Generalizing Zeckendorf's Theorem to Homogeneous Linear Recurrences

Thomas C. Martinez
Harvey Mudd College

tmartinez@hmc.edu

19th International Fibonacci Conference

Joint Work with Clay Mizgerd, Chenyang Sun, and Steven J. Miller

Zeckendorf's Theorem

Theorem (Zeckendorf, 1972)

Every positive integer can be **uniquely** *written as the sum of non-consecutive Fibonacci numbers.*

Zeckendorf's Theorem

Theorem (Zeckendorf, 1972)

Every positive integer can be **uniquely** *written as the sum of non-consecutive Fibonacci numbers.*

Example

$$118 = 89 + 21 + 8 = F_{10} + F_7 + F_5.$$

Positive Linear Recurrence Sequence

Definition

A **Positive Linear Recurrence Sequence** (PLRS) is a sequence $\{H_n\}$ satisfying

$$H_n = c_1 H_{n-1} + c_2 H_{n-2} + \cdots + c_L H_{n-L}$$

with non-negative integer coefficients c_i with $c_1, c_L \geq 1$ and specified initial values.

Positive Linear Recurrence Sequence

Definition

A **Positive Linear Recurrence Sequence** (PLRS) is a sequence $\{H_n\}$ satisfying

$$H_n = c_1 H_{n-1} + c_2 H_{n-2} + \cdots + c_L H_{n-L}$$

with non-negative integer coefficients c_i with $c_1, c_L \geq 1$ and specified initial values.

Convention

To make it easier to write, we will define the coefficient tuple of H_n to be

$$[c_1, c_2, \ldots, c_L]$$

PLRS Legal Decomposition

Definition

Let $\{H_n\}$ be a PLRS and N a positive integer. Then,

$$N = \sum_{i=1}^{m} a_i H_{m+1-i} = (a_1, \ldots, a_m)$$

is a **legal decomposition** if $a_1 > 0$, the other $a_i \geq 0$, and one of the following conditions hold:

- We have $m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.
- There exists $s \in \{1, \ldots, L\}$ such that $a_1 = c_1, a_2 = c_2, \ldots, a_s < c_s$, and $\{b_n\}_{i=1}^{m-s}$ (with $b_i = a_{s+i}$ either legal or empty.)

PLRS Legal Examples

Example

Consider the PLRS with coefficient tuple

$$[4, 3, 0, 3].$$

PLRS Legal Examples

Example

Consider the PLRS with coefficient tuple

$$[4, 3, 0, 3].$$

Examples of NOT legal decompositions:

- $N = (5, 0, 0, 0, 0)$.
- $N = (4, 3, 1, 0, 0)$.
- $N = (4, 3, 0, 3, 0)$.

PLRS Legal Examples

Example

Consider the PLRS with coefficient tuple

$$[4, 3, 0, 3].$$

Examples of NOT legal decompositions:

- $N = (5, 0, 0, 0, 0)$.
- $N = (4, 3, 1, 0, 0)$.
- $N = (4, 3, 0, 3, 0)$.

Examples of legal decompositions:

- $N = (4, 3, 0, 1, 0)$.
- $N = (1, 4, 1, 0, 3)$.

Generalized Zeckendorf's Theorem

Theorem (KKMW, 2010)

Let $\{H_n\}$ be a PLRS. Then there exists a **unique legal decomposition** for every positive integer N.

Motivating Question

Question

What if $c_1 = 0$?

Definition

An s-**deep Zero Linear Recurrence Sequence** (ZLRS) is a sequence $\{G_n\}$ satisfying

$$G_n = c_1 G_{n-1} + c_2 G_{n-2} + \ldots + c_{s+1} G_{n-s-1} + \ldots + c_L G_{n-L}$$

with non-negative integer coefficients c_i with $c_{s+1}, c_L \geq 1$, $c_i = 0$ for all $1 \leq i \leq s$, and $L \geq s \geq 0$.

s-deep Zero Linear Recurrence Sequence

Definition

An s-**deep Zero Linear Recurrence Sequence** (ZLRS) is a sequence $\{G_n\}$ satisfying

$$G_n = c_1 G_{n-1} + c_2 G_{n-2} + \ldots + c_{s+1} G_{n-s-1} + \ldots + c_L G_{n-L}$$

with non-negative integer coefficients c_i with $c_{s+1}, c_L \geq 1$, $c_i = 0$ for all $1 \leq i \leq s$, and $L \geq s \geq 0$. Moreover, let S be the set of indices of positive coefficients. We need $\gcd\{S\} = 1$.

