From the Manhattan Project to Elliptic Curves

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Introduction

Goals of the Talk

- Discuss applications of zeros of L-functions.
- Explain old and new models for these zeros.
- Highlight power of data and conjectures.

Joint with many colleagues and students over the years:

- Faculty: Eduardo Dueñez, Frank W. K. Firk, Chris Hughes, Jon Keating, Nina Snaith, Siman Wong.
- Graduate Students: Scott Arms, Duc Khiem Huynh, Alvaro Lozano-Robledo, Tim Novikoff, Anthony Sabelli.
- Undergraduates: John Goes, Chris Hammond, Steven Jackson, Gene Kopp, Murat Kologlu, Adam Massey, David Montague, Ralph Morrison, Kesinee Ninsuwan, Ryan Peckner, Thuy Pham, John Sinsheimer.

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k\alpha$ mod 1.
- Spacings b/w Zeros of L-functions.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.
2. Develop an explicit formula relating what we want to study to something we understand.
3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Classical
Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

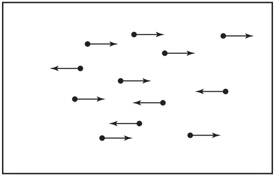
$$H\psi_n \ = \ E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

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$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

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$$\text{k}^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \ = \ \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \ = \ \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\begin{aligned} \mathrm{Trace}(A^2) &= \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \sim N^2 \\ \sum_{i=1}^{N} \lambda_i(A)^2 &\sim N^2 \end{aligned}$$

Gives $N\,\mathrm{Ave}(\lambda_i(A)^2) \sim N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ji}^2\cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty}a_{ij}^2p(a_{ij})da_{ij}\quad\cdot\prod_{\substack{(k,l)\neq(i,j)\\ k<l}}\int_{a_{kl}=-\infty}^{\infty}p(a_{kl})da_{kl}\;=\;1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

L-functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

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Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

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Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right]\left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right] \cdots$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function (cont)

$$\begin{aligned}\zeta(s) &= \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1 \\ \pi(x) &= \#\{p : p \text{ is prime}, p \le x\}\end{aligned}$$

Properties of $\zeta(s)$ and Primes:

Riemann Zeta Function (cont)

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Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.

Riemann Zeta Function (cont)

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Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.
- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L-functions

$$L(s,f) \ = \ \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \ = \ \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

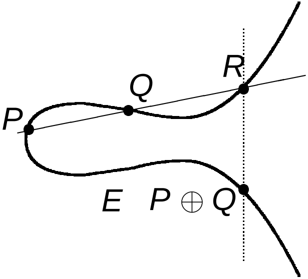
$$\Lambda(s,f) \ = \ \Lambda_\infty(s,f)L(s,f) \ = \ \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

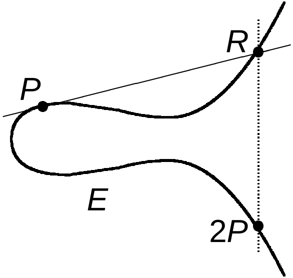
All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Elliptic Curves: Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Elliptic curve L-function

$E : y^2 = x^3 + ax + b$, associate L-function

$$L(s, E) \;=\; \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

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Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s,E)$ at $s = 1/2$.

Properties of zeros of *L*-functions

- infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from *L*-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Katz-Sarnak Density Conjectures

Measures of Spacings: *n*-Level Density and Families

Let g_i be even Schwartz functions whose Fourier Transform is compactly supported, $L(s,f)$ an L-function with zeros $\frac{1}{2}+\mathrm{i}\gamma_f$ and conductor Q_f:

$$D_{n,f}(g) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\gamma_{f,j_1}\frac{\log Q_f}{2\pi}\right)\cdots g_n\left(\gamma_{f,j_n}\frac{\log Q_f}{2\pi}\right)$$

- Properties of *n*-level density:
  - ⋄ Individual zeros contribute in limit
  - ⋄ Most of contribution is from low zeros
  - ⋄ Average over similar *L*-functions (family)

n-Level Density

n-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of L-functions ordered by conductors, g_k an even Schwartz function: $D_{n,\mathcal{F}}(g) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right)\cdots g_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N\to\infty$, n-level density converges to

$$\int g(\overrightarrow{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{x})d\overrightarrow{x} \;=\; \int \widehat{g}(\overrightarrow{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{u})d\overrightarrow{u}.$$

Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

1-Level Densities

Let $\mathcal{G}$ be one of the classical compact groups: Unitary, Symplectic, Orthogonal (or SO(even), SO(odd)).
If $\text{supp}(\widehat{g}) \subset (-1, 1)$, 1-level density of $\mathcal{G}$ is

$$\widehat{g}(0) \quad - \quad c_{\mathcal{G}} \, \frac{g(0)}{2},$$

where

$$c_{\mathcal{G}} \;=\; \begin{cases} 0 & \mathcal{G} \text{ is Unitary} \\ 1 & \mathcal{G} \text{ is Symplectic} \\ -1 & \mathcal{G} \text{ is Orthogonal.} \end{cases}$$

Some Results

- **Orthogonal:**
  ⋄ Iwaniec-Luo-Sarnak, Hughes-Miller: n-level density for $H_k^{\pm}(N)$, N square-free.
  ⋄ Miller, Young: families of elliptic curves.
  ⋄ Güloğlu: 1-level for $\{\mathrm{Sym}^r f : f \in H_k(1)\}$, r odd.
- **Symplectic:**
  ⋄ Gao, Rubinstein: n-level densities for $L(s, \chi_d)$.
  ⋄ Güloğlu: 1-level for $\{\mathrm{Sym}^r f : f \in H_k(1)\}$, r even.
  ⋄ Fouvry-Iwaniec, Miller-Peckner: 1-level for number field L-functions.
- **Unitary:**
  ⋄ Hughes-Rudnick, Miller: families of primitive Dirichlet characters.

Identifying the Symmetry Groups

- Often an analysis of the monodromy group in the function field case suggests the answer.
- All simple families studied to date are built from GL_1 or GL_2 *L*-functions.
- Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula.
- How to identify symmetry group in general? One possibility is by the signs of the functional equation:
- **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise SO(even). (False!)

Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake parameters $\{\alpha_{\pi,i}(p)\}_{i=1}^n$;
  $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi,i}(p)p^{-s})^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu}\widehat{g}\left(\frac{\nu\log p}{\log Q_\pi}\right)\frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are

$$\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} = \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Theorem (Dueñez-Miller)

If $\mathcal{F}$ and $\mathcal{G}$ are *nice* families of L-functions, then $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

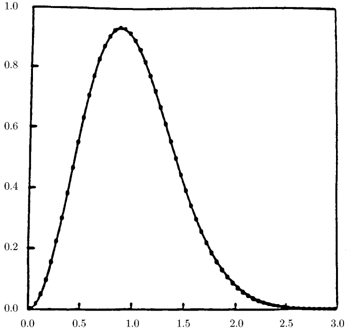
Correspondences

Similarities between *L*-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longrightarrow$	Neutron
Support of test function	$\longleftrightarrow$	Neutron Energy.

Theory and Models

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko) versus RMT prediction.

Orthogonal Random Matrix Models

RMT: $SO(2N)$: $2N$ eigenvalues in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

Independent Model:

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interaction Model: Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$: $1 \le j,k \le N-r$:

$$d\varepsilon_{2r}(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Indep):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Comparing the RMT Models

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi} \gamma_{E_t,j}\right) = \int \varphi(x) \rho_{\mathcal{G}}(x) dx + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd.} \end{cases}$$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

Sketch of Proof

- Explicit Formula: Relates sums over zeros to sums over primes.
- Averaging Formulas: Orthogonality of characters, Petersson formula.
- Control of conductors: Monotone.

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

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$$= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right)$$

$$= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.
$$

Explicit Formula (Contour Integration)

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\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
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&= \sum_p\frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p\frac{\log p}{p^\sigma}\int\phi(s)e^{-it\log p}ds.
$$

Knowledge of zeros gives info on coefficients.

