From the Manhattan Project to Number Theory: How Nuclear Physics Helps Us Understand Primes

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Introduction

Fundamental Problem: Spacing Between Events

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- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.

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- Spacings b/w Primes.
- Bus routes in Cuernavaca, Mexico.
- Scandinavian trees?

Background Material: Linear Algebra

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Example:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Background Material: Probability

Probability Density

A random variable X has a probability density $p(x)$ if

- $p(x) \geq 0$;
- $\int_{-\infty}^{\infty} p(x)dx = 1$;
- $\mathrm{Prob}(X \in [a,b]) = \int_a^b p(x)dx$.

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Examples:

1. Exponential: $p(x) = e^{-x/\lambda}/\lambda$ for $x \geq 0$;
2. Normal: $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$;
3. Uniform: $p(x) = \frac{1}{b-a}$ for $a \leq x \leq b$ and 0 otherwise.

Background Material: Central Limit Theorem

k^{th} moment: $\int_{-\infty}^{\infty} x^k p(x)dx$.

Central Limit Theorem

Let $X_1, X_2, \dots$ be independent, identically distributed random variables with mean μ, standard deviation σ and finite higher moments. Then

$$Y_n = \frac{\frac{X_1+\cdots+X_N}{N} - \mu}{\sigma/\sqrt{N}}, \quad \lim_{N\to\infty} Y_N = N(0,1).$$

- Universality.
- Rate of onvergence depends on higher moments.

Classical
Random Matrix Theory

Origins of Random Matrix Theory

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Fundamental Equation:

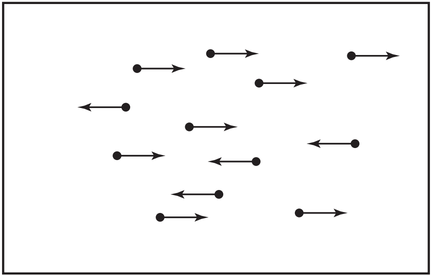
$$H\psi_n \ = \ E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins (continued)



- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

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This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij} = \alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Eigenvalue Trace Lemma

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Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \ = \ \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \ = \ \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

Density of States

Wigner's Semi-Circle Law

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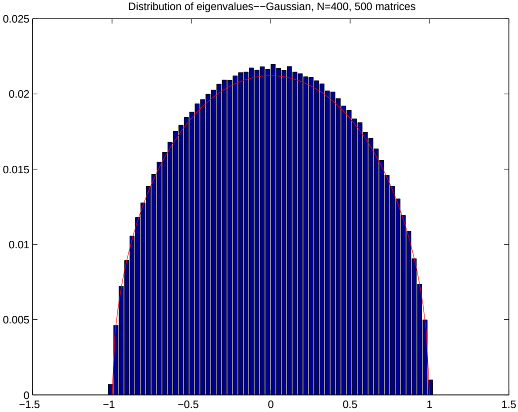
$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Numerical example: Gaussian density



500 Matrices: Gaussian 400×400

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Numerical example: Cauchy density $p(x) = 1/\pi(1 + x^2)$

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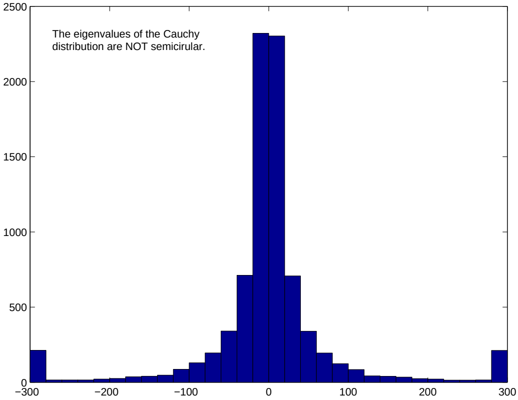

Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

Spacings between events

GOE Conjecture

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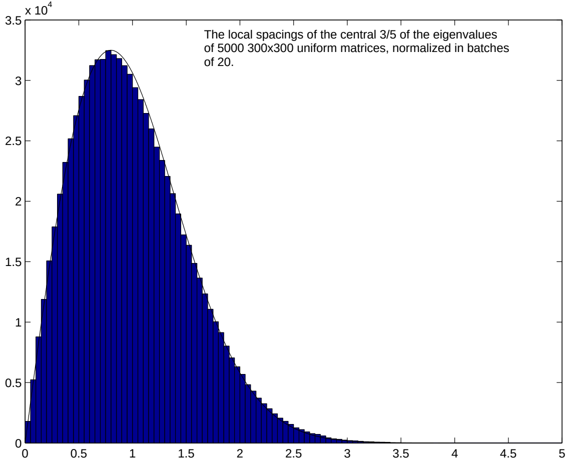
Only known if p is a Gaussian.

