression.
- Chebyshev’s bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ ‘most’ of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from $L$-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.
Distribution of zeros

\[ \zeta(s) \neq 0 \text{ for } \Re(s) = 1: \pi(x), \pi_{a,q}(x). \]

- **GRH:** error terms.

- **GSH:** Chebyshev’s bias.

- **Analytic rank, adjacent spacings:** \( h(D) \).
Explicit Formula (Contour Integration)

\[-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1}\]
Explicit Formula (Contour Integration)

\[- \frac{\zeta'(s)}{\zeta(s)} = - \frac{d}{ds} \log \zeta(s) = - \frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \]

\[= \frac{d}{ds} \sum_p \log (1 - p^{-s}) \]

\[= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s). \]
Explicit Formula (Contour Integration)

\[- \frac{\zeta'(s)}{\zeta(s)} = - \frac{d}{ds} \log \zeta(s) = - \frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \]

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\[= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s). \]

Contour Integration:

\[\int - \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \, ds \quad \text{vs} \quad \sum_p \log p \int \left( \frac{x}{p} \right)^s \frac{ds}{s}. \]
Explicit Formula (Contour Integration)

\[- \frac{\zeta'(s)}{\zeta(s)} = - \frac{d}{ds} \log \zeta(s) = - \frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \]

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Contour Integration:

\[\int - \frac{\zeta'(s)}{\zeta(s)} \phi(s) ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) p^{-s} ds.\]
Explicit Formula (Contour Integration)

\[- \frac{\zeta'(s)}{\zeta(s)} = - \frac{d}{ds} \log \zeta(s) = - \frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \]

= \frac{d}{ds} \sum_p \log (1 - p^{-s})

= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).

Contour Integration (see Fourier Transform arising):

\[ \int - \frac{\zeta'(s)}{\zeta(s)} \phi(s) ds \text{ vs } \sum_p \log p \int \phi(s) e^{-\sigma \log p} e^{-it \log p} ds. \]

Knowledge of zeros gives info on coefficients.
Explicit Formula: Example

**Dirichlet L-functions:** Let $\phi$ be an even Schwartz function and $L(s, \chi) = \sum_n \chi(n)/n^s$ a Dirichlet $L$-function from a non-trivial character $\chi$ with conductor $m$ and zeros $\rho = \frac{1}{2} + i\gamma_\chi$. Then

$$\sum_{\rho} \phi \left( \gamma_\chi \frac{\log(m/\pi)}{2\pi} \right) = \int_{-\infty}^{\infty} \phi(y) dy$$

$$-2 \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) \frac{\chi(p)}{p^{1/2}}$$

$$-2 \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( 2 \frac{\log p}{\log(m/\pi)} \right) \frac{\chi^2(p)}{p} + O\left( \frac{1}{\log m} \right).$$
Katz-Sarnak
Density Conjectures
Let $g_i$ be even Schwartz functions whose Fourier Transform is compactly supported, $L(s, f)$ an $L$-function with zeros $\frac{1}{2} + i\gamma_f$ and conductor $Q_f$:

$$D_{n,f}(g) = \sum_{j_1, \ldots, j_n \atop j_i \neq \pm j_k} g_1 \left( \gamma_{f,j_1} \frac{\log Q_f}{2\pi} \right) \cdots g_n \left( \gamma_{f,j_n} \frac{\log Q_f}{2\pi} \right)$$

- Properties of $n$-level density:
  - Individual zeros contribute in limit
  - Most of contribution is from low zeros
  - Average over similar $L$-functions (family)
**n-Level Density**

**n-level density**: \( \mathcal{F} = \bigcup \mathcal{F}_N \) a family of \( L \)-functions ordered by conductors, \( g_k \) an even Schwartz function: 

\[
D_{n,\mathcal{F}}(g) = \lim_{N \to \infty} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{j_1, \ldots, j_n} g_1 \left( \frac{\log Q_f}{2\pi} \gamma_{j_1;f} \right) \cdots g_n \left( \frac{\log Q_f}{2\pi} \gamma_{j_n;f} \right)
\]

As \( N \to \infty \), \( n \)-level density converges to

\[
\int g(\vec{x}) \rho_{n,\mathcal{G}(\mathcal{F})}(\vec{x}) d\vec{x} = \int \hat{g}(\vec{u}) \hat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\vec{u}) d\vec{u}.
\]

**Conjecture (Katz-Sarnak)**

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.
1-Level Densities

Let \( \mathcal{G} \) be one of the classical compact groups: Unitary, Symplectic, Orthogonal (or SO(even), SO(odd)). If \( \text{supp}(\hat{g}) \subset (-1, 1) \), 1-level density of \( \mathcal{G} \) is

\[
\hat{g}(0) - c_{\mathcal{G}} \frac{g(0)}{2},
\]

where

\[
c_{\mathcal{G}} = \begin{cases} 
0 & \text{if } \mathcal{G} \text{ is Unitary} \\
1 & \text{if } \mathcal{G} \text{ is Symplectic} \\
-1 & \text{if } \mathcal{G} \text{ is Orthogonal.}
\end{cases}
\]
Identifying the Symmetry Groups

- Often suggested by monodromy group in the function field.

