The Quadratic Formula: From Baseball to Gambling

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https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

JOURNAL OF NUMBER THEORY

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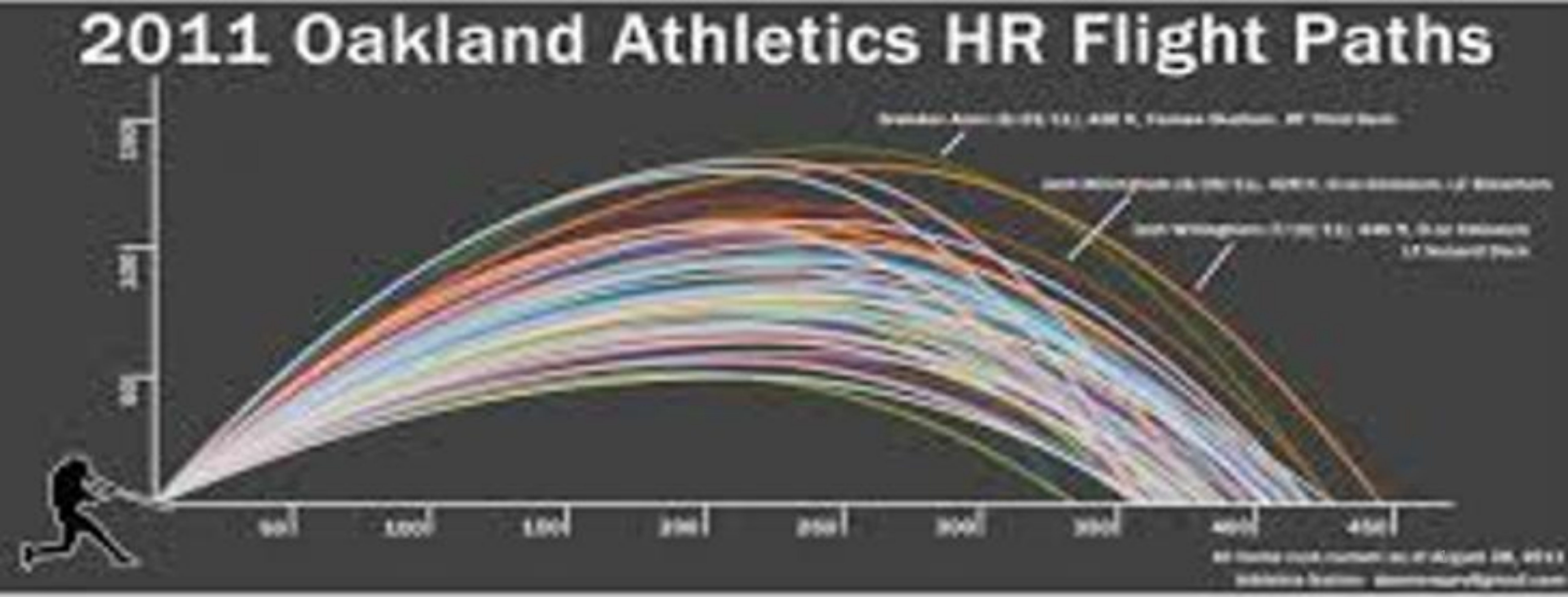


Goals

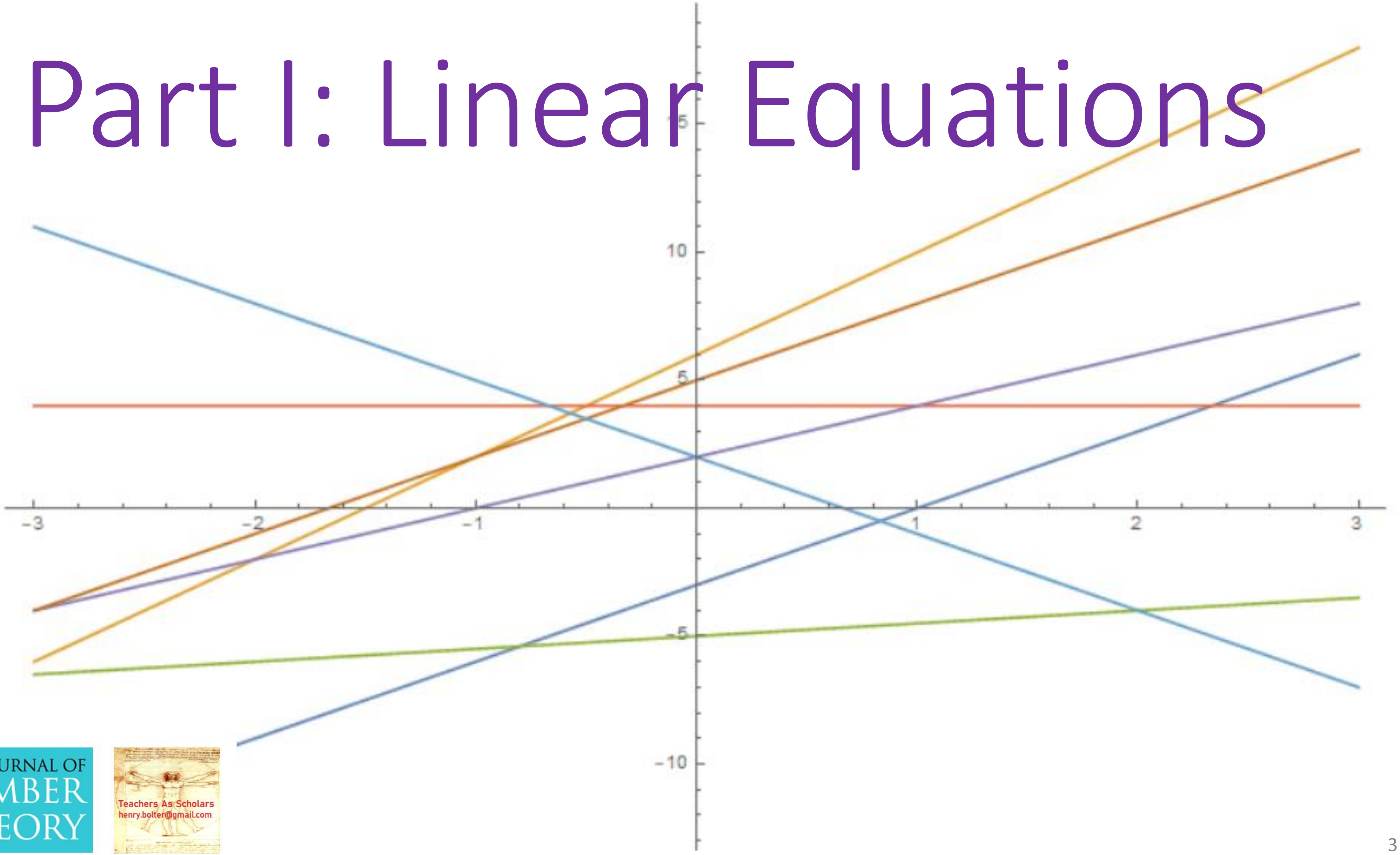
Learn how to solve polynomial equations and see applications of polynomials and ways to find their roots.

These range from

- flight paths of baseballs, cannonballs, and planes with no engines,
- gambling (roulette)
- Cryptography.

As always, we start simple and build up.

Part I: Linear Equations



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Linear Equations

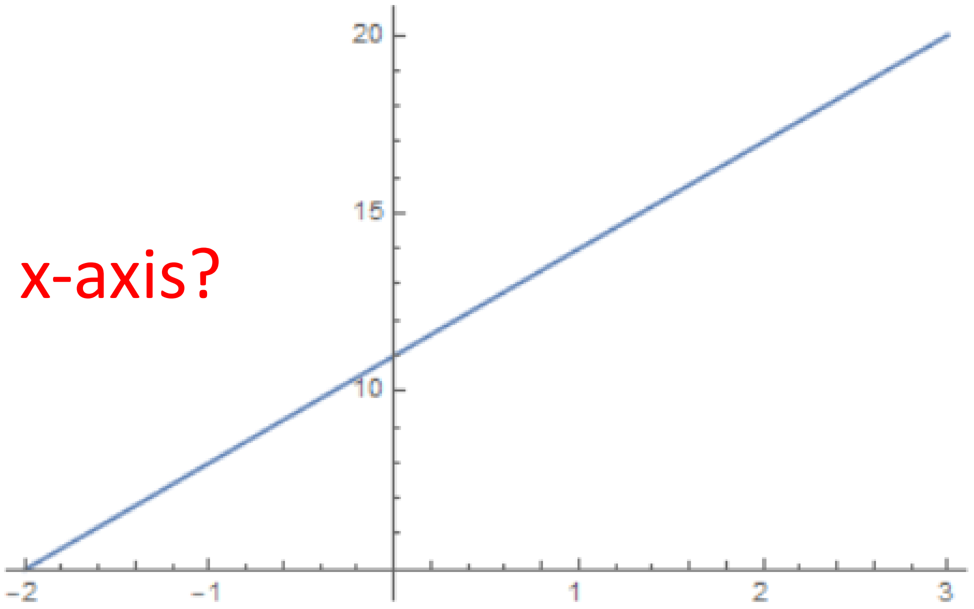
We start with linear equations: $f(x) = a x + b$, often written $y = a x + b$.

For example, if we have $y = 3 x + 11$, when $x = 2$ then $y = 3 * 2 + 11 = 17$.

The plots of these are straight lines.

Will every straight line cross the x-axis?

Pause and think about this.



Linear Equations

Not every line hits the x-axis: Consider y = 2 (or anything!)

However, if y = a x + b and a is not zero, it will cross the x-axis.

Where will y = a x + b hit the x-axis? In other words, what value of x yields y equals zero? Try to do y = 3 x + 11 first, then do the more general y = a x + b.

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Linear Equations

Not every line hits the x-axis: Consider $y = 2$ (or anything!)

However, if $y = a x + b$ and a is not zero, it will cross the x-axis.

Where will $y = a x + b$ hit the x-axis? In other words, what value of x yields y equals zero? Try to do $y = 3 x + 11$ first, then do the more general $y = a x + b$.

For $y = 3 x + 11$: Want $3 x + 11 = 0$ so $3 x = -11$ or $x = - 11/3$.

For $y = a x + b$: Want $a x + b = 0$ so $a x = -b$ so $x = -b/a$ (see now why need a to be non-zero).

Linear Equations

Imagine we know where a linear equation hits the x-axis; does that uniquely determine the line, or could there be multiple lines that hit the x-axis in the same spot?

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Linear Equations

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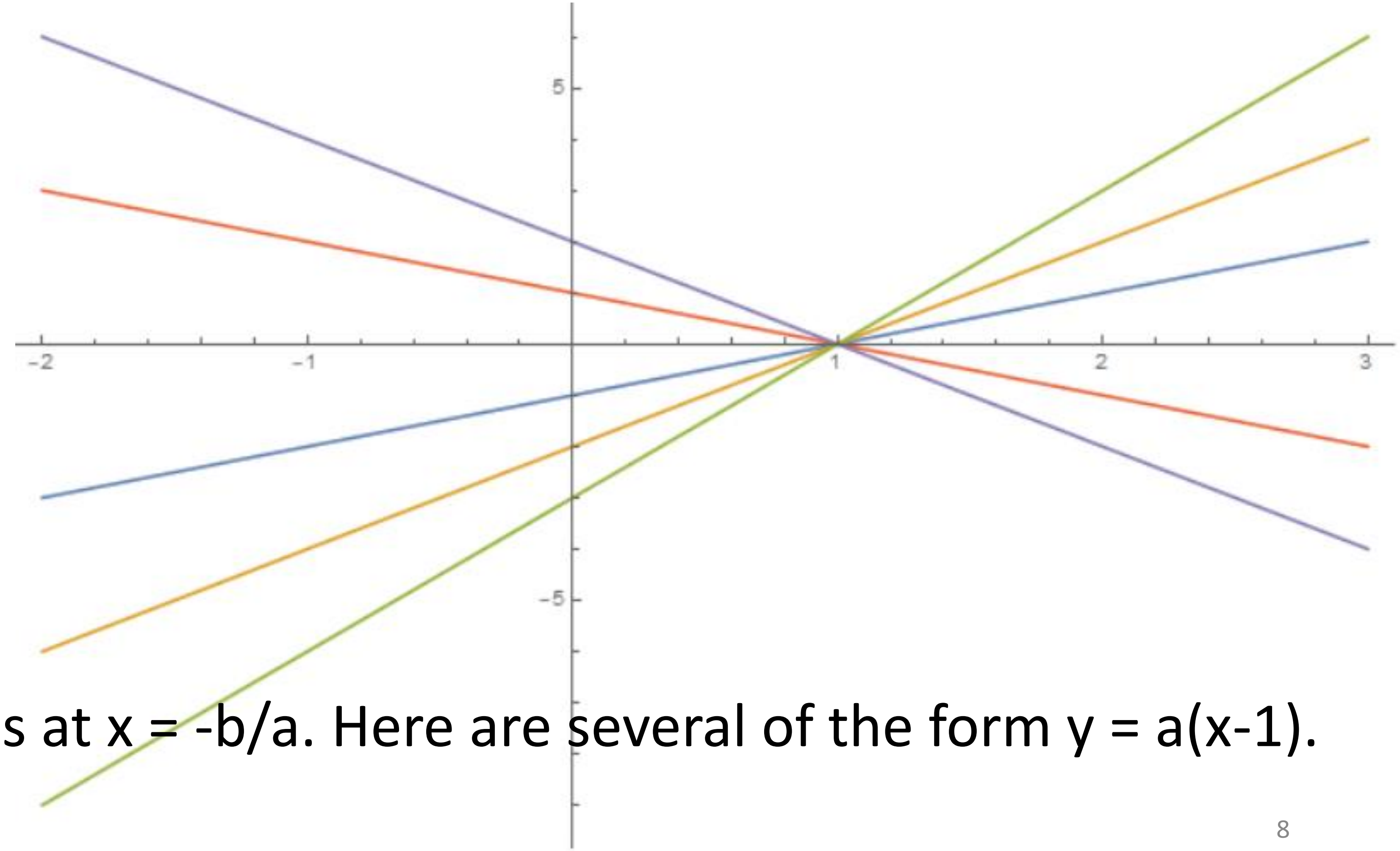


We can factor!

