

Mathematics of Card Tricks

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Magic!



Introduction

Have someone choose a card at random, look at it, and then put it at the top of the pile. Ask two other people to each give you a number; make the smaller number b and the larger number a (if they're the same, it doesn't matter). Deal out $a + b$ many cards. Now repeat three times the following:

1. Deal out a cards.
2. Put the remaining b cards on top of the just dealt a cards.

Deal the top card, face down to the person who chose the initial card, and ask them if this is theirs. **Ta-da!**

Card Trick

Main Theorem. Given deck of playing cards, choose $a, b \in \mathbb{N}$ such that $a + b < 52$ and $a \geq b$. Assume that you have the information of the card on top of the deck and deal out $a + b$ cards. Then after applying the following steps 3 times serially,

1. Deal out a cards.
2. Put the remaining b cards on top of the just dealt a cards.

The card which you started with will appear on top again.

Card Trick

Proof. The identity of each card is of no interest to our investigation, so we label them all with indices in order, going from 1 on top to $a + b$ at the bottom. To start, we will encode the effect of the move on a single index into a piecewise permutation:

$$\sigma(i) = \begin{cases} i - a & \text{if } i > a \\ a + b + 1 - i & \text{if } i \leq a \end{cases}$$

Also define the index reversing (*dealing out*) permutation for $1 \leq i \leq a + b$:

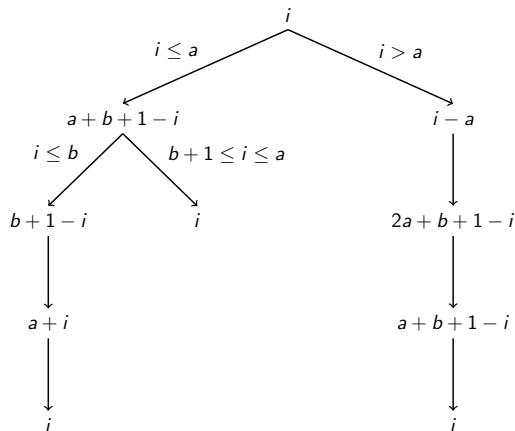
$$\tau(i) : i \mapsto a + b + 1 - i$$

Then we can compute $\sigma^3(\tau(1))$ to show that the top card is returned after 3 moves:

$$\sigma^3(\tau(1)) = \sigma^3(a + b) = \sigma^2(b) = \sigma(a + 1) = 1. \quad \square$$

Card Trick

We can also find the orbits for given cards, using our piecewise permutation:



Card Trick

Writing in the cycle notation,

$$\sigma(i) = \begin{cases} (i, a + b + 1 - i, b + 1 - i, a + i) & \text{if } 1 \leq i \leq b \\ (i, a + b + 1 - i) & \text{if } b + 1 \leq i \leq a \\ (i, i - a, 2a + b + 1 - i, a + b + 1 - i) & \text{if } a + 1 \leq i \leq a + b \end{cases}$$

Least common multiple of every cycle = 4

Card Trick

Corollary. Let $a, b \in \mathbb{N}$ and $a \geq b$, $a + b < 52$. Then for $k \geq 0$ with $4 \mid k$ the first a cards are in their initial order and the remaining b cards are in reverse order.

Proof. If $4 \mid k$, $\sigma^{3+k}(\tau(i)) = \sigma^3(\tau(i))$

$$\sigma^3(\tau(i)) = \begin{cases} i, & 1 \leq i \leq a, \\ 2a + b + 1 - i, & a + 1 \leq i \leq a + b. \end{cases}$$



Card Trick

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Observe that if $i \leq a$, we do a reflection. Else, we subtract a until $i \leq a$.

Card Trick

What about the $b > a$ case?

$$\sigma(i) = \begin{cases} i - a & \text{if } i > a \\ a + b + 1 - i & \text{if } i \leq a \end{cases}$$

Observe that if $i \leq a$, we do a reflection. Else, we subtract a until $i \leq a$. Say $b = qa + r$, and assume $i \in [1, a]$.

$$i \mapsto a + b + 1 - i \xrightarrow{q \text{ times } -a} a + b + 1 - i - qa = a + r + 1 - i$$

So, $\sigma^{(q+1)}(i) = a + r + 1 - i$. Now we have the following cases.

Card Trick

Case 1. $1 \leq i \leq r$

Then $\sigma^{(q+1)}(i) = a + r + 1 - i > a$. Thus, we need to apply our move once again to make sure it is less or equal to a ,

$$\sigma^{(q+2)}(i) = r + 1 - i \implies \sigma^{2(q+2)}(i) = i$$

Card Trick

Case 2. $r + 1 \leq i \leq a$

Then $\sigma^{(q+1)}(i) = a + r + 1 - i \leq a$. Thus,

$$\sigma^{(q+1)}(i) = a + r + 1 - i \implies \sigma^{2(q+1)}(i) = i$$

New Card Trick

Theorem. (Polymath Jr. 2026) Given deck of playing cards, choose $a, b \in \mathbb{N}$ such that $a + b < 52$ and $b > a$. Assume that you have the information of the card on top of the deck and take $a + b$ cards starting from top. Then after applying the following steps $2(q + 1)(q + 2)$ times serially, where $q = \lfloor \frac{b}{a} \rfloor$

1. Deal out a cards.
2. Put the remaining b cards on top of the just dealt a cards.

The deck will return to it's initial ordering.

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1. Deal out a cards.
2. Put the remaining b cards on top of the just dealt a cards.

The deck will return to it's initial ordering. **WIP:** $2(q + 1)(q + 2)$ is not always the minimal number to return the top card to top of the deck again!

Card Trick



How else can we investigate this trick?

Card Trick

A permutation matrix can be written in this way:

$$M = \begin{pmatrix} 0_{b \times a} & I_{b \times b} \\ J_{a \times a} & 0_{a \times b} \end{pmatrix}_{(a+b) \times (a+b)}$$

Card Trick

More explicitly,

$$M = \left(\begin{array}{c|cccc} & \overbrace{0 & 0 & \dots & 0}^a & \overbrace{1 & 0 & \dots & 0}^b \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ \hline 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 \end{array} \right) \left. \begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array} \right\} \begin{array}{l} b \\ \\ \\ \\ a \end{array}$$

Card Trick

Deal out a cards, put b cards on top of dealt a cards, deal out c cards on top of it, then deal out d cards again... up to your imagination!

Next Steps

1. What if we have more variables $c, d, e \dots$?
2. Given our matrix M , group of M^n 's will be a subgroup of S_{a+b} .
3. What if we choose a and b as Fibonacci numbers; can we catch a pattern?

Further Readings



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Persi Diaconis, Ronald L. Graham, and William M. Kantor,
The mathematics of perfect shuffles, *Advances in Applied Mathematics* **4** (1983), no. 2, 175–196.

Thank you!
Any questions?