- Tools: Explicit Formula, Summation Formula.

- How to identify symmetry group in general? One possibility is by the signs of the functional equation: **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise $\text{SO}(\text{even})$. (False!)

Explicit Formula

- $\pi$: cuspidal automorphic representation on $\text{GL}_n$.
- $Q_\pi > 0$: analytic conductor of $L(s, \pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake params: $\{\alpha_{\pi,i}(p)\}_{i=1}^n; \lambda_{\pi}(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s, \pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi,i}(p)p^{-s})^{-1}$.
- $\sum_j g\left(\gamma_{\pi,j} \frac{\log Q_\pi}{2\pi}\right) = \hat{g}(0) - 2 \sum_{p,\nu} \hat{g}\left(\frac{\nu \log p}{\log Q_\pi}\right) \frac{\lambda_{\pi}(p^\nu) \log p}{p^\nu/2 \log Q_\pi}$.
Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: \( c_L = 0 \) (resp, 1 or -1) if family \( L \) has unitary (resp, symplectic or orthogonal) symmetry.
Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: \( c_\mathcal{L} = 0 \) (resp, 1 or -1) if family \( \mathcal{L} \) has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for \( \pi_1,p \times \pi_2,p \) are \( \{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} = \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{1 \leq i \leq n}^{1 \leq j \leq m}. \)
Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: \( c_\mathcal{L} = 0 \) (resp, 1 or -1) if family \( \mathcal{L} \) has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for \( \pi_1, p \times \pi_2, p \) are

\[
\{ \alpha_{\pi_1 \times \pi_2}(k) \}_{k=1}^{nm} = \{ \alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j) \}_{1 \leq i \leq n, 1 \leq j \leq m}.
\]

Theorem (Dueñez-Miller)

If \( \mathcal{F} \) and \( \mathcal{G} \) are nice families of \( L \)-functions, then

\[
c_{\mathcal{F} \times \mathcal{G}} = c_\mathcal{F} \cdot c_\mathcal{G}.
\]

Breaks analysis of compound families into simple ones.

*The effect of convolving families of \( L \)-functions on the underlying group symmetries* (with Eduardo Dueñez),


1-Level Density

Assuming conductors constant in family \( \mathcal{F} \), have to study

\[
\nu^{\text{th}} \text{ moment: } \lambda_f(p^\nu) = \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu
\]

\[
S_1(\mathcal{F}) = -2 \sum_p \hat{g} \left( \frac{\log p}{\log R} \right) \frac{\log p}{\sqrt{p \log R}} \left[ \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p) \right]
\]

\[
S_2(\mathcal{F}) = -2 \sum_p \hat{g} \left( 2 \frac{\log p}{\log R} \right) \frac{\log p}{p \log R} \left[ \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p^2) \right]
\]

The corresponding classical compact group determined by

\[
\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p^2) = c_\mathcal{F} = \begin{cases} 
0 & \text{Unitary} \\
1 & \text{Symplectic} \\
-1 & \text{Orthogonal}
\end{cases}
\]
Takeaways

Very similar to Central Limit Theorem.

- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.
- First moment zero save for families of elliptic curves.
- Higher moments control convergence and can depend on arithmetic of family.
Correspondences

Similarities between $L$-Functions and Nuclei:

Zeros $\leftrightarrow$ Energy Levels

Schwartz test function $\rightarrow$ Neutron

Support of test function $\leftrightarrow$ Neutron Energy.
Main Tools

1. **Control of conductors**: Usually monotone, gives scale to study low-lying zeros.

2. **Explicit Formula**: Relates sums over zeros to sums over primes.

3. **Averaging Formulas**: Orthogonality of characters, Legendre Sums, Petersson Formula, Kuznestov Formula
Applications of $n$-level density

Bounding the order of vanishing at the central point.

Average rank $\cdot \phi(0) \leq \int \phi(x) W_{G(F)}(x) dx$ if $\phi$ non-negative.
Applications of $n$-level density

Bounding the order of vanishing at the central point.
Average rank · $\phi(0) \leq \int \phi(x) W_{G(F)}(x) dx$ if $\phi$ non-negative.

**Theorem (Miller, Hughes-Miller)**

*Using $n$-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any $r$ there is a $c_r$ such that probability of at least $r$ zeros at the central point is at most $c_n r^{-n}$.*

Better results from 2-level than Iwaniec-Luo-Sarnak for $r \geq 5$.

Cuspidal Newforms
(with Chris Hughes, Geoffrey Iyer and Nicholas Triantafillou)
Modular Form Preliminaries

\[ \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\} \]

\( f \) is a weight \( k \) holomorphic cuspform of level \( N \) if

\[ \forall \gamma \in \Gamma_0(N), \ f(\gamma z) = (cz + d)^k f(z). \]

- Fourier Expansion: \( f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i nz} \),

\[ L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}. \]

- Petersson Norm: \( \langle f, g \rangle = \int_{\Gamma_0(N) \backslash \mathbb{H}} f(z) \overline{g(z)} y^{k-2} \, dx \, dy \).