$y = a x + b$

is the same as

$y = a(x + b/a)$, so the root is at $x = -b/a$. Here are several of the form $y = a(x-1)$.

Linear Equations

Two points determine a line.

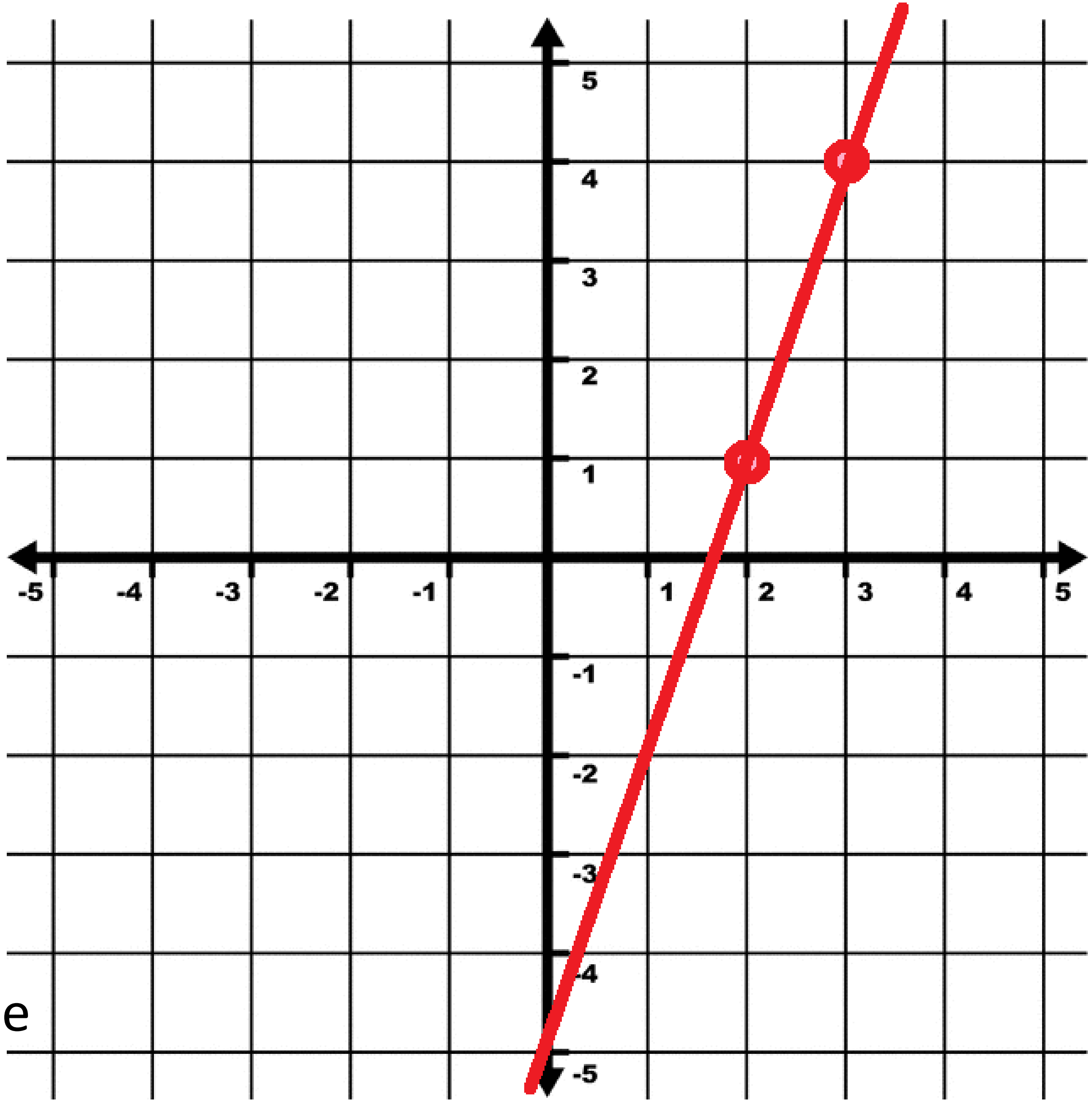
Here have (2,1) and (3,4)

Slope m = (change in y) / (change in x)

So m = (4-1) / (3-2) = 3.

If (x,y) on the line must give the same slope, so (y-1) / (x-2) = 3. Thus
y-1 = 3(x-2) or y-1 = 3x-6 or y = 3x-5.

The y-intercept is -5; this is where the line crosses the y-axis.



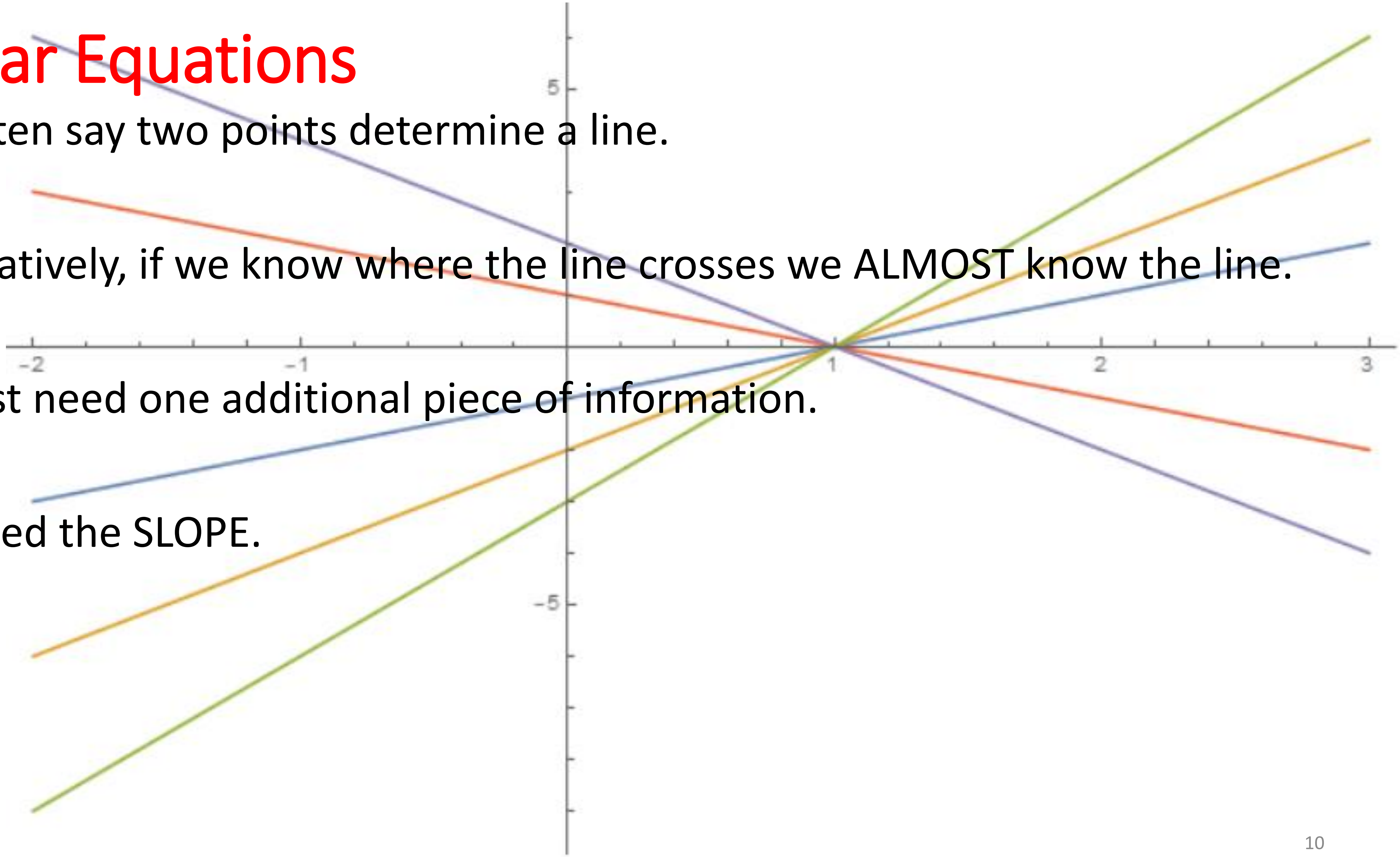
Linear Equations

We often say two points determine a line.

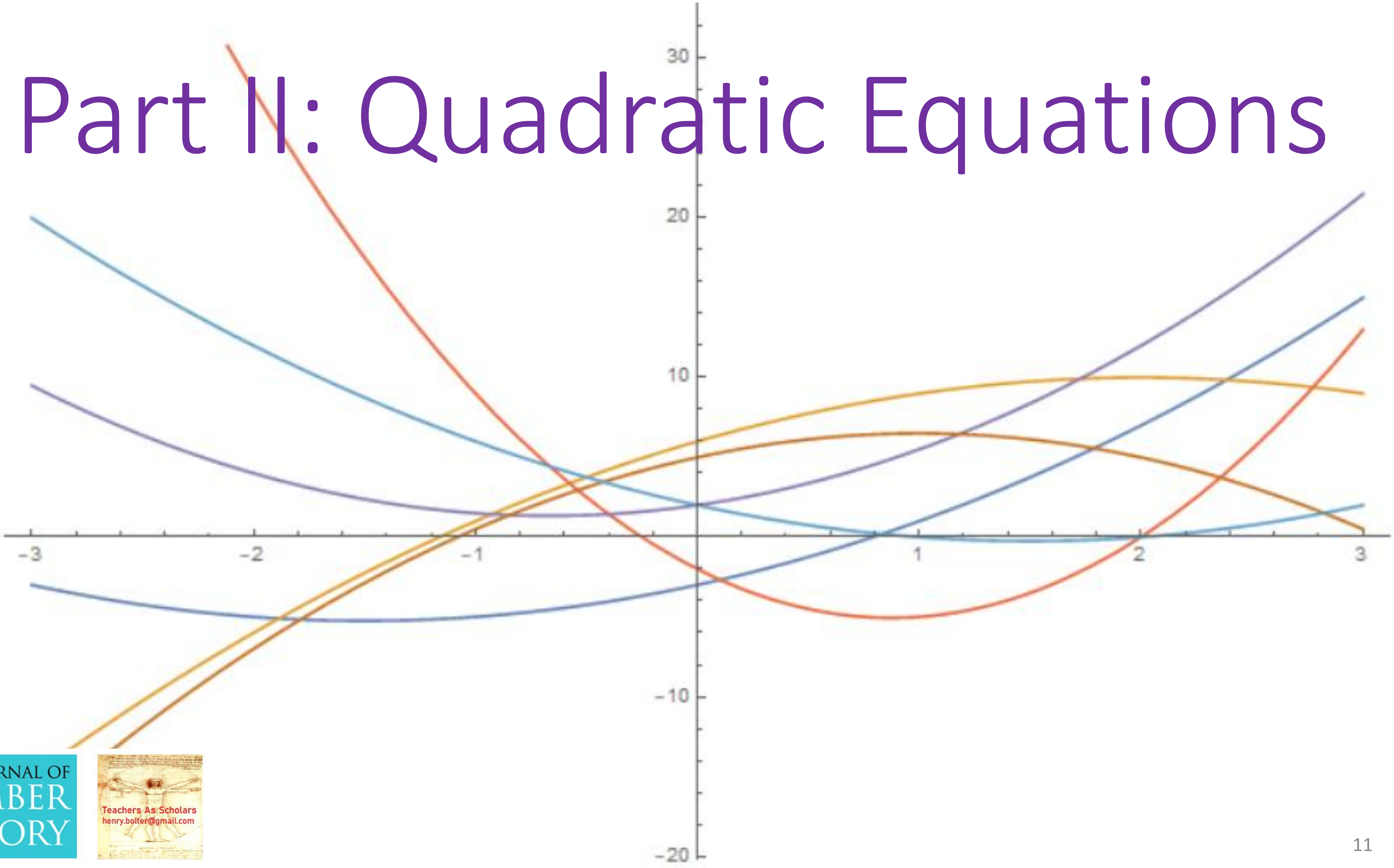
Alternatively, if we know where the line crosses we ALMOST know the line.

We just need one additional piece of information.

We need the SLOPE.



Part II: Quadratic Equations



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Quadratic Equations

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

If $y = 3x^2 – 5x + 7$, then when $x=2$, y equals $3*2^2 – 5*2 + 7 = 12 – 10 + 7 = 9$.

We say this is a **polynomial** of degree 2 as 2 is the highest power of x; a **cubic** would have its highest power 3, and so on.

How many points are needed to uniquely determine a quadratic?