s-deep Zero Linear Recurrence Sequence

Definition

An s-**deep Zero Linear Recurrence Sequence** (ZLRS) is a sequence $\{G_n\}$ satisfying

$$G_n = c_1 G_{n-1} + c_2 G_{n-2} + ... + c_{s+1} G_{n-s-1} + ... + c_L G_{n-L}$$

with non-negative integer coefficients c_i with $c_{s+1}, c_L \geq 1$, $c_i = 0$ for all $1 \leq i \leq s$, and $L \geq s \geq 0$. Moreover, let S be the set of indices of positive coefficients. We need $\gcd\{S\} = 1$.

Remark

The final condition is to prevent sequences like

$$G_n = G_{n-2} + G_{n-4}.$$

s-deep ZLRS Legal Decomposition

Definition

Let $\{G_n\}$ be an s-deep ZLRS and N a positive integer. Then

$$N = \sum_{i=1}^{m} a_i G_{m+1-i}$$

is a **legal decomposition** if $a_i \geq 0$ and one of the following conditions hold:

s-deep ZLRS Legal Decomposition

Definition

Let $\{G_n\}$ be an s-deep ZLRS and N a positive integer. Then

$$N = \sum_{i=1}^{m} a_i G_{m+1-i}$$

is a **legal decomposition** if $a_i \geq 0$ and one of the following conditions hold:

1. We have $a_1 = 1$ and $a_i = 0$ for $2 \leq i \leq m$.
2. We have $s < m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.
3. There exists $t \in \{s+1, \ldots, L\}$ such that

$$a_1 \ = \ c_1, \ \ a_2 \ = \ c_2, \ \ \ldots, \ \ a_{t-1} \ = \ c_{t-1}, \ \ a_t \ < \ c_t,$$

$a_{t+1}, \ldots, a_{t+\ell} = 0$ for some $\ell \geq 0$, and $\{b_i\}_{i=1}^{m-t-\ell}$ (with $b_i = a_{t+\ell+i}$) is legal.

Examples

Example

Consider the 2-deep ZLRS with coefficient tuple

$$[0, 0, 4, 3, 0, 3].$$

Examples

Example

Consider the 2-deep ZLRS with coefficient tuple

$$[0, 0, 4, 3, 0, 3].$$

Suppose $G_5 < N < G_6$. Examples of NOT legal decompositions:

- $N = [4, 2, 0, 0, 0]$.
- $N = [0, 0, 5, 0, 0]$.

Examples

Example

Consider the 2-deep ZLRS with coefficient tuple

$$[0, 0, 4, 3, 0, 3].$$

Suppose $G_5 < N < G_6$. Examples of NOT legal decompositions:

- $N = [4, 2, 0, 0, 0]$.
- $N = [0, 0, 5, 0, 0]$.

Examples of legal decompositions:

- $N = [0, 0, 4, 2, 0]$.
- If instead $N = G_5$, this decomposition $[1, 0, 0, 0, 0]$ would be legal.

Main Results

Theorem (MMMS, 2020)

Let $\{G_n\}$ be an s-deep ZLRS. Then there exists a legal decomposition for every positive integer N.

Theorem

Let $\{G_n\}$ be an s-deep ZLRS with $s \geq 1$. Then, uniqueness of decomposition is lost for at least one positive integer N.

Main Results

Theorem (MMMS, 2020)

Let $\{G_n\}$ be an s-deep ZLRS. Then there exists a legal decomposition for every positive integer N.

Theorem (?)

Let $\{G_n\}$ be an s-deep ZLRS with $s \geq 1$. Then, uniqueness of decomposition is lost for at least one positive integer N.

Initial Conditions

We construct two decompositions for a positive integer N. But first,

Important Facts about Initial Conditions

By construction, for every s-deep ZLRS $\{G_n\}$ with $s \geq 1$, we have

$$G_1 = 1 \text{ and } G_2 = 2.$$

Also, if $c_{s+1} = 1$, then

$$G_i = i \text{ for all } 3 \leq i \leq L.$$

Proof Sketch: Case 1

- Case 1: Suppose $c_{s+1} \geq 2$. Note that $G_1 = 1$ and $G_2 = 2$.