- Normalized coefficients:

\[ \psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \frac{1}{\|f\|}} a_f(n). \]
Modular Form Preliminaries: Petersson Formula

$B_k(N)$ an orthonormal basis for weight $k$ level $N$. Define

$$\Delta_{k,N}(m, n) = \sum_{f \in B_k(N)} \psi_f(m) \overline{\psi}_f(n).$$

**Petersson Formula**

$$\Delta_{k,N}(m, n) = 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m, n, c)}{c} J_{k-1}\left(4\pi \frac{\sqrt{mn}}{c}\right) + \delta(m, n).$$
Modular Form Preliminaries: Explicit Formula

\( \mathcal{F} \) a family of cuspidal newforms (weight \( k \), prime level \( N \) and possibly split by sign): 
\[ L(s, f) = \sum n \frac{\lambda_f(n)}{n^s}. \]
Then

\[
\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sum_{\gamma_f} \phi \left( \frac{\log R}{2\pi} \gamma_f \right) = \hat{\phi}(0) + \frac{1}{2} \phi(0) - \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} P(f; \phi) + O \left( \frac{\log \log R}{\log R} \right)
\]

\[ P(f; \phi) = \sum_{p \mid N} \lambda_f(p) \hat{\phi} \left( \frac{\log p}{\log R} \right) \frac{2 \log p}{\sqrt{p} \log R}. \]
Modular Form Preliminaries: Fourier Coefficient Review

\[ \lambda_f(n) = a_f(n)n^{\frac{k-1}{2}} \]

\[ \lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \ (d,M)=1}} \lambda_f \left( \frac{mn}{d} \right). \]

For a newform of level \( N \), \( \lambda_f(N) \) is trivially related to the sign of the form,

\[ \varepsilon_f = i^k \mu(N)\lambda_f(N)\sqrt{N}, \]

allowing us to split into even and odd families: \( 1 \pm \varepsilon_f \).
Theorem (ILS)

Let \( \Psi \) be an even Schwartz function with \( \text{supp}(\hat{\Psi}) \subset (-2, 2) \). Then

\[
\sum_{m \leq N^\varepsilon} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2, b)R(1, b)}{\varphi(b)} \int_{y=0}^\infty J_{k-1}(y) \hat{\Psi} \left( \frac{2 \log(by\sqrt{N}/4\pi m)}{\log R} \right) \frac{dy}{\log R} = -\frac{1}{2} \left[ \int_{-\infty}^\infty \varphi(x) \frac{\sin 2\pi x}{2\pi x} \, dx - \frac{1}{2} \varphi(0) \right] + O \left( \frac{k \log \log kN}{\log kN} \right),
\]

where \( R = k^2 N \) and \( \varphi \) is Euler’s totient function.

Key Kloosterman-Bessel integral from ILS

Ramanujan sum:

\[
R(n, q) = \sum_{a \mod q}^* e(an/q) = \sum_{d|(n,q)} \mu(q/d) d,
\]

where \( * \) restricts the summation to be over all \( a \) relatively prime to \( q \).
2-Level Density

\[
\int_{x_1=2}^{R^\sigma} \int_{x_2=2}^{R^\sigma} \hat{\phi} \left( \frac{\log x_1}{\log R} \right) \hat{\phi} \left( \frac{\log x_2}{\log R} \right) J_{k-1} \left( 4\pi \frac{\sqrt{m^2 x_1 x_2 N}}{c} \right) \frac{dx_1 dx_2}{\sqrt{x_1 x_2}}
\]

Change of variables and Jacobean:

\[
\begin{align*}
  u_2 &= x_1 x_2 \\
  u_1 &= x_1 \\
  x_2 &= \frac{u_2}{u_1} \\
  x_1 &= u_1
\end{align*}
\]

\[
\left| \frac{\partial x}{\partial u} \right| = \left| \begin{array}{cc}
  1 & 0 \\
  -\frac{u_2}{u_1} & \frac{1}{u_1}
\end{array} \right| = \frac{1}{u_1}.
\]

Left with

\[
\int \int \hat{\phi} \left( \frac{\log u_1}{\log R} \right) \hat{\phi} \left( \frac{\log \left( \frac{u_2}{u_1} \right)}{\log R} \right) \frac{1}{\sqrt{u_2}} J_{k-1} \left( 4\pi \frac{\sqrt{m^2 u_2 N}}{c} \right) \frac{du_1 du_2}{u_1}
\]
2-Level Density

Changing variables, $u_1$-integral is

$$\int_{w_1 = \frac{\log u_2}{\log R} - \sigma}^{\sigma} \hat{\phi}(w_1) \hat{\phi}\left(\frac{\log u_2}{\log R} - w_1\right) dw_1.$$

Support conditions imply

$$\psi_2\left(\frac{\log u_2}{\log R}\right) = \int_{w_1 = -\infty}^{\infty} \hat{\phi}(w_1) \hat{\phi}\left(\frac{\log u_2}{\log R} - w_1\right) dw_1.$$

Substituting gives

$$\int_{u_2 = 0}^{\infty} J_{k-1}\left(4\pi \frac{\sqrt{m^2 u_2 N}}{c}\right) \psi_2\left(\frac{\log u_2}{\log R}\right) \frac{du_2}{\sqrt{u_2}}$$
$n$-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$
\frac{b\sqrt{N}}{2\pi m} \int_{0}^{\infty} J_{k-1}(x) \Phi_n \left( \frac{2 \log(bx \sqrt{N}/4\pi m)}{\log R} \right) \frac{dx}{\log R}
$$

with $\Phi_n(x) = \phi(x)^n$.