1-Level Expansion

$$\begin{aligned}
D_{1,\mathcal{F}_N}(\phi) &= \frac{1}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\sum_j \phi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) \\
&= \frac{1}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\widehat{\phi}(0)+\phi_i(0) \\
&\quad - \frac{2}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\sum_p \frac{\log p}{\log C_{E_t}}\frac{1}{p}\,\widehat{\phi}\left(\frac{\log p}{\log C_{E_t}}\right)a_{E_t}(p) \\
&\quad - \frac{2}{|\mathcal{F}_N|}\sum_{E_t\in\mathcal{F}_N}\sum_p \frac{\log p}{\log C_{E_t}}\frac{1}{p^2}\,\widehat{\phi}\left(2\frac{\log p}{\log C_{E_t}}\right)a_{E_t}^2(p) \\
&\quad + O\left(\frac{\log\log C_{E_t}}{\log C_{E_t}}\right)
\end{aligned}$$

Input

For many families

$$\begin{aligned}(1): A_{1,\mathcal{F}}(p) &= -r + O(p^{-1})\\ (2): A_{2,\mathcal{F}}(p) &= p + O(p^{1/2})\end{aligned}$$

Rational Elliptic Surfaces (Rosen and Silverman): If rank r over $\mathbb{Q}(T)$:

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p\le X} -A_{1,\mathcal{F}}(p)\log p = r$$

Surfaces with $j(T)$ non-constant (Michel):

$$A_{2,\mathcal{F}}(p) = p + O\left(p^{1/2}\right).$$

Interesting Families and Testing RMT Predictions

Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank r over $\mathbb{Q}(T)$.

Know the right model for large conductors, want the correct model for finite conductors. Must explain:

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r + 2$.
2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

Data

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	701 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

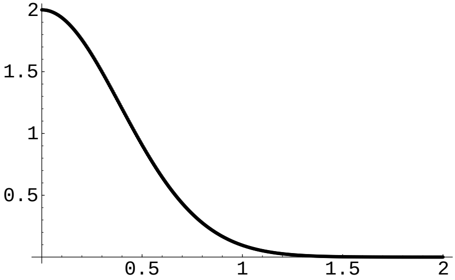
	64 Rank 2 Curves	23 Rank 4 Curves	t-Statistic
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

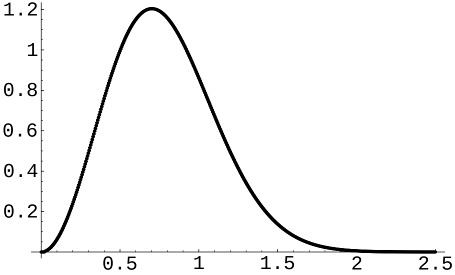
	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 0 Curves: 1st Normalized Zero above Central Point

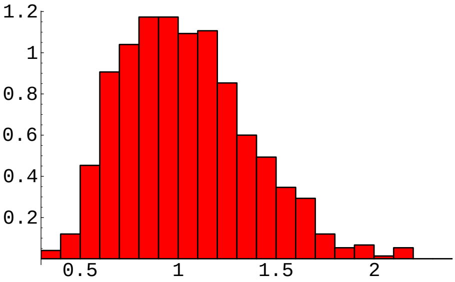


Figure 2a: 750 rank 0 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\mathrm{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$,
$\sigma_\mu = .32$

Rank 0 Curves: 1st Normalized Zero above Central Point

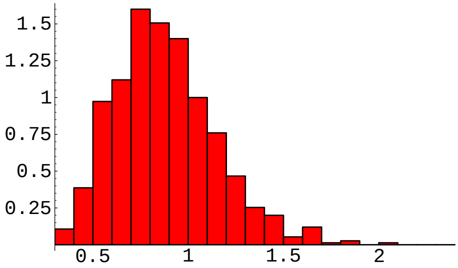


Figure 2b: 750 rank 0 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$,
$\sigma_\mu = .27$

