From the Manhattan Project to Number Theory: How Nuclear Physics Helps Us Understand Primes

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Introduction
Fundamental Problem: Spacing Between Events

**General Formulation:** Studying system, observe values at $t_1, t_2, t_3, \ldots$. 
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- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
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- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Bus routes in Cuernavaca, Mexico.
- Scandinavian trees?
Eigenvalue, Eigenvector

Say \( \vec{v} \neq \vec{0} \) is an eigenvector of \( A \) with eigenvalue \( \lambda \) if \( A \vec{v} = \lambda \vec{v} \).
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\[ A \vec{v} = \lambda \vec{v}. \]

Example:
\[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
= 3 \begin{pmatrix}
1 \\
1
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 & 2 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
-1
\end{pmatrix}
= -1 \begin{pmatrix}
1 \\
-1
\end{pmatrix}.
\]
Background Material: Probability

**Probability Density**

A random variable $X$ has a probability density $p(x)$ if

1. $p(x) \geq 0$;
2. $\int_{-\infty}^{\infty} p(x) dx = 1$;
3. $\text{Prob}(X \in [a, b]) = \int_{a}^{b} p(x) dx$. 
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**Examples:**

1. **Exponential:** $p(x) = e^{-x/\lambda} / \lambda$ for $x \geq 0$;
2. **Normal:** $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / 2\sigma^2}$;
3. **Uniform:** $p(x) = \frac{1}{b-a}$ for $a \leq x \leq b$ and $0$ otherwise.
Background Material: Central Limit Theorem

\[ k^{th \text{ moment}}: \int_{-\infty}^{\infty} x^k p(x) \, dx. \]

**Central Limit Theorem**

Let \( X_1, X_2, \ldots \) be independent, identically distributed random variables with mean \( \mu \), standard deviation \( \sigma \) and finite higher moments. Then

\[
Y_n = \frac{X_1 + \cdots + X_N}{N} - \mu \quad \frac{1}{\sigma/\sqrt{N}}, \quad \lim_{N \to \infty} Y_N = N(0, 1).
\]

- Universality.
- Rate of convergence depends on higher moments.
Classical Random Matrix Theory
Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.
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Heavy nuclei (Uranium: 200+ protons / neutrons) worse!
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Get some info by shooting high-energy neutrons into nucleus, see what comes out.
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**Fundamental Equation:**

\[ H \psi_n = E_n \psi_n \]

- \( H \): matrix, entries depend on system
- \( E_n \): energy levels
- \( \psi_n \): energy eigenfunctions
Origins (continued)

- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).
Random Matrix Ensembles

\[
A = \begin{pmatrix}
    a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\
    a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN}
\end{pmatrix}
= A^T, \quad a_{ij} = a_{ji}
\]
Random Matrix Ensembles

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Fix \( p \), define

\[
\text{Prob}(A) = \prod_{1 \leq i \leq j \leq N} p(a_{ij}).
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Random Matrix Ensembles

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Fix $p$, define

$$\text{Prob}(A) = \prod_{1 \leq i \leq j \leq N} p(a_{ij}).$$

This means

$$\text{Prob}(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]) = \prod_{1 \leq i \leq j \leq N} \int_{x_{ij} = \alpha_{ij}}^{\beta_{ij}} p(x_{ij}) \, dx_{ij}.$$
Eigenvalue Trace Lemma

Want to understand the eigenvalues of $A$, but it is the matrix elements that are chosen randomly and independently.
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**Eigenvalue Trace Lemma**

Let $A$ be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) = \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$
Density of States
Wigner’s Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all $A$, as $N \to \infty$

\[
\mu_{A,N}(x) \to \begin{cases} 
\frac{2}{\pi} \sqrt{1 - x^2} & \text{if } |x| \leq 1 \\
0 & \text{otherwise.}
\end{cases}
\]
Numerical example: Gaussian density

500 Matrices: Gaussian $400 \times 400$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
Numerical example: Cauchy density $\rho(x) = \frac{1}{\pi(1 + x^2)}$
Numerical example: Cauchy density \( p(x) = \frac{1}{\pi(1 + x^2)} \)

Cauchy Distribution: \( p(x) = \frac{1}{\pi(1 + x^2)} \)

The eigenvalues of the Cauchy distribution are NOT semicircular.
Spacings between events
GOE Conjecture:

As $N \to \infty$, the probability density of the spacing between consecutive normalized eigenvalues approaches a limit independent of $p$. 
GOE Conjecture:

As \( N \to \infty \), the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of \( p \).

Only known if \( p \) is a Gaussian.

\[
\text{GOE}(x) \approx \frac{\pi}{2} xe^{-\pi x^2/4}.
\]
Numerical Experiment: Uniform Distribution

Let \( p(x) = \frac{1}{2} \) for \( |x| \leq 1 \).

5000: 300 \times 300 uniform on \([-1, 1]\)
Let \( p(x) = \frac{1}{\pi (1 + x^2)} \).

The local spacings of the central 3/5 of the eigenvalues of 5000 100x100 Cauchy matrices, normalized in batches of 20.
Cauchy Distribution

Let \( p(x) = \frac{1}{\pi(1+x^2)} \).

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Random Graphs

Degree of a vertex = number of edges leaving the vertex. Adjacency matrix: \( a_{ij} = \text{number of edges b/w Vertex } i \text{ and Vertex } j \).

\[
A = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 2 \\
1 & 0 & 2 & 0
\end{pmatrix}
\]

These are Real Symmetric Matrices.
McKay’s Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for $d$-regular graphs

$$f(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)} \sqrt{4(d-1)-x^2} & |x| \leq 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$
McKay’s Law (Kesten Measure) with $d = 6$

Fat Thin: fat enough to average, thin enough to get something different than Semi-circle.
3-Regular, 2000 Vertices and GOE

Spacings between eigenvalues of 3-regular graphs and the GOE:
Introduction to $L$-Functions
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]
Riemann Zeta Function

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Unique Factorization: \( n = p_1^{r_1} \cdots p_m^{r_m} \).
Riemann Zeta Function

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Unique Factorization: \( n = p_1^{r_1} \cdots p_m^{r_m}. \)

\[ \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots \right] \left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 \right] = \sum_{n} \frac{1}{n^s}. \]
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \#\{p : p \text{ is prime}, p \leq x\} \]

Properties of \( \zeta(s) \) and Primes:
Riemann Zeta Function (cont)

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Properties of \(\zeta(s)\) and Primes:

- \(\lim_{s \to 1^+} \zeta(s) = \infty, \pi(x) \to \infty.\)
\[ \zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

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Properties of \( \zeta(s) \) and Primes:

- \( \lim_{s \to 1^+} \zeta(s) = \infty, \pi(x) \to \infty. \)
- \( \zeta(2) = \frac{\pi^2}{6}, \pi(x) \to \infty. \)
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\rho \text{ prime}} \left(1 - \frac{1}{\rho^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]

Functional Equation:

\[ \xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1 - s). \]

Riemann Hypothesis (RH):

All non-trivial zeros have \( \text{Re}(s) = \frac{1}{2} \); can write zeros as \( \frac{1}{2} + i\gamma \).

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices \( \overline{A}^T = A \).
Zeros of $\zeta(s)$ vs GUE

70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20}$th zero (from Odlyzko)
Bibliography