**Difficulty:** instead of $n$-dimensional integral of $\phi_1(x_1) \cdots \phi_n(x_n)$ have a 1-dimensional integral of a new test function. Harder combinatorics but can appeal to ILS.


*Moment Formulas for Ensembles of Classical Compact Groups* (with Geoffrey Iyer and Nicholas Triantafillou), preprint.
Elliptic Curves: First Zero Above Central Point
(with E. Dueñez, D. K. Huynh, J. P. Keating, N. Snaith)
Theoretical results: \( y^2 = x^3 + A(T)x + B(T) \)

**Theorem: M–’04**

For small support, one-param family of rank \( r \) over \( \mathbb{Q}(T) \):

\[
\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{E_t \in F_N} \sum_j \varphi \left( \frac{\log C_{E_t}}{2\pi} \gamma_{E_t,j} \right) = \int \varphi(x) \rho_G(x) dx + r\varphi(0)
\]

where \( G = \begin{cases} 
\text{SO(odd)} & \text{if half odd} \\
\text{SO(even)} & \text{if all even} \\
\text{weighted average} & \text{otherwise.}
\end{cases} \)

Supports Katz-Sarnak, B-SD, and Independent model in limit.

**Independent Model:**

\[
\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} \mathbf{I}_{2r \times 2r} & g \end{pmatrix} : g \in \text{SO}(2N - 2r) \right\}.
\]
Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank $r$ over $\mathbb{Q}(T)$. Natural sub-families:

- Curves of rank $r$.
- Curves of rank $r + 2$. 
Interesting Families

Let $E : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank $r$ over $\mathbb{Q}(T)$. Natural sub-families:

- Curves of rank $r$.
- Curves of rank $r + 2$.

**Question:** Does the sub-family of rank $r + 2$ curves in a rank $r$ family behave like the sub-family of rank $r + 2$ curves in a rank $r + 2$ family?

Equivalently, does it matter how one conditions on a curve being rank $r + 2$?
Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank $r$ one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r + 2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.
Excess Rank

One-parameter family, rank $r$ over $\mathbb{Q}(T)$.
Density Conjecture (Generic Family) $\Rightarrow$ 50% rank $r$, $r+1$.

For many families, observe
Percent with rank $r$ $\approx$ 32%
Percent with rank $r+1$ $\approx$ 48%
Percent with rank $r+2$ $\approx$ 18%
Percent with rank $r+3$ $\approx$ 2%

Problem: small data sets, sub-families, convergence rate log(conductor).
Data on Excess Rank

\[ y^2 + y = x^3 + Tx \]

Each set is 2000 curves, last has conductors of size $10^{17}$, (small on logarithmic scale).

<table>
<thead>
<tr>
<th>t-Start</th>
<th>Rk 0</th>
<th>Rk 1</th>
<th>Rk 2</th>
<th>Rk 3</th>
<th>Time (hrs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1000</td>
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<td>47.8</td>
<td>12.3</td>
<td>0.6</td>
<td>&lt;1</td>
</tr>
<tr>
<td>1000</td>
<td>38.4</td>
<td>47.3</td>
<td>13.6</td>
<td>0.6</td>
<td>&lt;1</td>
</tr>
<tr>
<td>4000</td>
<td>37.4</td>
<td>47.8</td>
<td>13.7</td>
<td>1.1</td>
<td>1</td>
</tr>
<tr>
<td>8000</td>
<td>37.3</td>
<td>48.8</td>
<td>12.9</td>
<td>1.0</td>
<td>2.5</td>
</tr>
<tr>
<td>24000</td>
<td>35.1</td>
<td>50.1</td>
<td>13.9</td>
<td>0.8</td>
<td>6.8</td>
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<tr>
<td>50000</td>
<td>36.7</td>
<td>48.3</td>
<td>13.8</td>
<td>1.2</td>
<td>51.8</td>
</tr>
</tbody>
</table>
RMT: Theoretical Results ($N \to \infty$)

1st normalized ev value above 1: SO(even)
RMT: Theoretical Results ($N \rightarrow \infty$)

1st normalized evaluae above 1: SO(odd)
Rank 2 Curves: 1st Norm. Zero above the Central Point

665 rank 2 curves from

\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]

\[ \log(\text{cond}) \in [10, 10.3125], \text{ median} = 2.29, \text{ mean} = 2.30 \]
Rank 2 Curves: 1st Norm. Zero above the Central Point

665 rank 2 curves from

\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \]

\[ \log(\text{cond}) \in [16, 16.5], \text{ median} = 1.81, \text{ mean} = 1.82 \]
Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

209 rank 0 curves from 14 rank 0 families, \( \log(\text{cond}) \in [3.26, 9.98] \), median = 1.35, mean = 1.36
Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

996 rank 0 curves from 14 rank 0 families, \( \log(\text{cond}) \in [15.00, 16.00] \), median = .81, mean = .86.
Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)

1st Normalized Zero above Central Point

35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\tilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_{\mu} = .41$
Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)

1st Normalized Zero above Central Point

34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\bar{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = 0.34$
The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.