Rank 2 Curves: 1st Norm. Zero above the Central Point

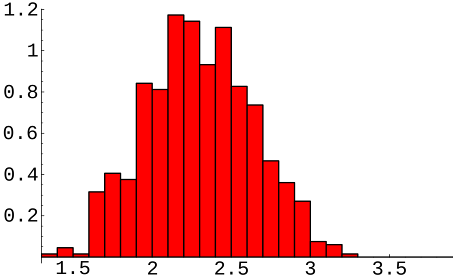


Figure 3a: 665 rank 2 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

Rank 2 Curves: 1st Norm. Zero above the Central Point

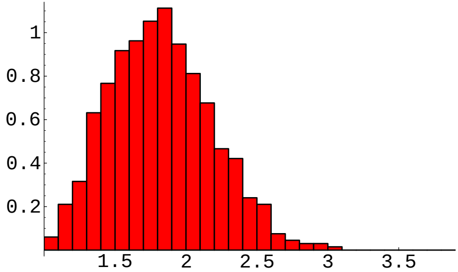


Figure 3b: 665 rank 2 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

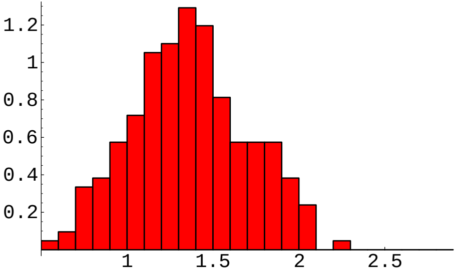


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

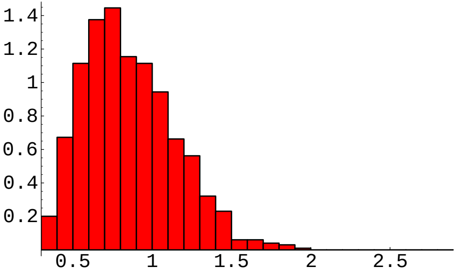


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

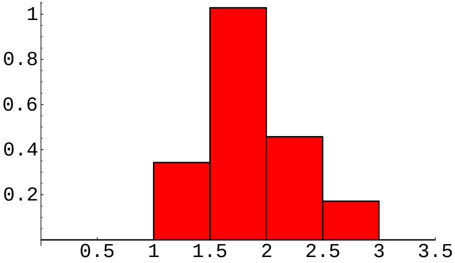


Figure 5a: 35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

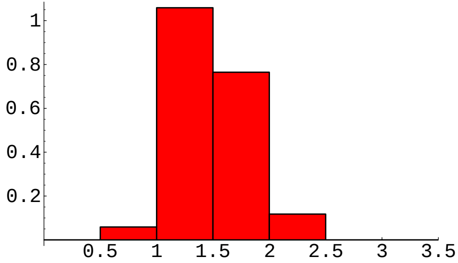


Figure 5b: 34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

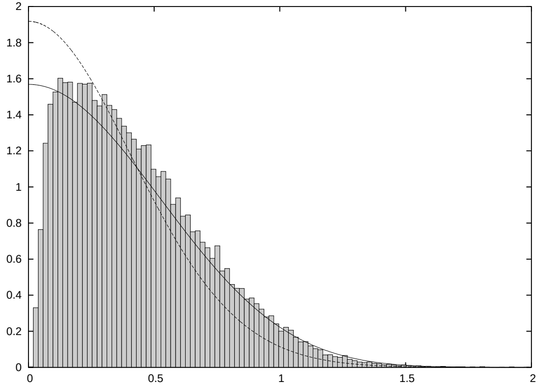
New Model for Finite Conductors

- Replace conductor N with $N_{\text{effective}}$.
  - ◇ Arithmetic info, predict with L-function Ratios Conj.
  - ◇ Do the number theory computation.

- Discretize Jacobi ensembles.
  - ◇ $L(1/2, E)$ discretized.
  - ◇ Study matrices in $\mathrm{SO}(2N_{eff})$ with $|\Lambda_A(1)| \geq ce^N$.