Proof Sketch: Case 1

- Case 1: Suppose $c_{s+1} \geq 2$. Note that $G_1 = 1$ and $G_2 = 2$.
- Consider $N = 2 + (c_L - 1)\, G_{s+3} + c_{L-1}\, G_{s+4} + \cdots + c_{s+1}\, G_{L+2}$.

Proof Sketch: Case 1

- Case 1: Suppose $c_{s+1} \geq 2$. Note that $G_1 = 1$ and $G_2 = 2$.
- Consider $N = 2 + (c_L - 1)\, G_{s+3} + c_{L-1}\, G_{s+4} + \cdots + c_{s+1}\, G_{L+2}$.
- If $G_{s+L+2} < N < G_{s+L+3}$, N has two legal decompositions. Namely,

$$(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 1, 0)$$

and

$$(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 0, 2).$$

Proof Sketch: Case 1

- Case 1: Suppose $c_{s+1} \geq 2$. Note that $G_1 = 1$ and $G_2 = 2$.
- Consider $N = 2 + (c_L - 1)\, G_{s+3} + c_{L-1}\, G_{s+4} + \cdots + c_{s+1}\, G_{L+2}$.
- If $G_{s+L+2} < N < G_{s+L+3}$, N has two legal decompositions. Namely,

$$(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 1, 0)$$

  and

$$(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 0, 2).$$

- Suffices to show that $G_{s+L+2} < N < G_{s+L+3}$, but not hard by the definition of N.

Proof Sketch: Case 2

- Case 2: Suppose $c_{s+1} = 1$. Note that $G_i = i$ for all $1 \leq i \leq L$.

Proof Sketch: Case 2

- Case 2: Suppose $c_{s+1} = 1$. Note that $G_i = i$ for all $1 \leq i \leq L$.
- Let c_{s+j} be the second positive coefficient. Note that $1 < j < L - s$. So,

$$1 < j + 1 < L - s + 1 \leq L.$$

Proof Sketch: Case 2

- Case 2: Suppose $c_{s+1} = 1$. Note that $G_i = i$ for all $1 \leq i \leq L$.
- Let c_{s+j} be the second positive coefficient. Note that $1 < j < L - s$. So,

$$1 < j + 1 < L - s + 1 \leq L.$$

- Consider $N = (j + 1) + (c_L - 1)\, G_{j+2+s} + c_{L-1}\, G_{j+3+s} + \cdots + c_{s+1}\, G_{j+1+L}$.

Proof Sketch: Case 2

- Case 2: Suppose $c_{s+1} = 1$. Note that $G_i = i$ for all $1 \leq i \leq L$.
- Let c_{s+j} be the second positive coefficient. Note that $1 < j < L - s$. So,

$$1 < j + 1 < L - s + 1 \leq L.$$

- Consider $N = (j+1) + (c_L - 1)\, G_{j+2+s} + c_{L-1}\, G_{j+3+s} + \cdots + c_{s+1}\, G_{j+1+L}$.
- If $G_{j+1+L+S} < N < G_{j+2+L+s}$, N has two legal decompositions. Namely,

$$(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 1, 0, \ldots, 0),$$

  where the 1 is at position $j+1$ and

$$(0, \ldots, 0, c_{s+1}, c_{s+2}, \ldots, c_{L-1}, c_L - 1, 0, \ldots, 0, 0, 1, 0, \ldots, 0, 1),$$

  where the 1's are at positions j and 1.

Summary

- So, we have proved what we desired...
- But still a lot of unanswered questions!

Summary

- So, we have proved what we desired...
- But still a lot of unanswered questions!
- For example, Can we show something similar for infinitely many N?
- What is the distribution of the number of decompositions?
- What about allowing negative coefficients in our recurrence relation?

Summary

- So, we have proved what we desired...
- But still a lot of unanswered questions!
- For example, Can we show something similar for infinitely many N?
- What is the distribution of the number of decompositions?
- What about allowing negative coefficients in our recurrence relation?

Thanks for listening!