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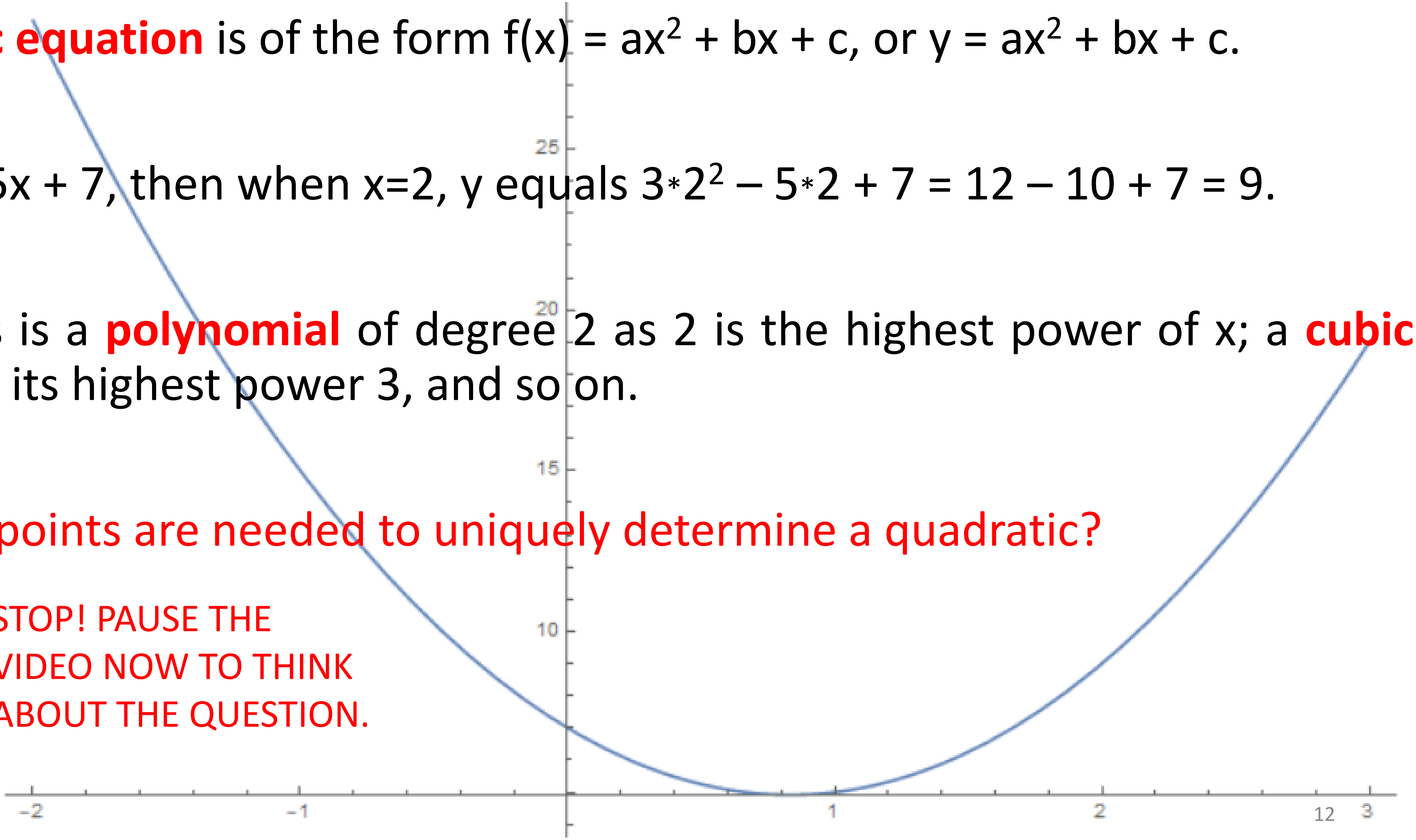

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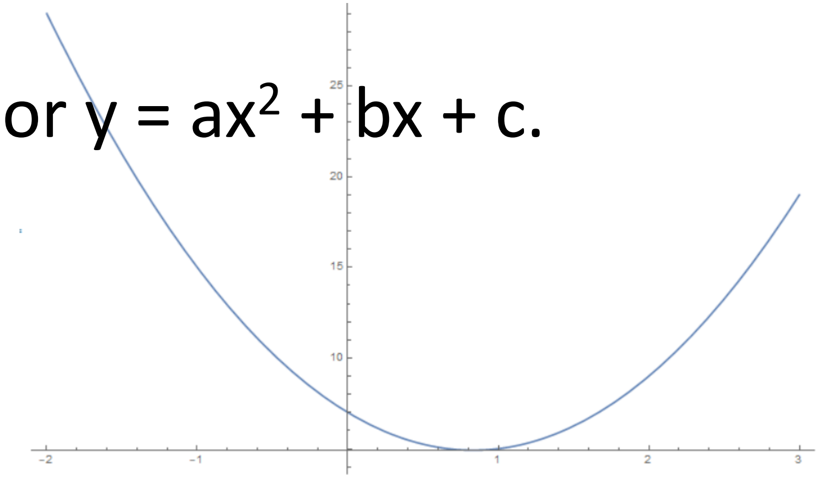

A line needed two points, a quadratic needs three.

Note there are three coefficients: a, b, c; by having three points we can figure out those three values. How would you do that?

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Quadratic Equations

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

Imagine know (0, 7), (1, 5) and (2,9) are on the quadratic. Then

- $7 = a * 0^2 + b * 0 + c$.
- $5 = a * 1^2 + b * 1 + c$.
- $9 = a * 2^2 + b * 2 + c$.

Quadratic Equations

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Imagine know (0, 7), (1, 5) and (2,9) are on the quadratic. Then

- $7 = a * 0^2 + b * 0 + c$, so $7 = c$.
- $5 = a * 1^2 + b * 1 + c$, so $5 = a + b + c$, but since $c=7$ we get $-2 = a+b$.
- $9 = a * 2^2 + b * 2 + c$, so $9 = 4a + 2b + c$, but since $c=7$ we get $2 = 4a+2b$.

Quadratic Equations

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- $5 = a * 1^2 + b * 1 + c$, so $5 = a + b + c$, but since $c=7$ we get $-2 = a+b$.
- $9 = a * 2^2 + b * 2 + c$, so $9 = 4a + 2b + c$, but since $c=7$ we get $2 = 4a+2b$.

We have two equations in two unknowns: we can solve! If we subtract two copies of the second from the first, the b-terms vanish:

$2 = 4a + 2b$

$-4 = 2a + 2b$

So $6 = 2a$ or $a = 3$; since $-2 = a + b$ we find $b = -2-a = -5$.

We could also have used $-2 = a+b$ to say $b = -2-a$, and substitute that into $2 = 4a+2b$, which would give $2 = 4a + 2(-2-a)$, or $2 = 4a – 4 – 2a$. Thus $6 = 2a$ and again $a = 3$.

Plotting Quadratic Equations

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

- What does a do?
- What's the difference between x^2, $2x^2$, $4x^2$ and $–x^2$, $-2x^2$ and $-4x^2$?

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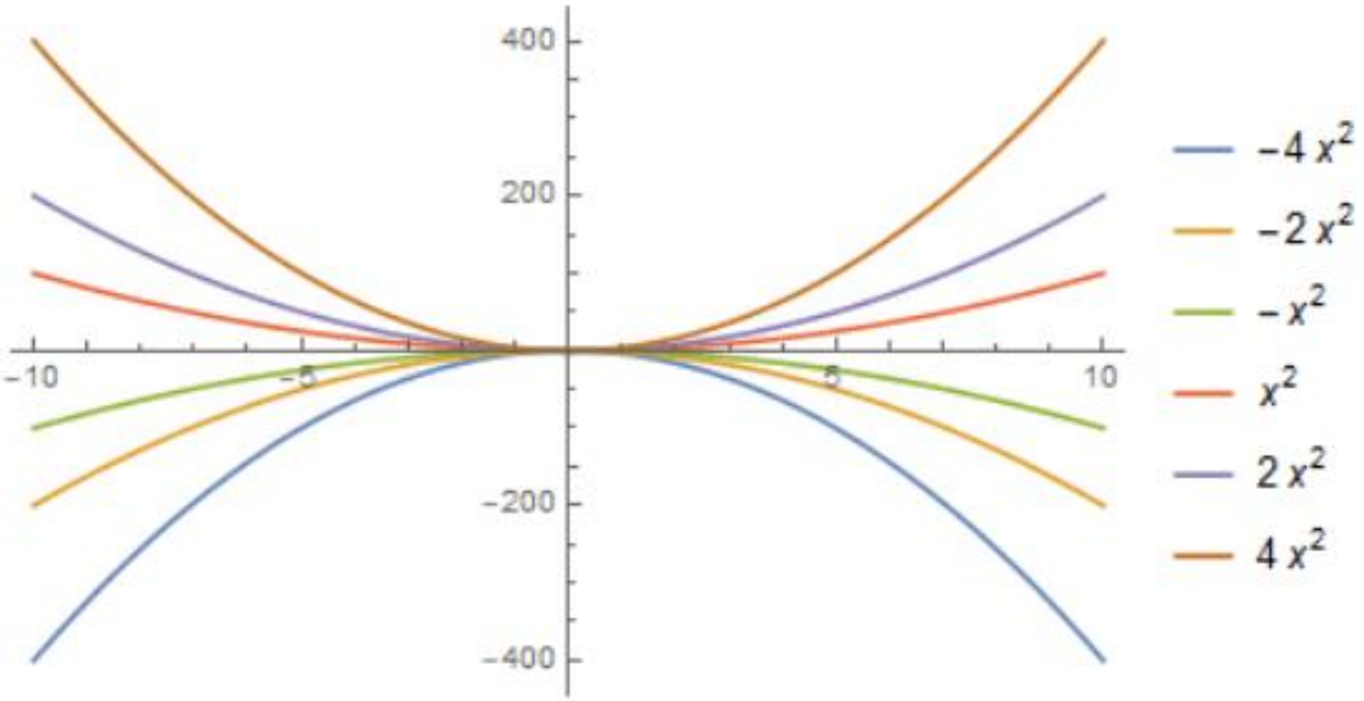
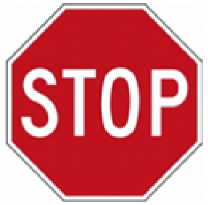
```
Plot[{-4 x^2, -2 x^2, -x^2, x^2, 2 x^2, 4 x^2}, {x, -10, 10}, PlotLegends → "Expressions"]
```



Plotting Quadratic Equations

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

- What does c do? What's the difference between $2x^2$ and $2x^2 + 1$ and $2x^2 - 1$?



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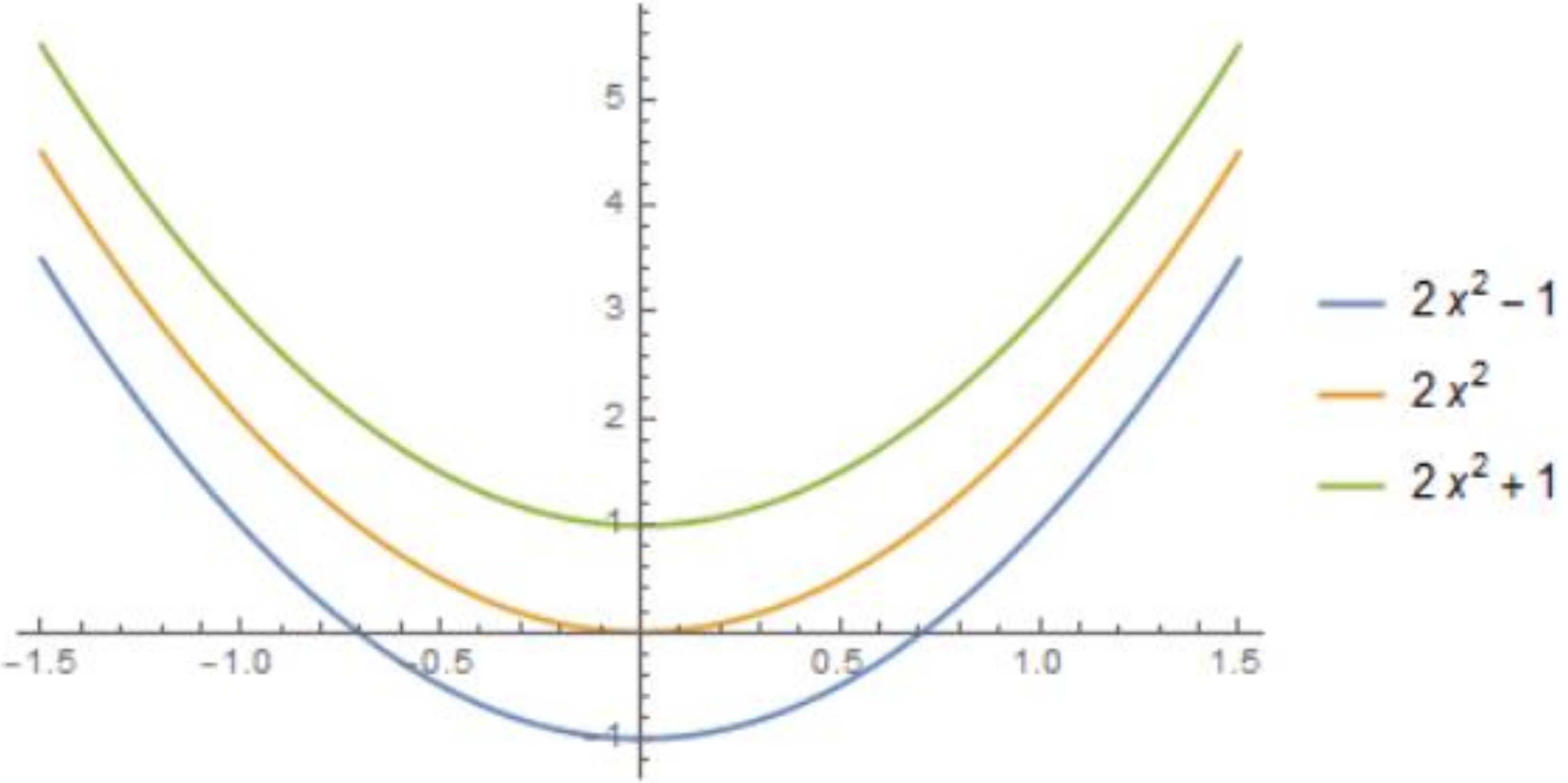
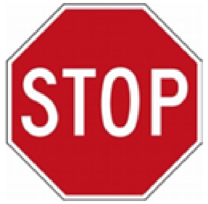
```
Plot[{2 x^2 - 1, 2 x^2, 2 x^2 + 1}, {x, -1.5, 1.5}, PlotLegends → "Expressions"]
```



Plotting Quadratic Equations

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

- What does b do?
- This is a bit harder to see.
- Later will see can rewrite as $a(x – h)^2 + d$ for some d. Now what does h do?
- Compare $(x-2)^2 + 1$, $(x-0)^2 + 1$, $(x+2)^2 + 1$.....



