Generalizing Zeckendorf’s Theorem to Homogeneous Linear Recurrences

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Joint Work with Clay Mizgerd, Chenyang Sun, and Steven J. Miller
Zeckendorf’s Theorem

Theorem (Zeckendorf, 1972)

*Every positive integer can be uniquely written as the sum of non-consecutive Fibonacci numbers.*
Zeckendorf’s Theorem

Theorem (Zeckendorf, 1972)

Every positive integer can be uniquely written as the sum of non-consecutive Fibonacci numbers.

Example

$$118 = 89 + 21 + 8 = F_{10} + F_{7} + F_{5}.$$
Definition

A **Positive Linear Recurrence Sequence** (PLRS) is a sequence \( \{H_n\} \) satisfying

\[
H_n = c_1 H_{n-1} + c_2 H_{n-2} + \cdots + c_L H_{n-L}
\]

with non-negative integer coefficients \( c_i \) with \( c_1, c_L \geq 1 \) and specified initial values.
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Convention
To make it easier to write, we will define the coefficient tuple of \( H_n \) to be

\[
[c_1, c_2, \ldots, c_L]
\]
Definition

Let \( \{H_n\} \) be a PLRS and \( N \) a positive integer. Then,

\[
N = \sum_{i=1}^{m} a_i H_{m+1-i} = (a_1, \ldots, a_m)
\]

is a **legal decomposition** if \( a_1 > 0 \), the other \( a_i \geq 0 \), and one of the following conditions hold:

- We have \( m < L \) and \( a_i = c_i \) for \( 1 \leq i \leq m \).
- There exists \( s \in \{1, \ldots, L\} \) such that \( a_1 = c_1, a_2 = c_2, \ldots, a_s < c_s \), and \( \{b_n\}_{i=1}^{m-s} \)
  (with \( b_i = a_{s+i} \) either legal or empty.)
Example
Consider the PLRS with coefficient tuple

\[ [4, 3, 0, 3]. \]
Example

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\[ [4, 3, 0, 3]. \]

Examples of NOT legal decompositions:

- \( N = (5, 0, 0, 0, 0). \)
- \( N = (4, 3, 1, 0, 0). \)
- \( N = (4, 3, 0, 3, 0). \)
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Examples of legal decompositions:

- \( N = (4, 3, 0, 1, 0). \)
- \( N = (1, 4, 1, 0, 3). \)
Theorem (KKMW, 2010)

Let \( \{H_n\} \) be a PLRS. Then there exists a unique legal decomposition for every positive integer \( N \).
Motivating Question

**Question**

What if $c_1 = 0$?
An $s$-deep Zero Linear Recurrence Sequence (ZLRS) is a sequence $\{G_n\}$ satisfying

$$G_n = c_1 G_{n-1} + c_2 G_{n-2} + \ldots + c_{s+1} G_{n-s-1} + \ldots + c_L G_{n-L}$$

with non-negative integer coefficients $c_i$ with $c_{s+1}, c_L \geq 1,$ $c_i = 0$ for all $1 \leq i \leq s,$ and $L \geq s \geq 0.$
An **$s$-deep Zero Linear Recurrence Sequence (ZLRS)** is a sequence $\{G_n\}$ satisfying

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with non-negative integer coefficients $c_i$ with $c_{s+1}, c_L \geq 1$, $c_i = 0$ for all $1 \leq i \leq s$, and $L \geq s \geq 0$. Moreover, let $S$ be the set of indices of positive coefficients. We need $\gcd\{S\} = 1$. 
s-deep Zero Linear Recurrence Sequence

**Definition**

An *s*-deep *Zero Linear Recurrence Sequence* (ZLRS) is a sequence \( \{G_n\} \) satisfying

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G_n = c_1 G_{n-1} + c_2 G_{n-2} + \ldots + c_{s+1} G_{n-s-1} + \ldots + c_L G_{n-L}
\]

with non-negative integer coefficients \( c_i \) with \( c_{s+1}, c_L \geq 1 \), \( c_i = 0 \) for all \( 1 \leq i \leq s \), and \( L \geq s \geq 0 \). Moreover, let \( S \) be the set of indices of positive coefficients. We need \( \gcd\{S\} = 1 \).

**Remark**

The final condition is to prevent sequences like

\[
G_n = G_{n-2} + G_{n-4}.
\]
Definition

Let \( \{ G_n \} \) be an \( s \)-deep ZLRS and \( N \) a positive integer. Then

\[
N = \sum_{i=1}^{m} a_i G_{m+1-i}
\]

is a \textbf{legal decomposition} if \( a_i \geq 0 \) and one of the following conditions hold:
Definition

Let \( \{G_n\} \) be an \( s \)-deep ZLRS and \( N \) a positive integer. Then

\[
N = \sum_{i=1}^{m} a_i G_{m+1-i}
\]

is a **legal decomposition** if \( a_i \geq 0 \) and one of the following conditions hold:

1. We have \( a_1 = 1 \) and \( a_i = 0 \) for \( 2 \leq i \leq m \).
2. We have \( s < m < L \) and \( a_i = c_i \) for \( 1 \leq i \leq m \).
3. There exists \( t \in \{s + 1, \ldots, L\} \) such that

\[
\begin{align*}
  a_1 &= c_1, \quad a_2 = c_2, \quad \ldots, \quad a_{t-1} = c_{t-1}, \quad a_t < c_t, \\
  a_{t+1}, \ldots, a_{t+\ell} &= 0 \text{ for some } \ell \geq 0, \text{ and } \{b_i\}_{i=1}^{m-t-\ell} \text{ (with } b_i = a_{t+\ell+i} \text{) is legal.}
\end{align*}
\]
Example