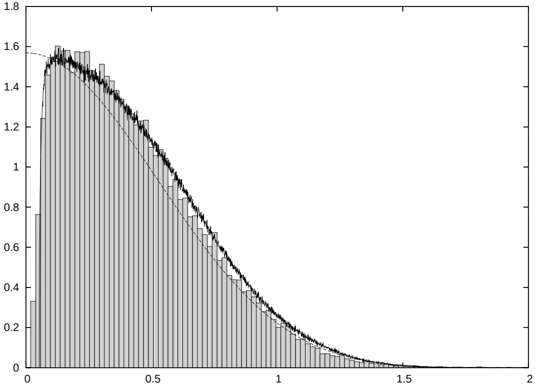
- Painlevé VI differential equation solver.
  - ◇ Use explicit formulas for densities of Jacobi ensembles.
  - ◇ Key input: Selberg-Aomoto integral for initial conditions.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



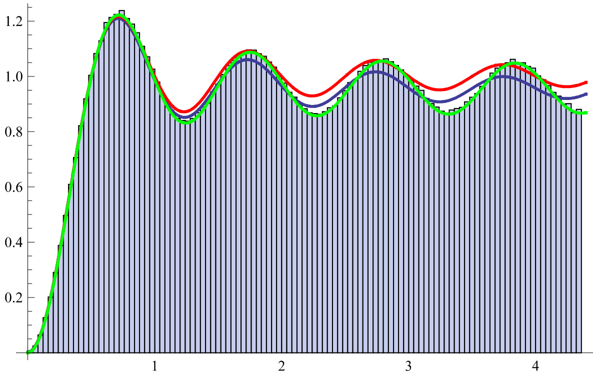
Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \le 1$, about 4 million).

$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.

$\diamond$ Green: all lower order terms.

Appendix

RH and the Prime Number Theorem

From $\zeta(s) = \sum n^{-s} = \prod_p \left(1 - p^{-s}\right)^{-1}$, logarithmic derivative is

$$-\frac{\zeta'(s)}{\zeta(s)} \ = \ \sum \frac{\Lambda(n)}{n^s},$$

where $\Lambda(n) = \log p$ if $n = p^k$ and is 0 otherwise.

Take Mellin transform, integrate and shift contour. Find

$$\sum_{n \le x} \Lambda(n) \ = \ x - \sum_{\rho} \frac{x^\rho}{\rho},$$

where $\rho = 1/2 + i\gamma$ runs over non-trivial zeros of $\zeta(s)$.

Partial summation gives Prime Number Theorem (to first order, there are $x/\log x$ primes at most x) if $\Re\mathfrak{e}\,\rho < 1$.

The smaller $\max \Re\mathfrak{e}(\rho)$ is, the better the error term in the Prime Number Theorem. The Riemann Hypothesis (RH) says $\Re\mathfrak{e}(\rho) = 1/2$.

Primes in Arithmetic Progression

To study number primes $p \equiv a \bmod q$, use

$$L(s,\chi) = \sum \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Key sum: $\frac{1}{\phi(q)} \sum_{\chi \bmod q} \chi(n)$ is 1 if $n \equiv 1 \bmod q$ and 0 otherwise.

Similar arguments give

$$\sum_{p \equiv a \bmod q} \frac{\log p}{p^s} = -\frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{L'(s,\chi)}{L(s,\chi)} \chi(\overline{a}) + \mathrm{Good}(s).$$

Note: To understand $\{p \equiv a \bmod q\}$ need to understand *all* $L(s,\chi)$; see benefit of studying a family.

GSH and Chebyshev's Bias

$\pi_{3,4}(x) \geq \pi_{1,4}(x)$ and $\pi_{2,3}(x) \geq \pi_{1,3}(x)$ 'most' of the time. Use analytic density:

$$\mathrm{Den}_{\mathrm{an}}(S) \ = \ \limsup \frac{1}{\log T} \int_{S\cap[2,T]} \frac{dt}{t}.$$

Have $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ with analytic density .9959 (first flip at 26861); $\pi_{2,3}(x) \geq \pi_{1,3}(x)$ with analytic density .9990 (first flip $\approx 6 \cdot 10^{11}$).

Non-residues beat residues. Key ingredient Generalized Simplicity Hypothesis (GSH): the zeros of $L(s,\chi)$ are linearly independent over $\mathbb{Q}$.

Structure of zeros important: GSH used to show a flow on a torus is full (becomes equidistributed).

Class Number

Class number: measures failure of unique factorization (order of ideal class group).