As the conductors increased, the repulsion decreased.

Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i.e., shifted by the same amount).
Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j = \text{imaginary part of } j^{\text{th}} \text{ normalized zero above the central point}$;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>863 Rank 0 Curves</th>
<th>701 Rank 2 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Median</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_2 - z_1$</td>
<td>1.28</td>
<td>1.30</td>
<td>-1.60</td>
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<tr>
<td><strong>Mean</strong></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>$z_2 - z_1$</td>
<td>1.30</td>
<td>1.34</td>
<td></td>
</tr>
<tr>
<td><strong>StDev</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_2 - z_1$</td>
<td>0.49</td>
<td>0.51</td>
<td></td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_3 - z_2$</td>
<td>1.22</td>
<td>1.19</td>
<td></td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_3 - z_2$</td>
<td>1.24</td>
<td>1.22</td>
<td>0.80</td>
</tr>
<tr>
<td><strong>StDev</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_3 - z_2$</td>
<td>0.52</td>
<td>0.47</td>
<td></td>
</tr>
<tr>
<td><strong>Median</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_3 - z_1$</td>
<td>2.54</td>
<td>2.56</td>
<td>-0.38</td>
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<tr>
<td><strong>Mean</strong></td>
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<tr>
<td>$z_3 - z_1$</td>
<td>2.55</td>
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<td><strong>StDev</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z_3 - z_1$</td>
<td>0.52</td>
<td>0.52</td>
<td></td>
</tr>
</tbody>
</table>
Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j = \text{imaginary part of the } j^{\text{th}} \text{ norm zero above the central point}$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>64 Rank 2 Curves</th>
<th>23 Rank 4 Curves</th>
<th>t-Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median</td>
<td>$z_2 - z_1$</td>
<td>1.26</td>
<td>1.27</td>
</tr>
<tr>
<td>Mean</td>
<td>$z_2 - z_1$</td>
<td>1.36</td>
<td>1.29</td>
</tr>
<tr>
<td>StDev</td>
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<tr>
<td>StDev</td>
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</tr>
<tr>
<td>Median</td>
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</tr>
<tr>
<td>StDev</td>
<td>$z_3 - z_1$</td>
<td>0.44</td>
<td>0.42</td>
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</tbody>
</table>
Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j = $ imaginary part of the $j^{th}$ norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

<table>
<thead>
<tr>
<th></th>
<th>701 Rank 2 Curves</th>
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<td><strong>StDev</strong> $z_3 - z_1$</td>
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New Model for Finite Conductors

- Replace conductor $N$ with $N_{\text{effective}}$.
  - Arithmetic info, predict with $L$-function Ratios Conj.
  - Do the number theory computation.

- Excised Orthogonal Ensembles.
  - $L(1/2, E)$ discretized.
  - Char. polys $\Lambda_A(\theta) = \det(I - e^{i\theta} A^{-1})$ model $L(1/2 + it, E)$.
  - Study matrices in $SO(2N_{\text{eff}})$ with $|\Lambda_A(0)| \geq ce^N$.

- Painlevé VI differential equation solver.
  - Use explicit formulas for densities of Jacobi ensembles.
  - Key input: Selberg-Aomoto integral for initial conditions.
Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$.

Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with $N_{\text{eff}}$ (solid), standard $N_0$ (dashed).
Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$

Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): $N_{\text{eff}} = 2$ (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.


http://arxiv.org/pdf/1005.1298

Open Questions and References
Open Questions: Low-lying zeros of $L$-functions

- Generalize excised ensembles for higher weight $GL_2$ families where expect different discretizations.

- Obtain better estimates on vanishing at the central point by finding optimal test functions for the second and higher moment expansions.

- Further explore $L$-function Ratios Conjecture to predict lower order terms in families, compute these terms on number theory side.

See Dueñez-Huynh-Keating-Miller-Snaith, Miller, and the Ratios papers.
Open Questions: Non-determinantal Expansions

• More tractable for comparing number theory and random matrix theory.

• New combinatorial terms when support exceeds $\frac{1}{n-k}$.

• Goal: analyze these terms on both sides.

• Note: extending support for $L$-functions related to subtle arithmetic questions.

See Hughes-Miller and Iyer-Miller-Triantafillou.
Open Questions: Dirichlet $L$-functions

- Extend support for families of Dirichlet $L$-functions. Related to deep questions on distribution of primes in residue classes.

- Show agreement for extended support and RMT. For quadratic Dirichlet $L$-functions, with Levinson reduced $n$-level density agreement to a combinatorial / Fourier transform identity, whose proof currently eludes us.

See Fiorilli-Miller, Levinson-Miller.
Open Questions: Random Matrix Theory

- How the structure of the ensemble affects the density of eigenvalues and spacings b/w them. 1000 Toeplitz matrices (1000 × 1000), entries N(0, 1), spacings b/w middle 11 eigenvalues.