Consider the 2-deep ZLRS with coefficient tuple

\[0, 0, 4, 3, 0, 3].\]
Example
Consider the 2-deep ZLRS with coefficient tuple

$$[0, 0, 4, 3, 0, 3].$$

Suppose $G_5 < N < G_6$. Examples of NOT legal decompositions:

- $N = [4, 2, 0, 0, 0]$.
- $N = [0, 0, 5, 0, 0]$. 
Consider the 2-deep ZLRS with coefficient tuple

$$[0, 0, 4, 3, 0, 3].$$

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Examples of legal decompositions:

- $N = [0, 0, 4, 2, 0]$.
- If instead $N = G_5$, this decomposition $[1, 0, 0, 0, 0]$ would be legal.
Theorem (MMMS, 2020)

Let \( \{G_n\} \) be an \( s \)-deep ZLRS. Then there exists a legal decomposition for every positive integer \( N \).

Theorem

Let \( \{G_n\} \) be an \( s \)-deep ZLRS with \( s \geq 1 \). Then, uniqueness of decomposition is lost for at least one positive integer \( N \).
Main Results

**Theorem (MMMS, 2020)**

Let \( \{G_n\} \) be an \( s \)-deep ZLRS. Then there exists a legal decomposition for every positive integer \( N \).

**Theorem (?)**

Let \( \{G_n\} \) be an \( s \)-deep ZLRS with \( s \geq 1 \). Then, uniqueness of decomposition is lost for at least one positive integer \( N \).
We construct two decompositions for a positive integer $N$. But first,

**Important Facts about Initial Conditions**

By construction, for every $s$-deep ZLRS $\{G_n\}$ with $s \geq 1$, we have

$$G_1 = 1 \text{ and } G_2 = 2.$$  

Also, if $c_{s+1} = 1$, then

$$G_i = i \text{ for all } 3 \leq i \leq L.$$
Proof Sketch: Case 1

- Case 1: Suppose $c_{s+1} \geq 2$. Note that $G_1 = 1$ and $G_2 = 2$. 
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- Consider $N = 2 + (c_L - 1) G_{s+3} + c_{L-1} G_{s+4} + \cdots + c_{s+1} G_{L+2}$.
Case 1: Suppose \( c_{s+1} \geq 2 \). Note that \( G_1 = 1 \) and \( G_2 = 2 \).

Consider \( N = 2 + (c_L - 1)G_{s+3} + c_{L-1}G_{s+4} + \cdots + c_{s+1}G_{L+2} \).

If \( G_{s+L+2} < N < G_{s+L+3} \), \( N \) has two legal decompositions. Namely,

\[
(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 1, 0)
\]

and

\[
(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 0, 2).
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Proof Sketch: Case 1

• Case 1: Suppose $c_{s+1} \geq 2$. Note that $G_1 = 1$ and $G_2 = 2$.

• Consider $N = 2 + (c_L - 1)G_{s+3} + c_{L-1}G_{s+4} + \cdots + c_{s+1}G_{L+2}$.

• If $G_{s+L+2} < N < G_{s+L+3}$, $N$ has two legal decompositions. Namely,

$$(0, \ldots, 0, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 1, 0)$$

and

$$(0, \ldots, 0, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 0, 2).$$

• Suffices to show that $G_{s+L+2} < N < G_{s+L+3}$, but not hard by the definition of $N$. 
Case 2: Suppose $c_{s+1} = 1$. Note that $G_i = i$ for all $1 \leq i \leq L$. 
Proof Sketch: Case 2

- Case 2: Suppose \( c_{s+1} = 1 \). Note that \( G_i = i \) for all \( 1 \leq i \leq L \).
- Let \( c_{s+j} \) be the second positive coefficient. Note that \( 1 < j < L - s \). So,

\[
1 < j + 1 < L - s + 1 \leq L.
\]
Proof Sketch: Case 2

- Case 2: Suppose $c_{s+1} = 1$. Note that $G_i = i$ for all $1 \leq i \leq L$.
- Let $c_{s+j}$ be the second positive coefficient. Note that $1 < j < L - s$. So,

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- Consider $N = (j + 1) + (c_L - 1) G_{j+2+s} + c_{L-1} G_{j+3+s} + \cdots + c_{s+1} G_{j+1+L}$.  

Proof Sketch: Case 2

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• Consider $N = (j + 1) + (c_L - 1) G_{j+2+s} + c_{L-1} G_{j+3+s} + \cdots + c_{s+1} G_{j+1+L}$.

• If $G_{j+1+L+s} < N < G_{j+2+L+s}$, $N$ has two legal decompositions. Namely,

$$(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 1, 0, \ldots, 0),$$

where the 1 is at position $j + 1$ and

$$(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 0, 1, 0, \ldots, 0, 1),$$

where the 1’s are at positions $j$ and 1.
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- What is the distribution of the number of decompositions?
- What about allowing negative coefficients in our recurrence relation?
Summary

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• But still a lot of unanswered questions!
• For example, Can we show something similar for infinitely many $N$?
• What is the distribution of the number of decompositions?
• What about allowing negative coefficients in our recurrence relation?

Thanks for listening!