Imaginary quadratic field $Q(\sqrt{D})$, fundamental discriminant $D < 0$, I group of non-zero fractional ideals, P subgroup of principal ideals, $\mathcal{H} = I/P$ class group, $h(D) = \#\mathcal{H}$ the class number. Dirichlet proved

$$L(1, \chi_D) \ = \ \frac{2\pi h(D)}{w_D\sqrt{D}},$$

where χ_D the quadratic character and $w_D = 2$ if $D < -4$, 4 if $D = -4$ and 6 if $D = -3$.

Theorem: $h(D) = 1 \Leftrightarrow -D \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$.

Class Number and Distribution of Zeros I

Expect $\frac{\sqrt{|D|}}{\log\log|D|} \ll h(D) \ll \sqrt{|D|}\log\log|D|$. Siegel proved $h(D) > c(\epsilon)|D|^{1/2-\epsilon}$ (but ineffective).

Goldfeld, Gross-Zagier: f primitive cusp form of weight k, level N, trivial central character, suppose $m = \mathrm{ord}_{s=1/2}L(s,f)L(s,\chi_D) \geq 3$, $g = m-1$ or $m-2$ so that $(-1)^g = \omega(f)\omega(f_{\chi_D})$ (signs of fnal eqs). Then have effective bound

$$h(D) \gg (\log|D|)^{g-1}\prod_{p|D}\left(1+\frac{1}{p}\right)^{-3}\left(1+\frac{\lambda(p)\sqrt{p}}{p+1}\right)^{-1}.$$

Good result from using an elliptic curve that vanishes to order 3 at $s = 1/2$, application of many zeros at central point.

Class Number and Distribution of Zeros II

Assume a positive percent of zeros (or $cT\log T/(\log|D|)^A$) of zeros with $\gamma \le T$) of $\zeta(s)$ are at most $1/2-\epsilon$ of the average spacing from the next zero $\zeta(s)$. Then $h(D) \gg \sqrt{|D|}/(\log|D|)^B$, all constants computable.

See actual spacings between zeros are tied to number theory (have positive percent are less than half the average spacing if GUE Conjecture holds for adjacent spacings).

Instead of $1/2-\epsilon$, under RH have: .68 (Montgomery), .5179 (Montgomery-Odlyzko), .5171 (Conrey-Ghosh-Gonek), .5169 (Conrey-Iwaniec) (Montgomery says led to pair correlation conjecture by looking at gaps between zeros of $\zeta(s)$ and $h(D)$).

Ratios Conjecture

History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\, dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} = \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  - ◇ n-level correlations and densities;
  - ◇ mollifiers;
  - ◇ moments;
  - ◇ vanishing at the central point;

- Advantages:
  - ◇ RMT models often add arithmetic ad hoc;
  - ◇ predicts lower order terms, often to square-root level.

Inputs for 1-level density

- Approximate Functional Equation:

$$L(s,f) \;=\; \sum_{m\le x}\frac{a_m}{m^s} + \epsilon\mathbb{X}_L(s)\sum_{n\le y}\frac{a_n}{n^{1-s}};$$

$\diamond$ ϵ sign of the functional equation,
$\diamond$ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

- Explicit Formula: g Schwartz test function,

$$\sum_{f\in\mathcal{F}}\omega_f\sum_{\gamma} g\left(\gamma\frac{\log N_f}{2\pi}\right) \;=\; \frac{1}{2\pi i}\int_{(c)} - \int_{(1-c)} R'_{\mathcal{F}}(\cdots)g\left(\cdots\right)$$

$\diamond$ $R'_{\mathcal{F}}(r) = \frac{\partial}{\partial\alpha}R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=r}$.

Procedure

- Use approximate functional equation to expand numerator.
- Expand denominator by generalized Mobius function: cusp form

$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.
- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

Procedure: Steps in red are invalid!

- Use approximate functional equation to expand numerator.
- Expand denominator by generalized Mobius function: cusp form
$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$
where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.
- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

Symplectic Families

- Fundamental discriminants: d square-free and 1 modulo 4, or $d/4$ square-free and 2 or 3 modulo 4.
- Associated character χ_d:
  ◇ $\chi_d(-1) = 1$ say d even;
  ◇ $\chi_d(-1) = -1$ say d odd.
  ◇ even (resp., odd) if $d > 0$ (resp., $d < 0$).