$$\mathrm{GOE}(x) \approx \frac{\pi}{2} x e^{-\pi x^2/4}.$$

Numerical Experiment: Uniform Distribution

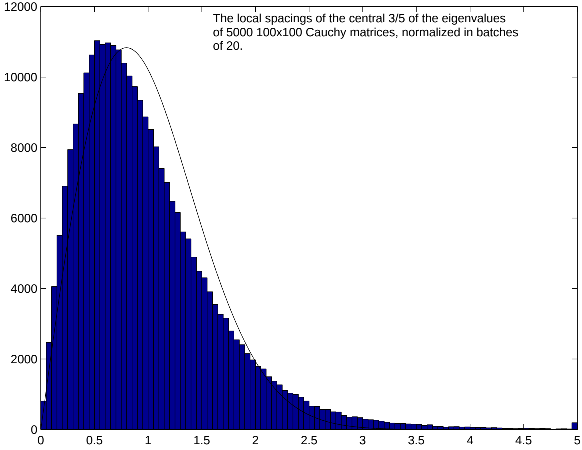
Let $p(x) = \frac{1}{2}$ for $|x| \le 1$.



5000: 300×300 uniform on $[-1, 1]$

Cauchy Distribution

Let $p(x) = \frac{1}{\pi(1+x^2)}$.



5000: 100×100 Cauchy

Cauchy Distribution

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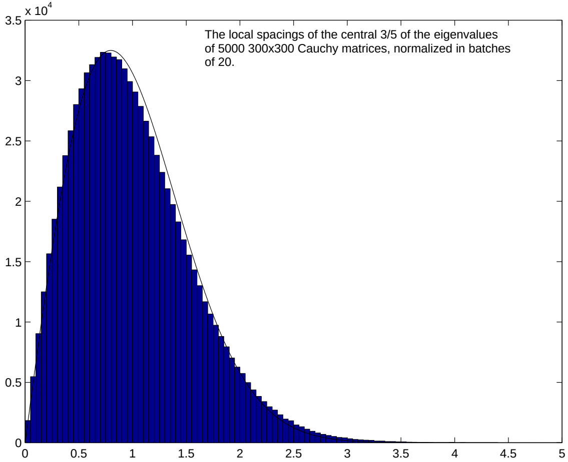
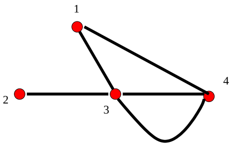


5000: 300×300 Cauchy

Random Graphs



Degree of a vertex $=$ number of edges leaving the vertex.
Adjacency matrix: a_{ij} $=$ number edges b/w Vertex i and Vertex j.

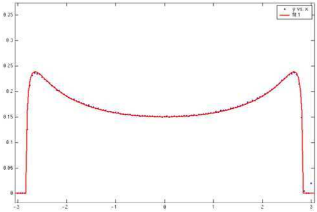
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.

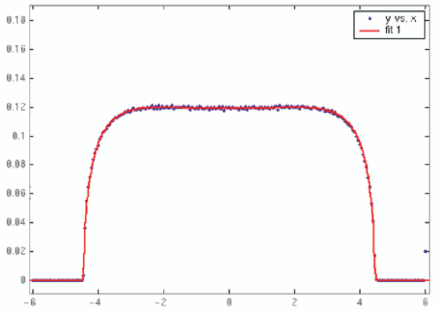
McKay's Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for d-regular graphs

$$f(x) \;=\; \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



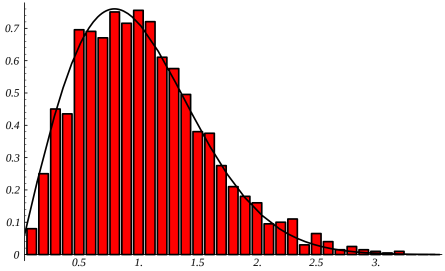
McKay's Law (Kesten Measure) with $d = 6$



Fat Thin: fat enough to average, thin enough to get something different than Semi-circle.

3-Regular, 2000 Vertices and GOE

Spacings between eigenvalues of 3-regular graphs and the GOE:



Introduction to L-Functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

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$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right] \left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2\right.$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function (cont)

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Properties of $\zeta(s)$ and Primes:

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Properties of $\zeta(s)$ and Primes:

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Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.
- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

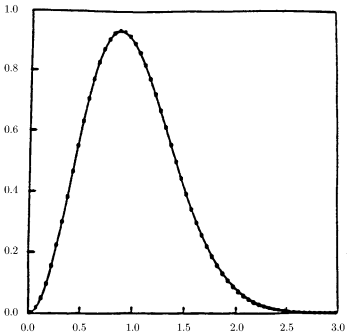
$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko)

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