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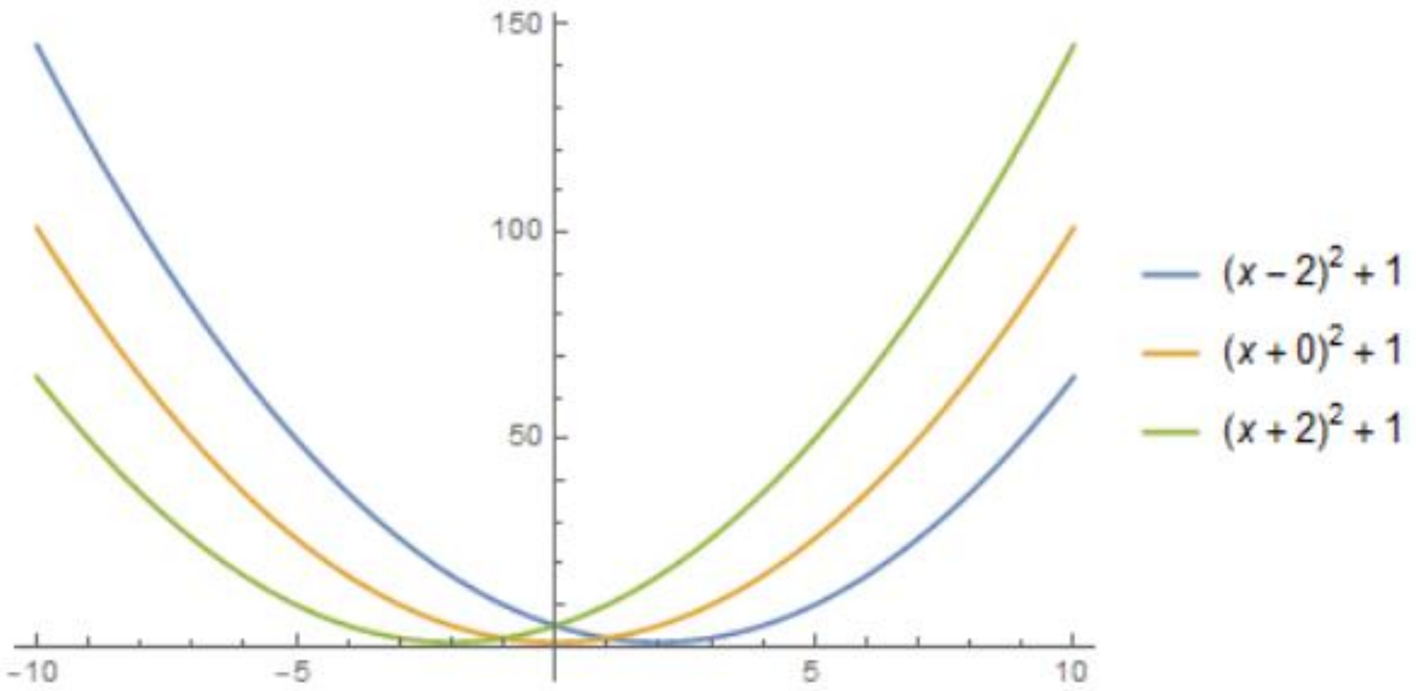
Plotting Quadratic Equations

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

- What does b do? Harder to see.
- Rewrite as $a(x – h)^2 + d$ for some d. Compare $(x-2)^2 + 1$, $(x-0)^2 + 1$, $(x+2)^2 + 1$..... (this is called vertex form)

```
Plot[{(x - 2)^2 + 1, (x - 0)^2 + 1, (x + 2)^2 + 1}, {x, -10, 10}, PlotLegends → "Expressions"]
```



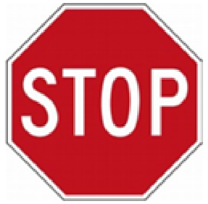
Roots of Quadratic Equations

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

How do we find where a quadratic equation is zero, in other words, where it hits the x-axis?

Thus we want to solve $ax^2 + bx + c = 0$.

Before doing the general case, let’s try a simpler quadratic. What’s the simplest quadratic you can think of? What would its roots be? How many might it have?



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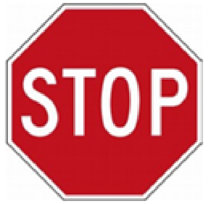
Simplest is $y = x^2$, so solving $y = 0$ means solving $x^2 = 0$; the only solution is $x=0$.

We should really view this as a **double root**; a polynomial of degree n has n roots (not necessarily distinct); thus a linear equation has one root, a quadratic has two roots, and so on. This is the Fundamental Theorem of Algebra, and is beyond the scope of this lecture.

Building on the success of studying $y = x^2$, what is the next simplest?



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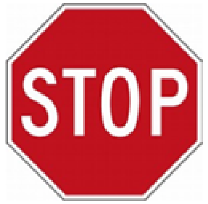
Simplest is $y = x^2$, next would be either $y = ax^2$ or $y = x^2 + c$.

If want $ax^2 = 0$ we see again it's just a double root at $x = 0$. The other is more interesting. If $y = x^2 + c$, then when does $y = 0$?



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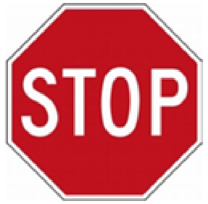
So $x^2 + c = 0$, thus $x^2 = -c$ so $x = \sqrt{-c}\ or\ -\sqrt{-c}$.

For what values of c will this have "interesting" solutions?



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If want $ax^2 = 0$ we see again it's just a double root at $x = 0$. The other is more interesting. If $y = x^2 + c$, then when does $y = 0$?

So $x^2 + c = 0$, thus $x^2 = -c$ so $x = \sqrt{-c}\ \ or\ \ -\sqrt{-c}$.

For what values of c will this have "interesting" solutions? If $c > 0$ then we have to take the square-root of a negative number! There is no real number that squares to -1; we introduce a new number i (for **imaginary**) and say $i^2 = \sqrt{-1}$.

Roots of Quadratic Equations

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Simplest is $y = x^2$, next would be either $y = ax^2$ or $y = x^2 + c$.

We can combine these two cases and consider $y = ax^2 + c$; when does this equal 0?

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We can combine these cases and consider $y = ax^2 + c$; when does this equal 0?

We want $ax^2 + c = 0$ so $ax^2 = -c$ so $x^2 = -c/a$.

We now take the square-root and find $x = \sqrt{-\frac{c}{a}}\ or\ -\sqrt{-\frac{c}{a}}$.

What should we study next?

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Roots of Quadratic Equations

A **quadratic equation** is of the form f(x) = $ax^2 + bx + c$, or $y = ax^2 + bx + c$.

We can look at $y = a(x - h)^2 + c$. Note this is not in the same form as above, but we can solve it VERY easily; we will then reduce the above general case to this.

We want $a(x-h)^2 + c = 0$ so $a(x-h)^2 = -c$ so $(x-h)^2 = -c/a$.

We now take the square-root, and find $x - h = \sqrt{-\frac{c}{a}}$ or $-\sqrt{-\frac{c}{a}}$. We often write this as $x - h = \pm\sqrt{-\frac{c}{a}}$. The ± indicates there are two terms, one with a positive and one with a negative.

Thus $x = h + \sqrt{-\frac{c}{a}}$ or $x = h - \sqrt{-\frac{c}{a}}$. Or, equivalently, $x = h \pm \sqrt{-\frac{c}{a}}$.

We see it is easy if there is no x term.

Roots of Quadratic Equations: Completing the Square

A **quadratic equation** is of the form f(x) = $ax^2 + bx + c$, or $y = ax^2 + bx + c$.

$$ax^2 + bx + c \; = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \; - \frac{b^2}{4a^2}\right) + c$$

We have done one of the most powerful things a mathematician can do: NOTHING!

But we did nothing in a clever way – we added zero, which doesn't change anything, but will allow us to re-write much of the above.

Trying to make it look like $a(x-h)^2 + d$. We saw that is easy to understand.....

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$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + \; c$$

We can simplify the above – note $(x + h)^2 = x^2 + 2hx + h^2$. Trying to make it look like $a(x-h)^2 + d$.

Can we find h such that 2h = b/a and $h^2 = b^2/4a^2$? Yes – take h = b/2a.

You've probably guessed – this is why we added 0 = $\frac{b^2}{4a^2} \; - \frac{b^2}{4a^2}$.

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$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

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$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \quad (how\ should\ we\ group\ to\ make\ this\ look\ like\ what\ we\ studied?)$$

Roots of Quadratic Equations: Completing the Square

A **quadratic equation** is of the form f(x) = ax^2 + bx + c, or y = ax^2 + bx + c. Let's solve f(x) = 0....

$$ax^2 + bx + c \;= a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \;- \frac{b^2}{4a^2}\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + \; c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + \; c \;= a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a} - c\right) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 \; - 4ac}{4a} \;= \; 0.$$

We now leave it to you to use what you have learned to solve this – it is in the form from before! You'll get $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Remember a(x-h)^2 + c = 0 meant $x = h \pm \sqrt{-\frac{c}{a}}$.

Roots of Quadratic Equations: Completing the Square

A **quadratic equation** is of the form f(x) = $ax^2 + bx + c$, or y = $ax^2 + bx + c$. Let's solve f(x) = 0....

The roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. We used $a(x-h)^2 + c = 0$ meant $x = h \pm \sqrt{-\frac{c}{a}}$.

Why did this work?

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Why did this work?

We know how to solve linear equations, we know how to solve simple quadratic equations like $x^2 = 0$. We are thus combining these two ideas, replacing one hard problem with two easier ones. Let $t^2=(x-h)^2$, then we have $at^2 - \left(\frac{b^2}{4a} - c\right) = 0$, so $t^2 = \left(\frac{b^2-4ac}{4a^2}\right)$. We can take the square root, and then as $t = x-h$ we get x is h plus the two solutions.

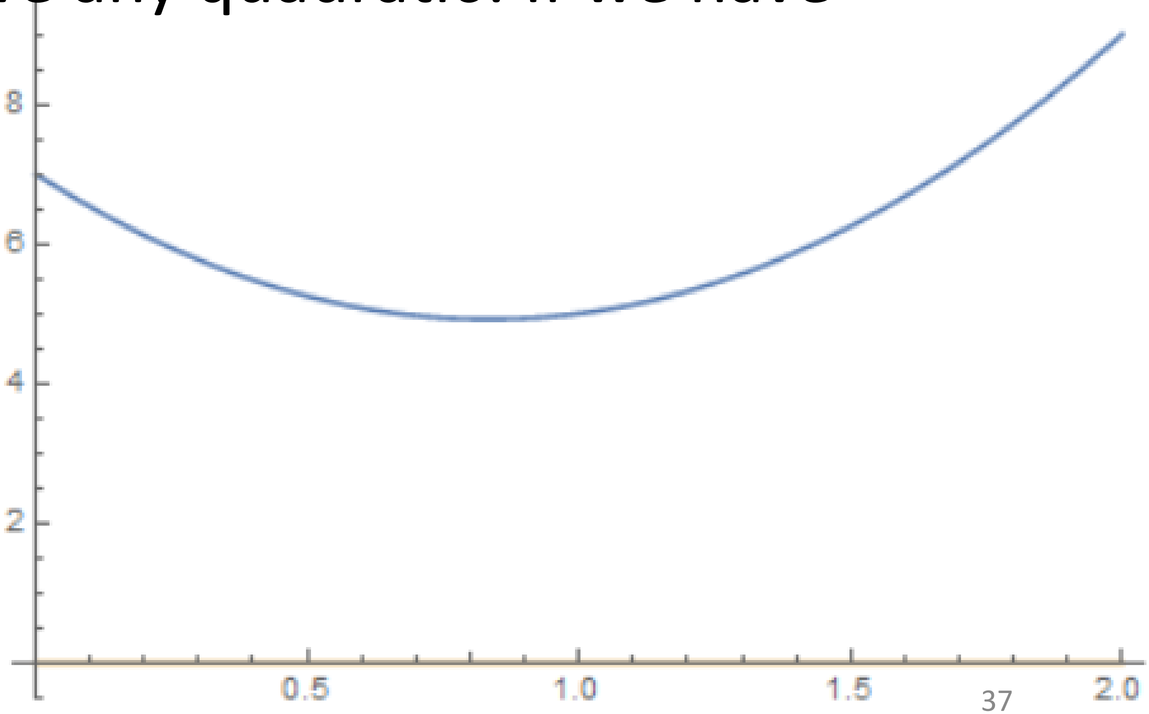
Solving Quadratic Equations

If $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$, the solutions to $f(x) = 0$ or $y = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula allows us to mechanically solve any quadratic! If we have $y = 3x^2 - 5x + 7 = 0$, the roots are just....

STOP STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION. STOP



Solving Quadratic Equations

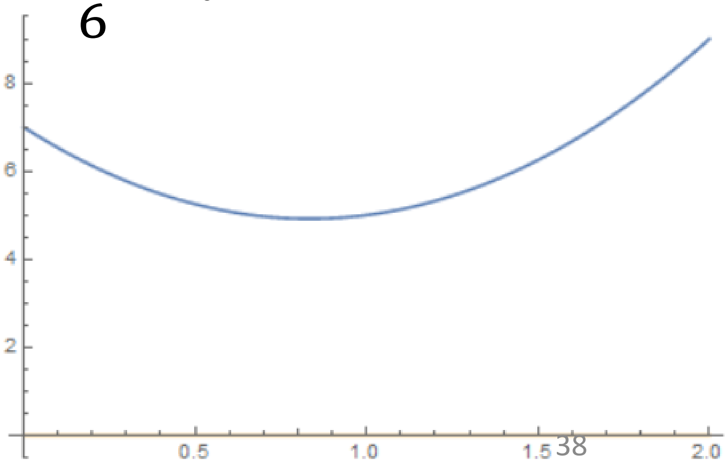
If $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$, the solutions to $f(x) = 0$ or $y = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula allows us to mechanically solve any quadratic! If we have

$y = 3\,x^2 - 5\,x + 7 = 0$, the roots are just

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4*3*7}}{2*3} = \frac{5 \pm \sqrt{25 - 4*3*7}}{2*3} = \frac{5 \pm \sqrt{25 - 84}}{6} = \frac{5 \pm \sqrt{-59}}{6}.$$



So our first example gives non-real roots, but looking at the plot, this isn't surprising….