- Determine the combinatorial impact of weights on the eigenvalue distribution in d-regular graphs.

See Random Matrix Theory papers.
Publications: Random Matrix Theory

   http://arxiv.org/abs/math/0312215

   http://arxiv.org/abs/math/0512146

   http://arxiv.org/abs/math/0611649


   http://arxiv.org/abs/1008.4812

   http://arxiv.org/abs/1112.3719

Publications: \(L\)-Functions


# Publications: Elliptic Curves


Publications: \textit{L-Function Ratio Conjecture}


Appendix: 
Dirichlet $L$-functions

The purpose of this appendix is to give details of a 1-level density calculation for a tractable family.
Dirichlet Characters \((m \text{ prime})\)

\((\mathbb{Z}/m\mathbb{Z})^*\) is cyclic of order \(m - 1\) with generator \(g\). Let \(\zeta_{m-1} = e^{2\pi i / (m-1)}\). The principal character \(\chi_0\) is given by

\[
\chi_0(k) = \begin{cases} 
1 & (k, m) = 1 \\
0 & (k, m) > 1.
\end{cases}
\]

The \(m - 2\) primitive characters are determined (by multiplicativity) by action on \(g\).

As each \(\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*\), for each \(\chi\) there exists an \(l\) such that \(\chi(g) = \zeta_l^{m-1}\). Hence for each \(l, 1 \leq l \leq m - 2\) we have

\[
\chi_l(k) = \begin{cases} 
\zeta_l^{m-1} & k \equiv g^a(m) \\
0 & (k, m) > 0
\end{cases}
\]
Dirichlet $L$-Functions

Let $\chi$ be a primitive character mod $m$. Let

$$c(m, \chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k / m}.$$

$c(m, \chi)$ is a Gauss sum of modulus $\sqrt{m}$.

$$L(s, \chi) = \prod_p (1 - \chi(p) p^{-s})^{-1}$$

$$\Lambda(s, \chi) = \pi^{-\frac{1}{2}(s+\epsilon)} \Gamma \left( \frac{s + \epsilon}{2} \right) m^{\frac{1}{2}(s+\epsilon)} L(s, \chi),$$

where

$$\epsilon = \begin{cases} 
0 & \text{if } \chi(-1) = 1 \\
1 & \text{if } \chi(-1) = -1 
\end{cases}$$
Explicit Formula

Let $\phi$ be an even Schwartz function with compact support $(-\sigma, \sigma)$, let $\chi$ be a non-trivial primitive Dirichlet character of conductor $m$.

$$
\sum \phi \left( \gamma \frac{\log(m/\pi)}{2\pi} \right) = \int_{-\infty}^{\infty} \phi(y) dy
$$

$$
- \sum_{p} \frac{\log p}{\log(m/\pi)} \, \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) \left[ \chi(p) + \bar{\chi}(p) \right] p^{-\frac{1}{2}}
$$

$$
- \sum_{p} \frac{\log p}{\log(m/\pi)} \, \hat{\phi} \left( 2 \frac{\log p}{\log(m/\pi)} \right) \left[ \chi^2(p) + \bar{\chi}^2(p) \right] p^{-1}
$$

$$
+ O \left( \frac{1}{\log m} \right).
$$
{\chi_0} \cup \{\chi_l\}_{1 \leq l \leq m-2} \text{ are all the characters mod } m.

Consider the family of primitive characters mod a prime } m \text{ (} m - 2 \text{ characters)}:

\[
\int_{-\infty}^{\infty} \phi(y) dy \\
- \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \hat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}} \\
- \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \hat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) [\chi^2(p) + \bar{\chi}^2(p)] p^{-1} \\
+ O\left(\frac{1}{\log m}\right).
\]

Note can pass Character Sum through Test Function.
Character Sums

\[ \sum_{\chi} \chi(k) = \begin{cases} 
  m - 1 & k \equiv 1(m) \\ 
  0 & \text{otherwise.}
\end{cases} \]
Character Sums

\[ \sum_{\chi} \chi(k) = \begin{cases} 
  m - 1 & k \equiv 1(m) \\
  0 & \text{otherwise.}
\end{cases} \]

For any prime \( p \neq m \)

\[ \sum_{\chi \neq \chi_0} \chi(p) = \begin{cases} 
  -1 + m - 1 & p \equiv 1(m) \\
  -1 & \text{otherwise.}
\end{cases} \]
Character Sums

\[ \sum_{\chi} \chi(k) = \begin{cases} 
  m - 1 & k \equiv 1(m) \\
  0 & \text{otherwise.} 
\end{cases} \]

For any prime \( p \neq m \)

\[ \sum_{\chi \neq \chi_0} \chi(p) = \begin{cases} 
  -1 + m - 1 & p \equiv 1(m) \\
  -1 & \text{otherwise.} 
\end{cases} \]