Will study following families:

◇ even fundamental discriminants at most X;
◇ $\{8d : 0 < d \leq X, d$ an odd, positive square-free fundamental discriminant$\}$.

Prediction from Ratios Conjecture

$$\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} g\left(\gamma_d\frac{\log X}{2\pi}\right) = \frac{1}{X^*\log X}\int_{-\infty}^{\infty} g(\tau)\sum_{d\le X}\left[\log\frac{d}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\pm\frac{i\pi\tau}{\log X}\right)\right]d\tau$$

$$+\frac{2}{X^*\log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau)\left[\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)+A_D'\left(\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right.$$

$$\left.-e^{-2\pi i\tau\log(d/\pi)/\log X}\frac{\Gamma\left(\frac{1}{4}-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4}+\frac{\pi i\tau}{\log X}\right)}\zeta\left(1-\frac{4\pi i\tau}{\log X}\right)A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right]d\tau + O(X^{-\frac{1}{2}+\epsilon}),$$

with

$$A_D(-r,r) = \prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac{1}{p}\right)^{-1}$$

$$A_D'(r;r) = \sum_p\frac{\log p}{(p+1)(p^{1+2r}-1)}.$$

Prediction from Ratios Conjecture

Main term is

$$\frac{1}{X^*}\sum_{d\leq X}\sum_{\gamma_d} g\left(\gamma_d \frac{\log X}{2\pi}\right) = \int_{-\infty}^{\infty} g(x)\left(1-\frac{\sin(2\pi x)}{2\pi x}\right)dx + O\left(\frac{1}{\log X}\right),$$

which is the 1-level density for the scaling limit of $\mathrm{USp}(2N)$. If $\mathrm{supp}(\widehat{g}) \subset (-1, 1)$, then the integral of $g(x)$ against $-\sin(2\pi x)/2\pi x$ is $-g(0)/2$.

Prediction from Ratios Conjecture

Assuming RH for $\zeta(s)$, for $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$:

$$\frac{-2}{X^* \log X} \sum_{d \le X} \int_{-\infty}^{\infty} g(\tau)\, e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau$$

$$= \; -\frac{g(0)}{2} + O(X^{-\frac{3}{4}(1-\sigma)+\epsilon});$$

the error term may be absorbed into the $O(X^{-1/2+\epsilon})$ error if $\sigma < 1/3$.

Main Results

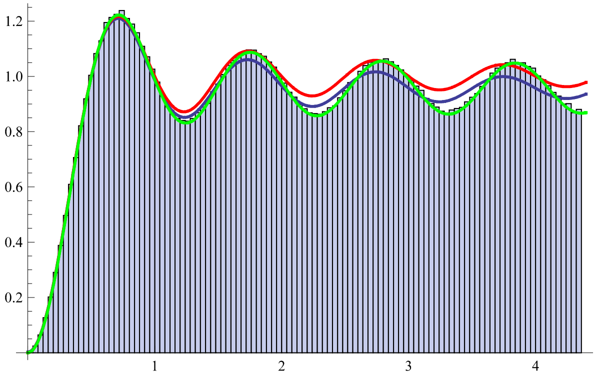
Theorem (M– '07)

Let $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma)$*, assume RH for* $\zeta(s)$*. 1-Level Density agrees with prediction from Ratios Conjecture*

- *up to* $O(X^{-(1-\sigma)/2+\epsilon})$ *for the family of quadratic Dirichlet characters with even fundamental discriminants at most* X*;*
- *up to* $O(X^{-1/2} + X^{-(1-\frac{3}{2}\sigma)+\epsilon} + X^{-\frac{3}{4}(1-\sigma)+\epsilon})$ *for our sub-family. If* $\sigma < 1/3$ *then agrees up to* $O(X^{-1/2+\epsilon})$*.*

Have similar results with students for other ensembles.

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).

$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.

$\diamond$ Green: all lower order terms.