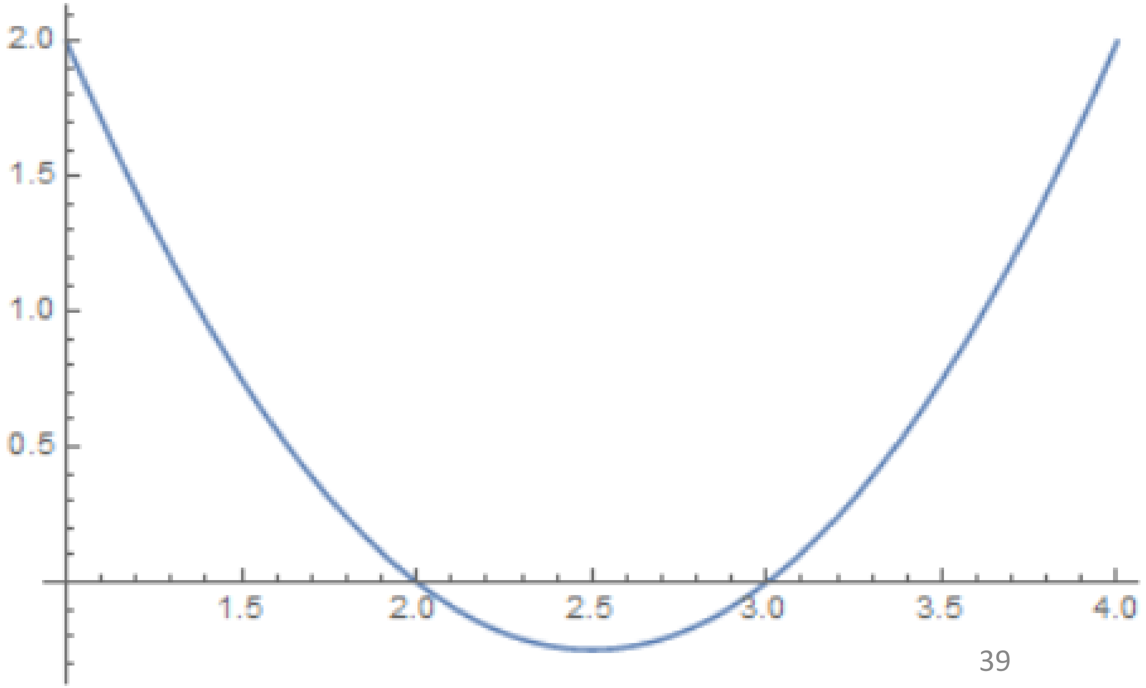
Solving Quadratic Equations

If $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$, the solutions to $f(x) = 0$ or $y = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If we have $y = x^2 - 5x + 6 = 0$, the roots are just….

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.



Solving Quadratic Equations

If $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$, the solutions to $f(x) = 0$ or $y = 0$ are

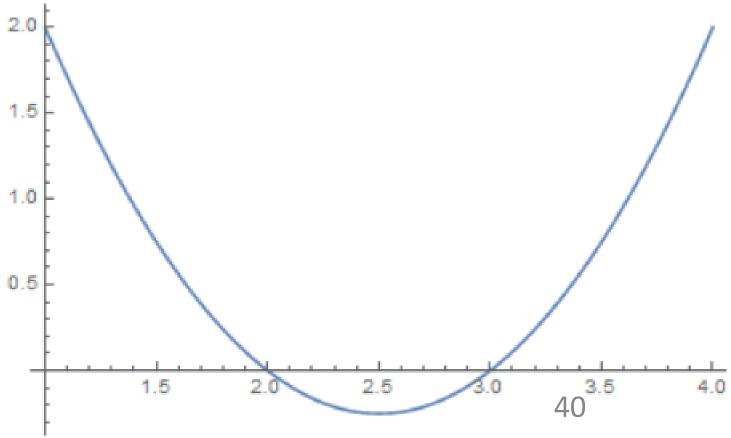
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If we have $y = x^2 - 5x + 6 = 0$, the roots are just

$$x = \frac{-5 \pm \sqrt{5^2 - 4*1*6}}{2*1} = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5 \pm \sqrt{1}}{2} = \frac{5 \pm 1}{2}, \text{so } x = \frac{6}{2} \text{ or } \frac{4}{2} \text{ (i.e., 3 or 2).}$$

Is there another way to find these answers?

They seem so nice!



Solving Quadratic Equations

If we have $y = x^2 - 5x + 6 = 0$, if the roots are r_1 and r_2 we are looking to factor it as $x^2 - 5x + 6 = (x - r_1)(x - r_2)$.

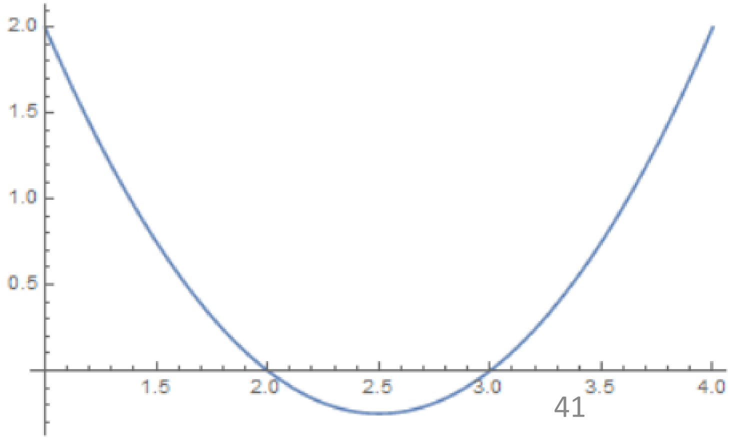
Expanding with FOIL, we get $x^2 - 5x + 6 = x^2 - r_2 x - r_1 x + r_1 r_2$.

So $x^2 - 5x + 6 = x^2 - (r_1 + r_2)x + r_1 r_2$.

We need two numbers r_1 and r_2 such that $r_1 + r_2 = 5$ and $r_1 r_2 = 6$.

This is an ***art***, you can often ***see*** the answer.....

Here we see one root is 2, one is 3.



Solving Quadratic Equations

If we have $y = x^2 – 7 x + 12 = 0$, if the roots are r_1 and r_2 we are looking to factor it as $x^2 – 7 x + 12 = (x – r_1) (x – r_2)$.

Expanding with FOIL, we get $x^2 – 7 x + 12 = x^2 – r_2 x – r_1 x + r_1 r_2$.

So $x^2 – 7 x + 12 = x^2 – (r_1 + r_2)x + r_1 r_2$.

We need two numbers r_1 and r_2 such that $r_1 + r_2 = 7$ and $r_1 r_2 = 12$.

This is an ***art***, you can often ***see*** the answer…..

Here we see one root is **???**, one is **???**.

Solving Quadratic Equations

If we have $y = x^2 – 7x + 12 = 0$, if the roots are r_1 and r_2 we are looking to factor it as $x^2 – 7x + 12 = (x – r_1)(x – r_2)$.

Expanding with FOIL, we get $x^2 – 7x + 12 = x^2 – r_2 x – r_1 x + r_1 r_2$.

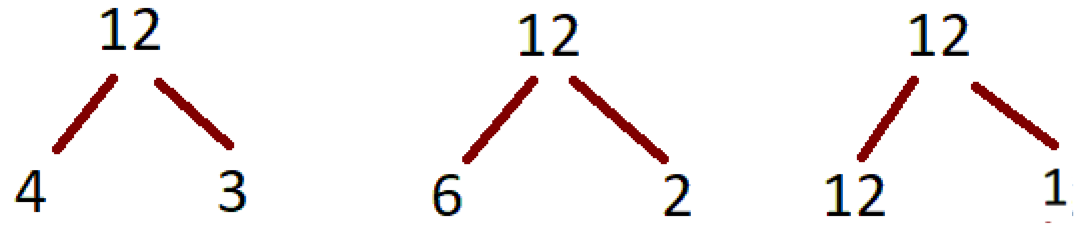
So $x^2 – 7x + 12 = x^2 – (r_1 + r_2)x + r_1 r_2$.

We need two numbers r_1 and r_2 such that $r_1 + r_2 = 7$ and $r_1 r_2 = 12$.

This is an ***art***, you can often ***see*** the answer…..

Here we see one root is **3**, one is **4**.

Drawing the factor trees helps….



Solving Quadratic Equations

If we have $y = x^2 + x - 12 = 0$, if the roots are r_1 and r_2 we are looking to factor it as $x^2 + x - 12 = (x – r_1)\ (x – r_2)$.

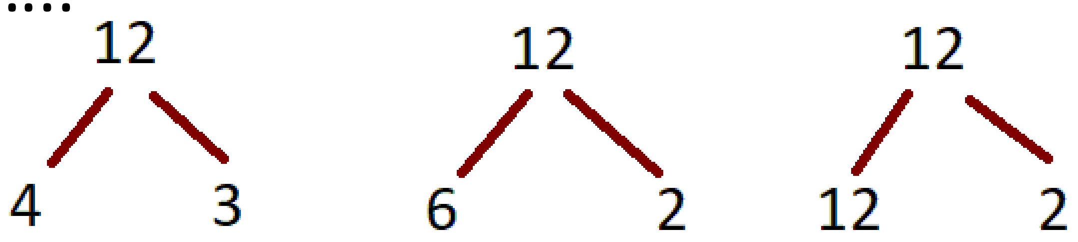
Expanding with FOIL, we get $x^2 + x - 12 = x^2 – r_2 x – r_1 x + r_1r_2$.

So $x^2 + x - 12 = x^2 – (r_1 + r_2)x + r_1r_2$.

We need two numbers r_1 and r_2 such that $r_1 + r_2 = -1$ and $r_1 r_2 = -12$.

This is an ***art***, you can often ***see*** the answer…..

Here we see one root is **???**, one is **???**.



Solving Quadratic Equations

If we have $y = x^2 + x - 12 = 0$, if the roots are r_1 and r_2 we are looking to factor it as $x^2 + x - 12 = (x - r_1)(x - r_2)$.

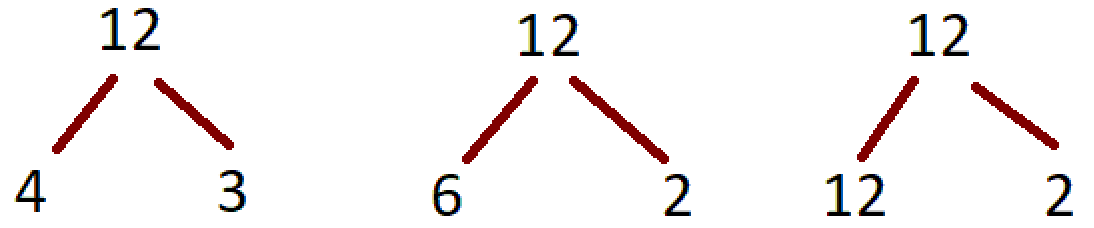
Expanding with FOIL, we get $x^2 + x - 12 = x^2 - r_2 x - r_1 x + r_1 r_2$.

So $x^2 + x - 12 = x^2 - (r_1 + r_2)x + r_1 r_2$.

We need two numbers r_1 and r_2 such that $r_1 + r_2 = -1$ and $r_1 r_2 = -12$.

This is an ***art***, you can often ***see*** the answer…..

Here we see one root is **-4**, one is **3**.

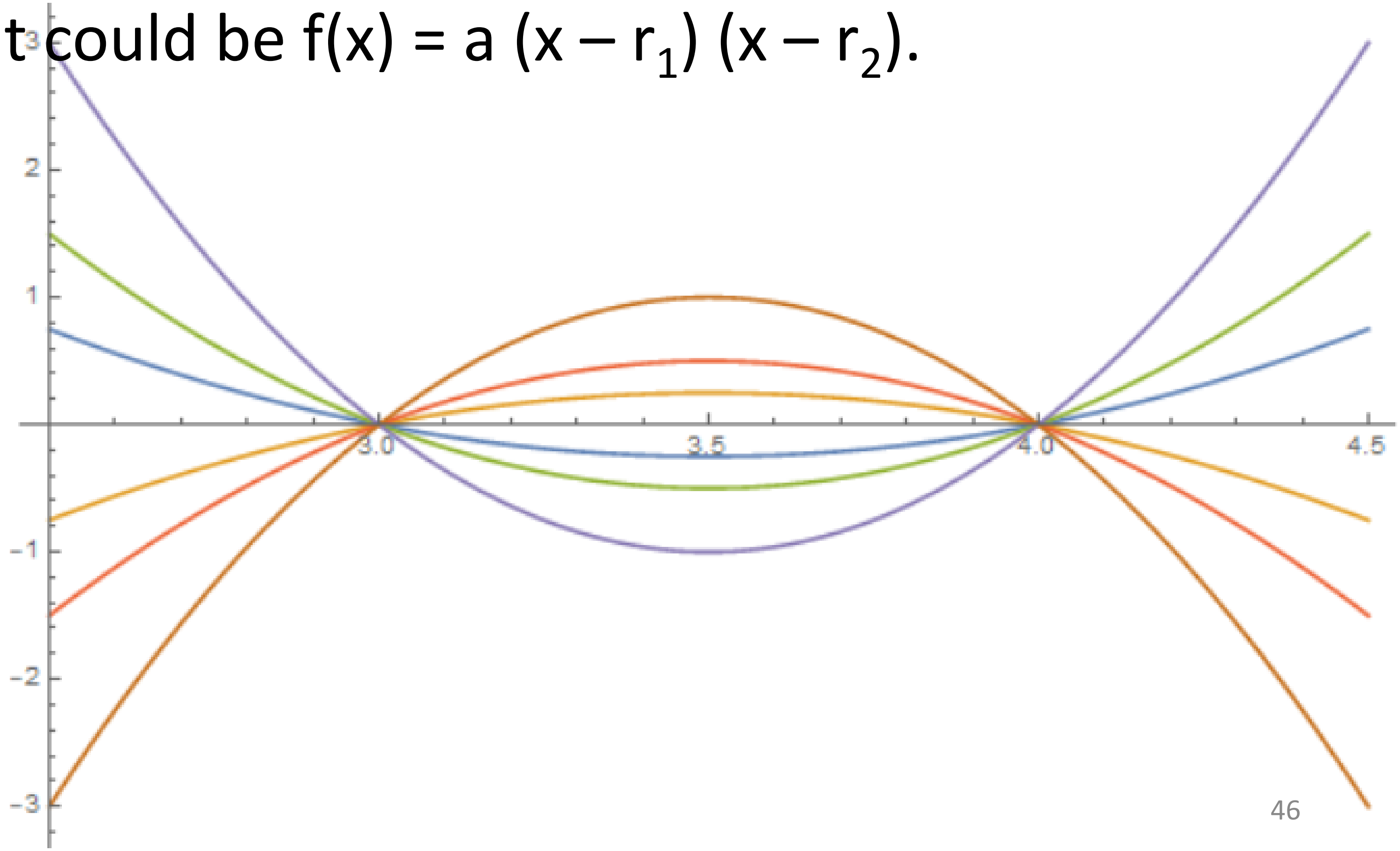


Quadratic Equations

Similar to lines, if we know the two roots of a quadratic we don't quite know it.