Substitute into

\[ \frac{1}{m - 2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}(\frac{\log p}{\log(m/\pi)}) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}} \]
First Sum: no contribution if $\sigma < 2$

\[
-\frac{2}{m-2} \sum_{p} \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}}
\]

\[
+ \frac{m-1}{m-2} \sum_{p \equiv 1(m)} \log p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}}
\]
First Sum: no contribution if $\sigma < 2$

\[
\begin{align*}
&\left(-\frac{2}{m-2}\right) \sum_{p}^{m^\sigma} \frac{\log p}{\log(m/\pi)} \hat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} \\
+ \ &\frac{2}{m-2} \sum_{p \equiv 1(m)}^{m^\sigma} \frac{\log p}{\log(m/\pi)} \hat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} \\
\ll \ &\frac{1}{m} \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} + \sum_{p \equiv 1(m)}^{m^\sigma} p^{-\frac{1}{2}}
\end{align*}
\]
First Sum: no contribution if $\sigma < 2$

\[
\begin{align*}
\frac{-2}{m-2} \sum_{p} & \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}} \\
+ & \frac{2(m-1)}{m-2} \sum_{p \equiv 1(m)} \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}} \\
\ll & \frac{1}{m} \sum_{p} p^{-\frac{1}{2}} + \sum_{p \equiv 1(m)} p^{-\frac{1}{2}} \ll \frac{1}{m} \sum_{k} m^{\sigma} k^{-\frac{1}{2}} + \sum_{k \equiv 1(m)} k^{-\frac{1}{2}}
\end{align*}
\]
First Sum: no contribution if $\sigma < 2$

\[
-\frac{2}{m - 2} \sum_{p}^{m^\sigma} \frac{\log p}{\log(m/\pi)} \hat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}}
\]

\[
+ \frac{2m - 1}{m - 2} \sum_{p \equiv 1(m)}^{m^\sigma} \frac{\log p}{\log(m/\pi)} \hat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m} \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} + \sum_{p \equiv 1(m)}^{m^\sigma} p^{-\frac{1}{2}} \ll \frac{1}{m} \sum_{k}^{m^\sigma} k^{-\frac{1}{2}} + \sum_{k \equiv 1(m)}^{k \geq m+1} k^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m} \sum_{k}^{m^\sigma} k^{-\frac{1}{2}} + \frac{1}{m} \sum_{k}^{m^\sigma} k^{-\frac{1}{2}} \ll \frac{1}{m} m^{\sigma/2}.
\]
Second Sum

\[
\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left(2 \frac{\log p}{\log(m/\pi)} \right) \frac{\chi^2(p) + \bar{\chi}^2(p)}{p}.
\]

\[
\sum_{\chi \neq \chi_0} [\chi^2(p) + \bar{\chi}^2(p)] = \begin{cases} 
2(m-2) & p \equiv \pm 1(m) \\
-2 & p \not\equiv \pm 1(m)
\end{cases}
\]

Up to \(O\left(\frac{1}{\log m}\right)\) we find that

\[
\ll \frac{1}{m-2} \sum_p p^{-1} + \frac{2m-2}{m-2} \sum_{p \equiv \pm 1(m)} p^{-1}
\]

\[
\ll \frac{1}{m-2} \sum_k k^{-1} + \sum_{k \equiv 1(m)} k^{-1} + \sum_{k \equiv -1(m)} k^{-1}
\]
Dirichlet Characters: \( m \) Square-free

Fix an \( r \) and let \( m_1, \ldots, m_r \) be distinct odd primes.

\[
m = m_1 m_2 \cdots m_r
\]

\[
M_1 = (m_1 - 1)(m_2 - 1) \cdots (m_r - 1) = \phi(m)
\]

\[
M_2 = (m_1 - 2)(m_2 - 2) \cdots (m_r - 2).
\]

\( M_2 \) is the number of primitive characters mod \( m \), each of conductor \( m \). A general primitive character mod \( m \) is given by \( \chi(u) = \chi_{l_1}(u)\chi_{l_2}(u) \cdots \chi_{l_r}(u) \). Let \( \mathcal{F} = \{ \chi : \chi = \chi_{l_1}\chi_{l_2} \cdots \chi_{l_r} \} \).

\[
\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}} \sum_{\chi \in \mathcal{F}} [\chi(p) + \bar{\chi}(p)]
\]

\[
\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( 2 \frac{\log p}{\log(m/\pi)} \right) p^{-1} \sum_{\chi \in \mathcal{F}} [\chi^2(p) + \bar{\chi}^2(p)]
\]
Characters Sums

\[ \sum_{i=1}^{m_i-2} \chi_{l_i}(p) = \begin{cases} m_i - 1 - 1 & p \equiv 1(m_i) \\ -1 & \text{otherwise.} \end{cases} \]

Define

\[ \delta_{m_i}(p, 1) = \begin{cases} 1 & p \equiv 1(m_i) \\ 0 & \text{otherwise.} \end{cases} \]

Then

\[ \sum_{\chi \in \mathcal{F}} \chi(p) = \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}(p) \cdots \chi_{l_r}(p) \]

\[ = \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \prod_{i=1}^{r} \left( -1 + (m_i - 1)\delta_{m_i}(p, 1) \right). \]
Expansion Preliminaries