If the roots are r_1 and r_2 then it could be $f(x) = a\,(x - r_1)\,(x - r_2)$.

Any choice of a will work.



Cubic Equations

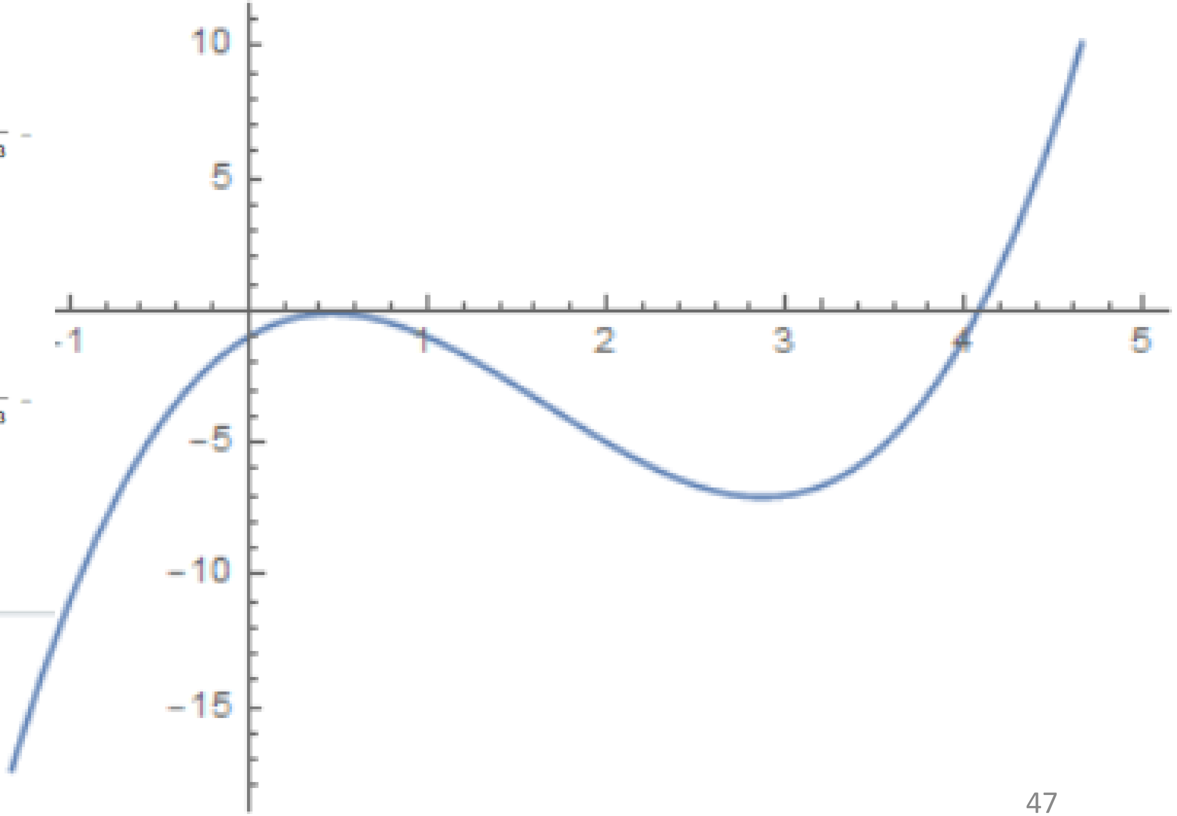
Amazingly there is a formula for cubics!

```
Solve[a x^3 + b x^2 + c x + d == 0, x]
```

$$\left\{\left\{x \to -\frac{b}{3a} - \frac{2^{1/3}(-b^2+3ac)}{3a\left(-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}\right)^{1/3}} + \frac{\left(-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}\right)^{1/3}}{3\times 2^{1/3}a}\right\},\right.$$

$$\left\{x \to -\frac{b}{3a} + \frac{(1+i\sqrt{3})(-b^2+3ac)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}\right)^{1/3}} - \frac{(1-i\sqrt{3})\left(-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}\right)^{1/3}}{6\times 2^{1/3}a}\right\},$$

$$\left.\left\{x \to -\frac{b}{3a} + \frac{(1-i\sqrt{3})(-b^2+3ac)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}\right)^{1/3}} - \frac{(1+i\sqrt{3})\left(-2b^3+9abc-27a^2d+\sqrt{4(-b^2+3ac)^3+(-2b^3+9abc-27a^2d)^2}\right)^{1/3}}{6\times 2^{1/3}a}\right\}\right\}$$



Cubic Equations

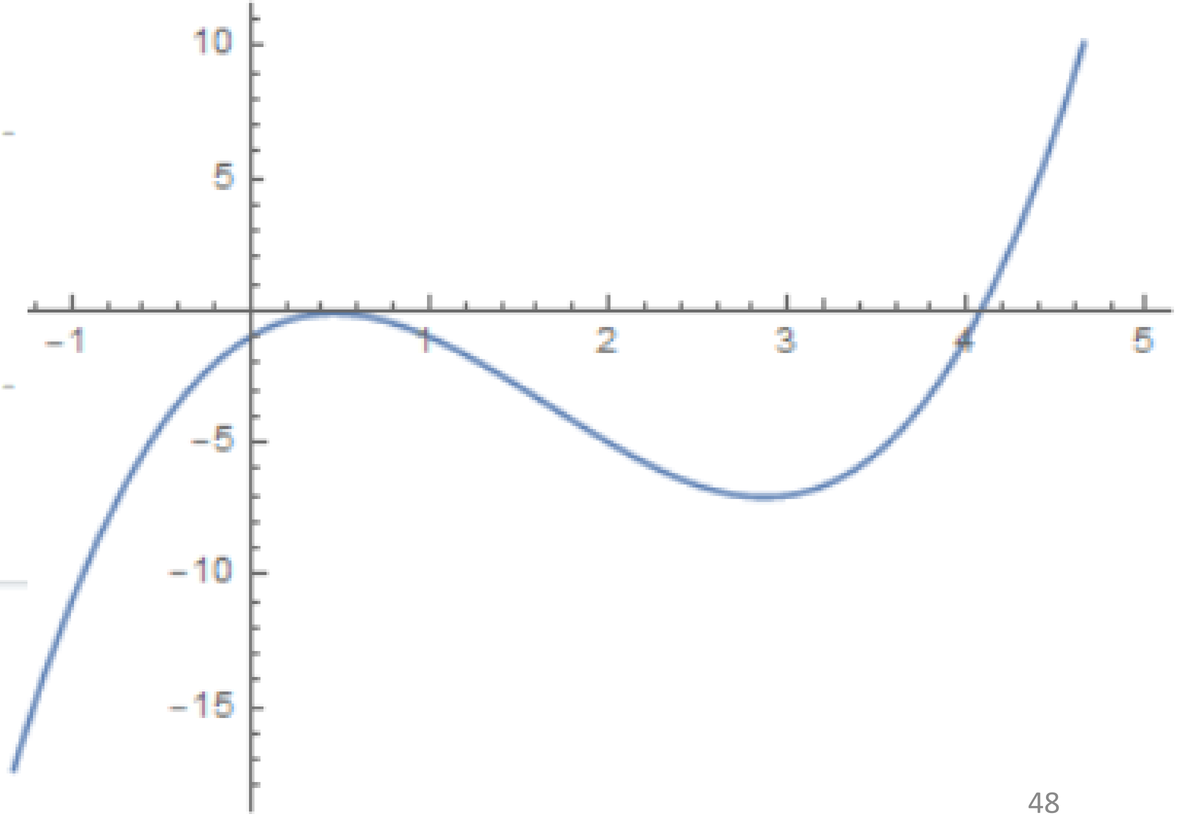
Amazingly there is a formula for cubics! Little cleaner if adjust to a=1.

```
Solve[x^3 + b x^2 + c x + d == 0, x]
```

$$\left\{\left\{x \to -\frac{b}{3} - \frac{2^{1/3}\left(-b^2+3c\right)}{3\left(-2b^3+9bc-27d+3\sqrt{3}\sqrt{-b^2c^2+4c^3+4b^3d-18bcd+27d^2}\right)^{1/3}} + \frac{\left(-2b^3+9bc-27d+3\sqrt{3}\sqrt{-b^2c^2+4c^3+4b^3d-18bcd+27d^2}\right)^{1/3}}{3\times 2^{1/3}}\right\},\right.$$

$$\left\{x \to -\frac{b}{3} + \frac{\left(1+i\sqrt{3}\right)\left(-b^2+3c\right)}{3\times 2^{2/3}\left(-2b^3+9bc-27d+3\sqrt{3}\sqrt{-b^2c^2+4c^3+4b^3d-18bcd+27d^2}\right)^{1/3}} - \frac{\left(1-i\sqrt{3}\right)\left(-2b^3+9bc-27d+3\sqrt{3}\sqrt{-b^2c^2+4c^3+4b^3d-18bcd+27d^2}\right)^{1/3}}{6\times 2^{1/3}}\right\},$$

$$\left.\left\{x \to -\frac{b}{3} + \frac{\left(1-i\sqrt{3}\right)\left(-b^2+3c\right)}{3\times 2^{2/3}\left(-2b^3+9bc-27d+3\sqrt{3}\sqrt{-b^2c^2+4c^3+4b^3d-18bcd+27d^2}\right)^{1/3}} - \frac{\left(1+i\sqrt{3}\right)\left(-2b^3+9bc-27d+3\sqrt{3}\sqrt{-b^2c^2+4c^3+4b^3d-18bcd+27d^2}\right)^{1/3}}{6\times 2^{1/3}}\right\}\right\}$$



Cubic Equations

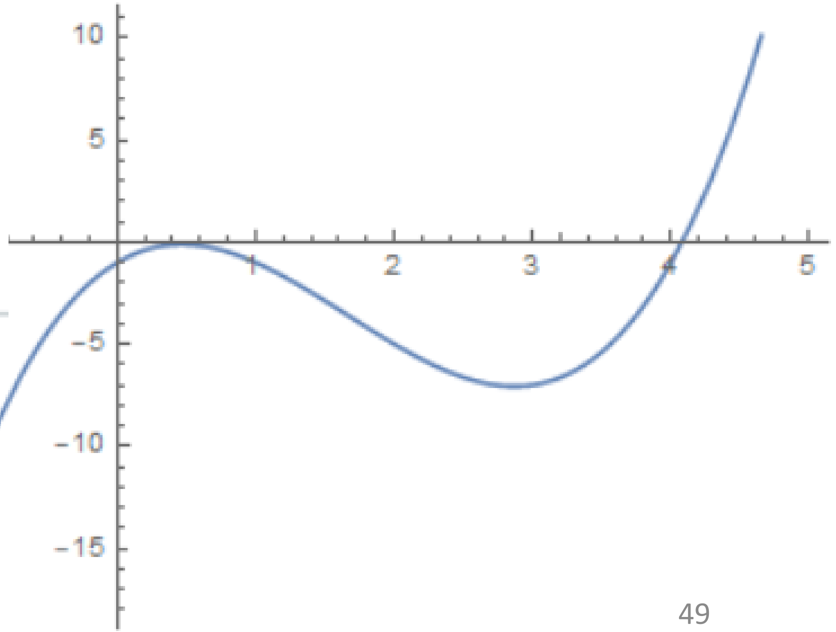
Amazingly there is a formula for cubics! Better: a=1, translate to b=0.

```
Solve[x^3 + c x + d == 0, x]
```

$$\left\{\left\{x \to -\frac{\left(\frac{2}{3}\right)^{1/3} c}{\left(-9 d + \sqrt{3}\sqrt{4 c^3 + 27 d^2}\right)^{1/3}} + \frac{\left(-9 d + \sqrt{3}\sqrt{4 c^3 + 27 d^2}\right)^{1/3}}{2^{1/3}\, 3^{2/3}}\right\},\right.$$

$$\left\{x \to \frac{\left(1 + i\sqrt{3}\right) c}{2^{2/3}\, 3^{1/3}\left(-9 d + \sqrt{3}\sqrt{4 c^3 + 27 d^2}\right)^{1/3}} - \frac{\left(1 - i\sqrt{3}\right)\left(-9 d + \sqrt{3}\sqrt{4 c^3 + 27 d^2}\right)^{1/3}}{2\, 2^{1/3}\, 3^{2/3}}\right\},$$

$$\left.\left\{x \to \frac{\left(1 - i\sqrt{3}\right) c}{2^{2/3}\, 3^{1/3}\left(-9 d + \sqrt{3}\sqrt{4 c^3 + 27 d^2}\right)^{1/3}} - \frac{\left(1 + i\sqrt{3}\right)\left(-9 d + \sqrt{3}\sqrt{4 c^3 + 27 d^2}\right)^{1/3}}{2\, 2^{1/3}\, 3^{2/3}}\right\}\right\}$$



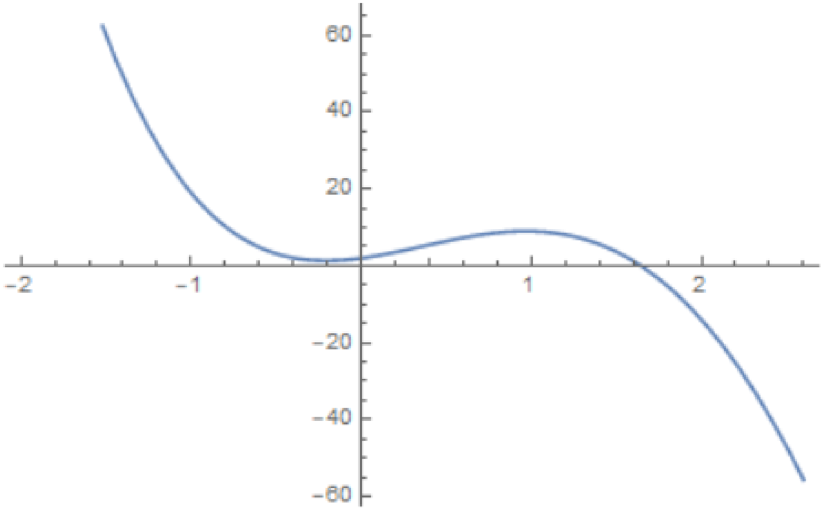
Quartic Equations

Amazingly there is a formula for quartics, but not for degree 5 and higher!

Doing $x^4 + cx^2 + dx + e = 0$. Here is ONE of the four roots….

$$x \to \frac{1}{2}\sqrt{-\frac{2c}{3} + \frac{2^{1/3}\left(c^2+12e\right)}{3\left(2c^3+27d^2-72ce+\sqrt{-4\left(c^2+12e\right)^3+\left(2c^3+27d^2-72ce\right)^2}\right)^{1/3}} + \frac{\left(2c^3+27d^2-72ce+\sqrt{-4\left(c^2+12e\right)^3+\left(2c^3+27d^2-72ce\right)^2}\right)^{1/3}}{3\times 2^{1/3}}} -$$

$$\frac{1}{2}\sqrt{-\frac{4c}{3} - \frac{2^{1/3}\left(c^2+12e\right)}{3\left(2c^3+27d^2-72ce+\sqrt{-4\left(c^2+12e\right)^3+\left(2c^3+27d^2-72ce\right)^2}\right)^{1/3}} - \frac{\left(2c^3+27d^2-72ce+\sqrt{-4\left(c^2+12e\right)^3+\left(2c^3+27d^2-72ce\right)^2}\right)^{1/3}}{3\times 2^{1/3}} - \frac{2d}{\sqrt{-\frac{2c}{3} + \frac{2^{1/3}\left(c^2+12e\right)}{3\left(2c^3+27d^2-72ce+\sqrt{-4\left(c^2+12e\right)^3+\left(2c^3+27d^2-72ce\right)^2}\right)^{1/3}} + \frac{\left(2c^3+27d^2-72ce+\sqrt{-4\left(c^2+12e\right)^3+\left(2c^3+27\cdots\right.}\right.}{3\,2^{1/3}}}}}$$



Where does it all go?

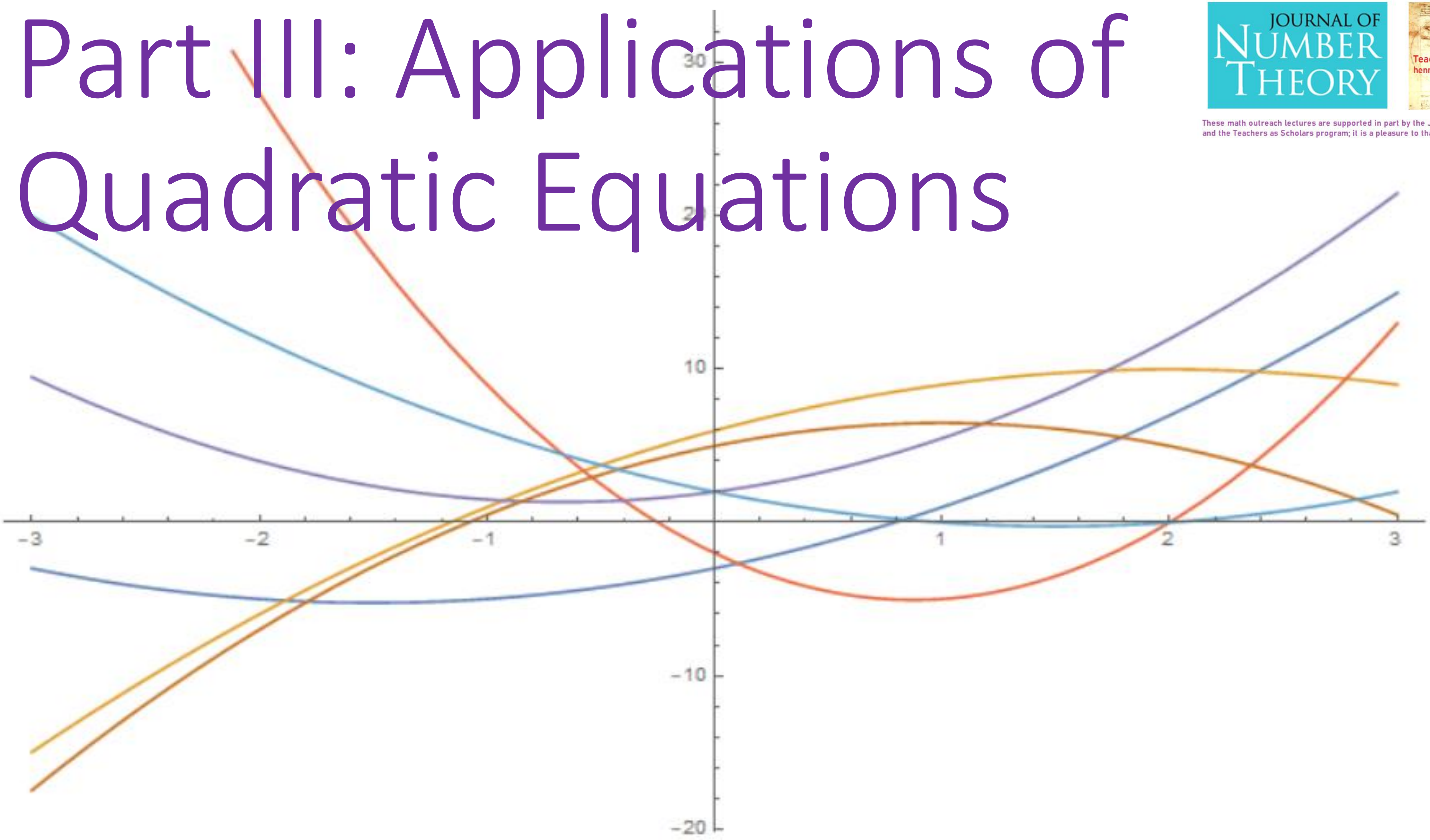
- The **Fundamental Theorem of Algebra** states that if you have a polynomial of degree d then it has exactly d roots.
- Some roots might be multiple roots: $x^4 - x^2 = x^2 (x^2 - 1) = x\, x\, (x-1)(x+1)$ has roots 0, 0, 1 and -1; so 0 is a double root.
- Amazingly, all the roots are complex numbers and can be written in the form a + ib where a and b are real numbers and i is the square-root of -1; this means the one number we introduced to solve $x^2 + 1 = 0$ is all we need to solve any polynomial!

Part III: Applications of Quadratic Equations

JOURNAL OF NUMBER THEORY

Teachers As Scholars
henry.bolter@gmail.com

These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

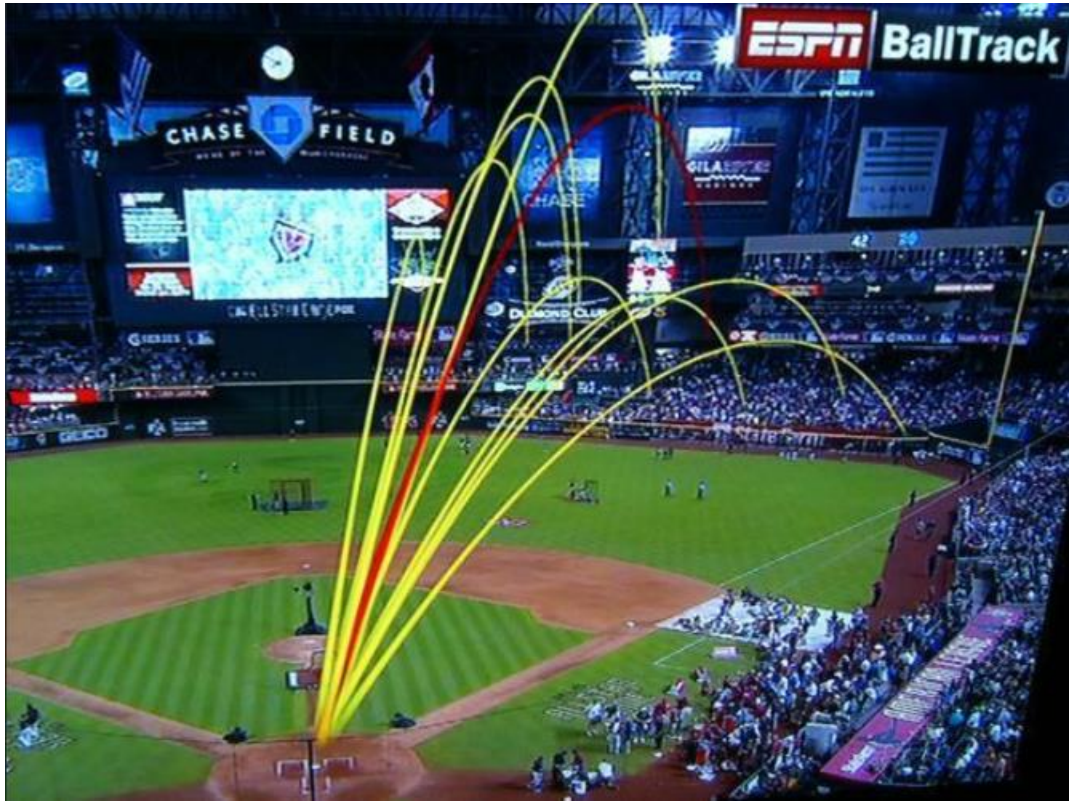


https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

First Application: Trajectories

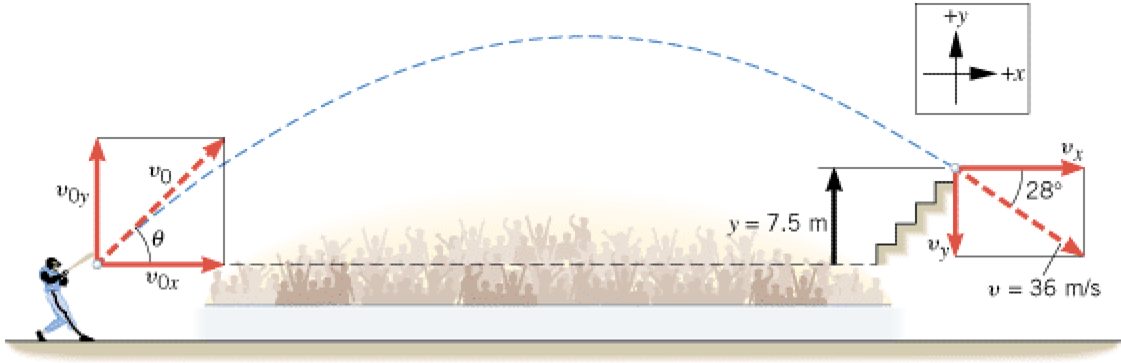
The paths of many objects follow a parabola.

- Baseballs.
- Cannonballs.
- Water foutains.



First Application: Trajectories

The paths of many objects follow a parabola.



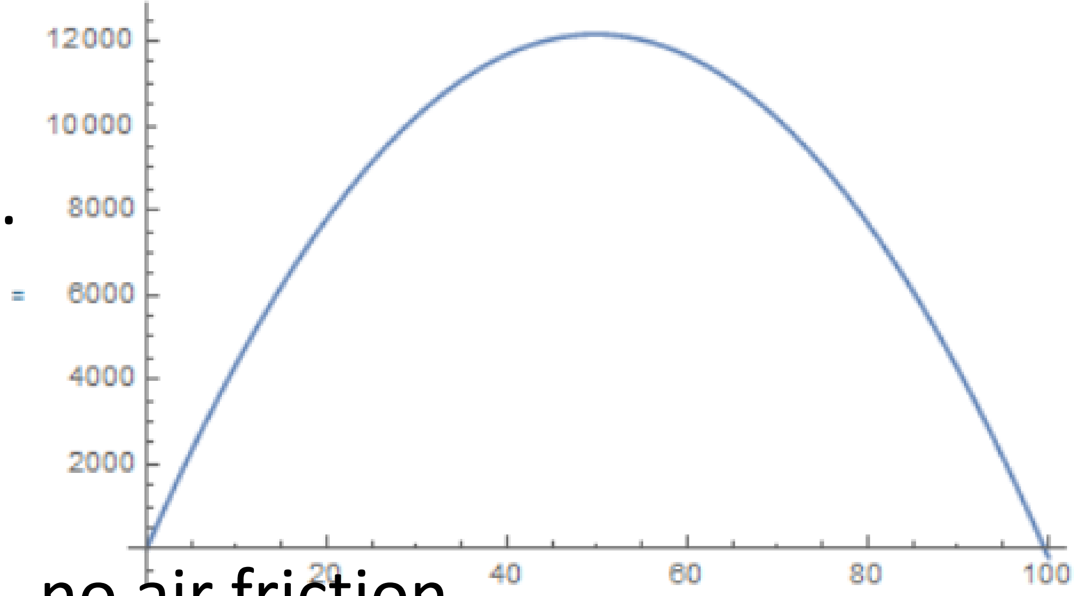
Consider a cannonball (or a batted baseball).

If the path is given by $f(x) = ax^2 + bx + c$, we can find where it hits the ground by solving for $f(x) = 0$; we just learned how to do that.

However, just as we saw earlier that there is more than one way to write a parabola, there's more than one way used to describe a cannonball's path.

First Application: Trajectories

Consider a cannonball (or a batted baseball).



We assume a constant force of gravity down, no air friction.