$k(s)$ is an $s$-tuple $(k_1, k_2, \ldots, k_s)$ with $k_1 < k_2 < \cdots < k_s$. This is just a subset of $(1, 2, \ldots, r)$, $2^r$ possible choices for $k(s)$.

$$\delta_{k(s)}(p, 1) = \prod_{i=1}^{s} \delta_{m_{k_i}}(p, 1).$$

If $s = 0$ we define $\delta_{k(0)}(p, 1) = 1 \ \forall p$. Then

$$\prod_{i=1}^{r} \left( -1 + (m_i - 1)\delta_{m_i}(p, 1) \right)$$

$$= \sum_{s=0}^{r} \sum_{k(s)} (-1)^{r-s} \delta_{k(s)}(p, 1) \prod_{i=1}^{s} (m_{k_i} - 1).$$
First Sum

\[ \ll \sum_{p} p^{-\frac{1}{2}} \frac{1}{M_2} \left( 1 + \sum_{s=1}^{r} \sum_{k(s)} \delta_{k(s)}(p, 1) \prod_{i=1}^{s} (m_{k_i} - 1) \right). \]

As \( m/M_2 \leq 3^r \), \( s = 0 \) sum contributes

\[ S_{1,0} = \frac{1}{M_2} \sum_{p} p^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2}} \sigma^{-1}, \]

hence negligible for \( \sigma < 2 \).
First Sum

\[ \ll \sum_{p} p^{-\frac{1}{2}} \frac{1}{M_2} \left( 1 + \sum_{s=1}^{r} \sum_{k(s)} \delta_k(s)(p, 1) \prod_{i=1}^{s} (m_{k_i} - 1) \right). \]

Now we study

\[ S_{1, k(s)} = \frac{1}{M_2} \prod_{i=1}^{s} (m_{k_i} - 1) \sum_{p} p^{-\frac{1}{2}} \delta_k(s)(p, 1) \]

\[ \ll \frac{1}{M_2} \prod_{i=1}^{s} (m_{k_i} - 1) \sum_{n \equiv 1(m_{k(s)})} n^{-\frac{1}{2}} \]

\[ \ll \frac{1}{M_2} \prod_{i=1}^{s} (m_{k_i} - 1) \frac{1}{\prod_{i=1}^{s} (m_{k_i})} \sum_{n} n^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2} \sigma - 1}. \]
First Sum

There are $2^r$ choices, yielding

$$S_1 \ll 6^r m^{\frac{1}{2} \sigma - 1},$$

which is negligible as $m$ goes to infinity for fixed $r$ if $\sigma < 2$. Cannot let $r$ go to infinity.

If $m$ is the product of the first $r$ primes,

$$\log m = \sum_{k=1}^{r} \log p_k$$

$$= \sum_{p \leq r} \log p \approx r$$

Therefore

$$6^r \approx m^{\log 6} \approx m^{1.79}.$$
Second Sum Expansions and Bounds

\[
\sum_{l_i=1}^{m_i-2} \chi^2_{l_i}(p) = \begin{cases} 
    m_i - 1 - 1 & \text{if } p \equiv \pm 1(m_i) \\
    -1 & \text{otherwise}
\end{cases}
\]

\[
\sum_{\chi \in \mathcal{F}} \chi^2(p) = \sum_{l_1=1}^{m_1-2} \chi_{l_1}^2(p) \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_r}^2(p)
\]

\[
= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p)
\]

\[
= \prod_{i=1}^{r} \left(-1 + (m_i - 1)\delta_{m_i}(p, 1) + (m_i - 1)\delta_{m_i}(p, -1)\right).
\]
Second Sum Expansions and Bounds

Handle similarly as before. Say

\[ p \equiv 1 \mod m_{k_1}, \ldots, m_{k_a} \]
\[ p \equiv -1 \mod m_{k_{a+1}}, \ldots, m_{k_b} \]

How small can \( p \) be?

+1 congruences imply \( p \geq m_1 \cdots m_{k_a} + 1 \).

-1 congruences imply \( p \geq m_{k_{a+1}} \cdots m_{k_b} - 1 \).

Since the product of these two lower bounds is greater than \( \prod_{i=1}^{b}(m_{k_i} - 1) \), at least one must be greater than

\[ \left( \prod_{i=1}^{b}(m_{k_i} - 1) \right)^{\frac{1}{2}}. \]

There are \( 3^r \) pairs, yielding

\[ \text{Second Sum} = \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} S_{2,k(s),j(s)} \ll 9^r m^{-\frac{1}{2}}. \]
Agrees with Unitary for $\sigma < 2$ for square-free $m \in [N, 2N]$ from:

**Theorem**

- $m$ square-free odd integer with $r = r(m)$ factors;
- $m = \prod_{i=1}^{r} m_i$;
- $M_2 = \prod_{i=1}^{r} (m_i - 2)$.

Then family $\mathcal{F}_m$ of primitive characters mod $m$ has

**First Sum** $\ll \frac{1}{M_2} 2^r m_2^{\frac{1}{2} \sigma}$

**Second Sum** $\ll \frac{1}{M_2} 3^r m_2^{\frac{1}{2}}$. 