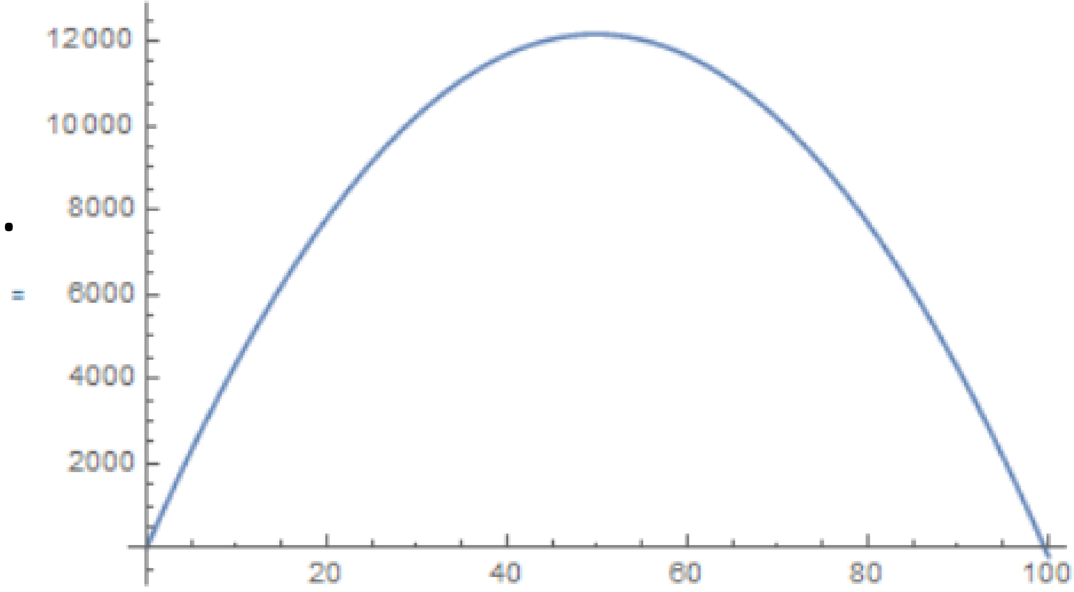
Will do the simple case of firing vertically upward.

- Initial speed is 488m/sec
- Initial height is 10m
- At time t, cannonball's height is $f(t) = 10 + 488\, t - 4.9\, t^2$.

Can you find the highest point?

First Application: Trajectories

Consider a cannonball (or a batted baseball).



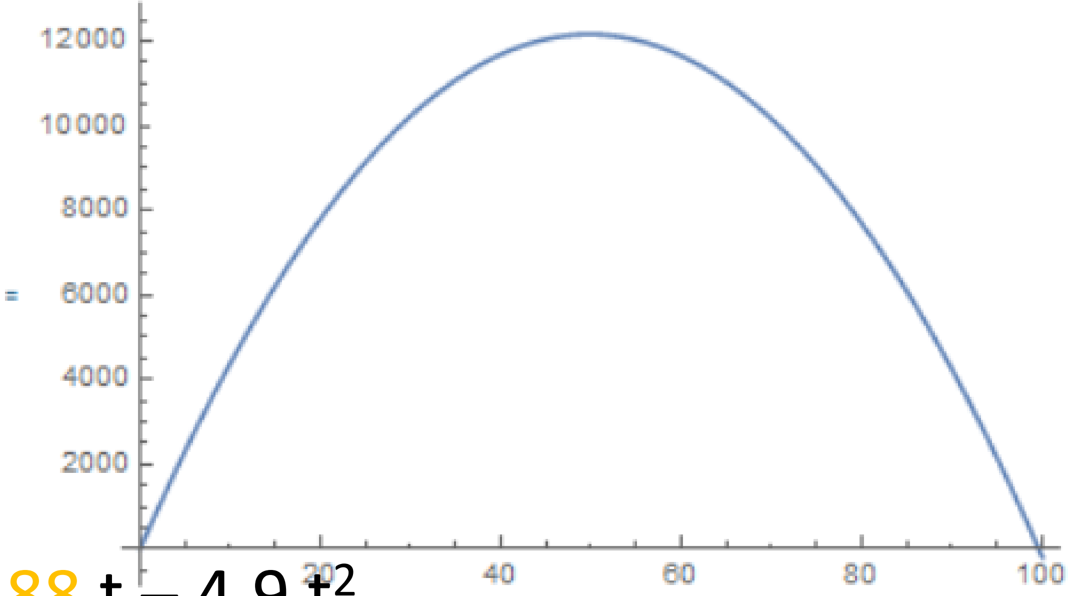
- Initial speed is 488m/sec
- Initial height is 10m
- At time t, cannonball's height is $f(t) = 10 + 488\,t - 4.9\,t^2$.

Write it as $a(t - h)^2 + d$; as a will be negative, greatest height at $t=h$.

What are a, h and d?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

First Application: Trajectories



Consider a cannonball (or a batted baseball).

- Initial speed is 488m/sec
- Initial height is 10m
- At time t, cannonball's height is $f(t) = 10 + 488\,t - 4.9\,t^2$.

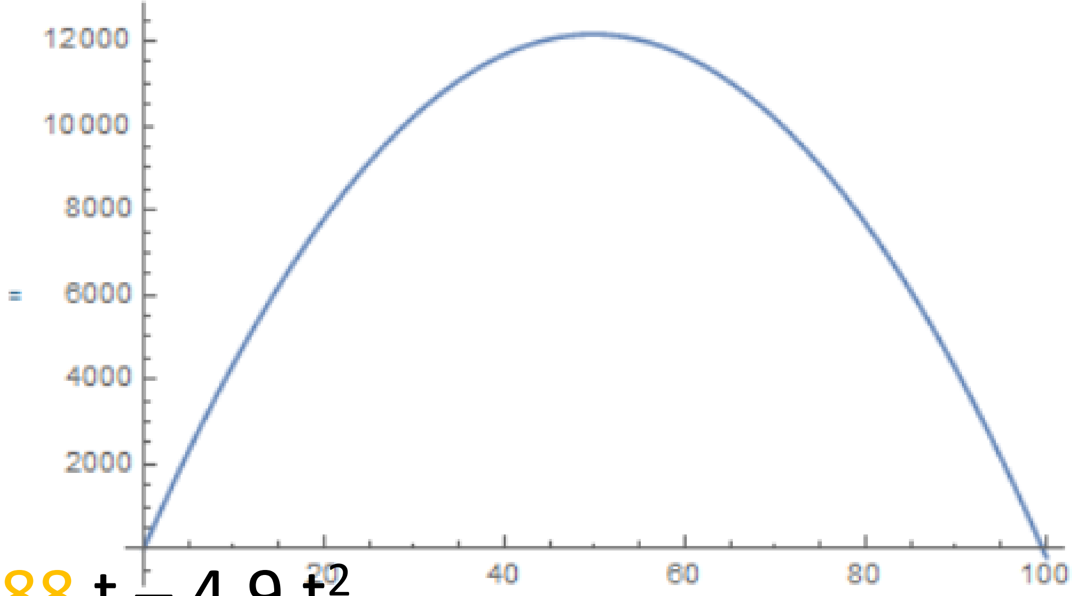
Write it as $a(t - h)^2 + d$; as a will be negative, greatest height at t=h. Will approximate fractions….

$f(t) = -4.9\,t^2 + 488\,t + 10 = -4.9\,(t^2 - 99.6\,t) + 10$

$= -4.9\,(t^2 - 99.6\,t + 2480.04 - 2480.04) + 10$

$= -4.9\,(t - 49.8)^2 + 10$. Thus maximum height at about 49.8 seconds after launch.

First Application: Trajectories



Consider a cannonball (or a batted baseball).

- Initial speed is 488m/sec
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- At time t, cannonball's height is $f(t) = 10 + 488\,t - 4.9\,t^2$.

$f(t) = -4.9\,t^2 + 488\,t + 10 = -4.9\,(t^2 - 99.6\,t) + 10$

$= -4.9\,(t^2 - 99.6\,t + 2480.04 - 2480.04) + 10$

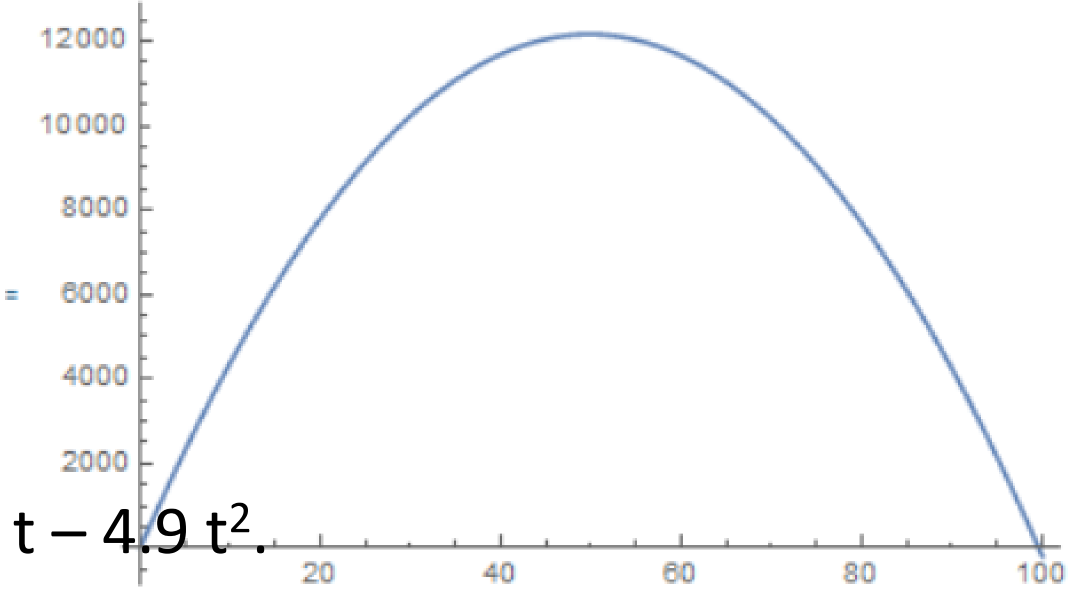
$= -4.9\,(t - 49.8)^2 + 10$. Thus maximum height at about 49.8 seconds after launch.

To find: $-4.9(t - h)^2 + d = -4.9\,t^2 + 488\,t + 10$

Or: $-4.9\,(t^2 - 2ht + h^2) + d = -4.9\,t^2 + 488\,t + 10$

Thus $-4.9t^2 + 9.8ht + (d-4.9h^2) + d = -4.9t^2 + 488\,t + 10$. How do we find h and d?

First Application: Trajectories



Consider a cannonball (or a batted baseball).

- Initial speed is 488m/sec
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- At time t, cannonball's height is $f(t) = 10 + 488\, t - 4.9\, t^2$.

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To find: $-4.9(t - h)^2 + d = -4.9\, t^2 + 488\, t + 10$

Or: $-4.9\, (t^2 - 2ht + h^2) + d = -4.9\, t^2 + 488\, t + 10$

Thus $-4.9t^2 + 9.8ht + (d - 4.9h^2) + d = -4.9t^2 + 488\, t + 10$. How do we find h and d?

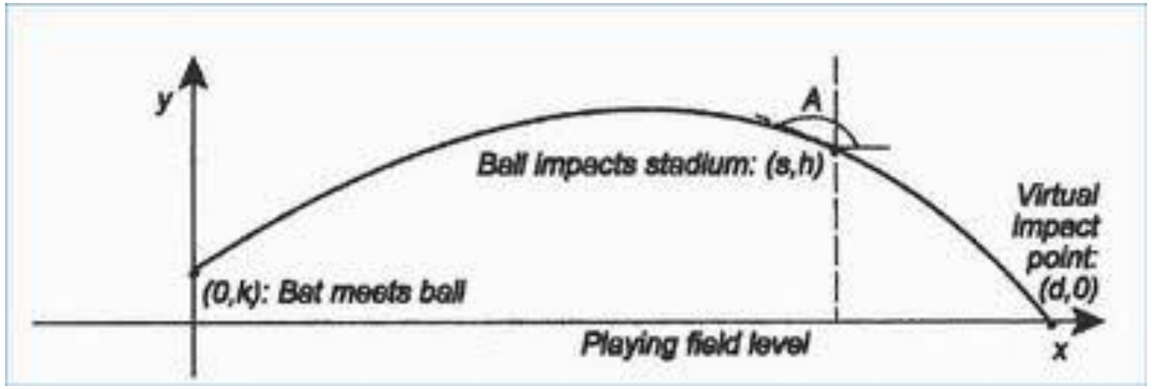
First $9.8h = 488$ so $h = 488/9.8$, then $d - 4.9h^2 = 10$ and thus $d = 10 + 4.9h^2$.

First Application: Trajectories

Consider a batted baseball.

https://calculushowto.com/how-to-describe-the-path-of-a-baseball-in-calculus/

Path of a baseball: Steps



Step 1: Define the variables used in both the parametric equations.

- Represent the height in feet by 'h',
- The angle in degrees by 'a',
- The initial velocity in feet per second by 'v'
- The time in seconds by 't'.

Step 2: Write an equation for the horizontal motion of the baseball as a function of time:

- $x(t) = v * Cos(a) * t$.

Step 3: Write an equation to describe the vertical motion of the baseball as a function of time:

- $y(t) = h + v * Sin(a) * t - 16 * t^2$.

Two steps. We figure out how long it is airborn until the y-coordinate is zero, and that gives the time to plug in to the formula for x. What do you think is the best angle?

First Application: Trajectories

Consider a batted baseball. Use 6 ft above ground, 176 ft/sec, 45 degree angle.

https://calculushowto.com/how-to-describe-the-path-of-a-baseball-in-calculus/

Vertical as a function of time: $y(t) = 6 + 124t - 16t^2$.

Horizontal as a function of time: $x(t) = 124t$.

The time of flight is when $y(t) = 0$, so solve $-16t^2 + 124t + 6 = 0$.

Use Quadratic Formula: Get $\frac{31 \pm \sqrt{985}}{8}$, so time is about 7.8 seconds.

Thus horizontal distance is about 967 feet.

Is this reasonable? If yes why, if not why not?

First Application: Trajectories

Consider a batted baseball. Use 6 ft above ground, 176 ft/sec, 45 degree angle.

https://calculushowto.com/how-to-describe-the-path-of-a-baseball-in-calculus/

Vertical as a function of time: $y(t) = 6 + 124t - 16t^2$.

Horizontal as a function of time: $x(t) = 124t$.

The time of flight is when $y(t) = 0$, so solve $-16t^2 + 124t + 6 = 0$.

Use Quadratic Formula: Get $\frac{31 \pm \sqrt{985}}{8}$, so time is about 7.8 seconds.

Thus horizontal distance is about 967 feet.

Seems high – no air resistance.

First Application: Trajectories

http://www.schoolphysics.co.uk/age16-19/Mechanics/Kinematics/text/Projectiles_and_air_resistance/index.html

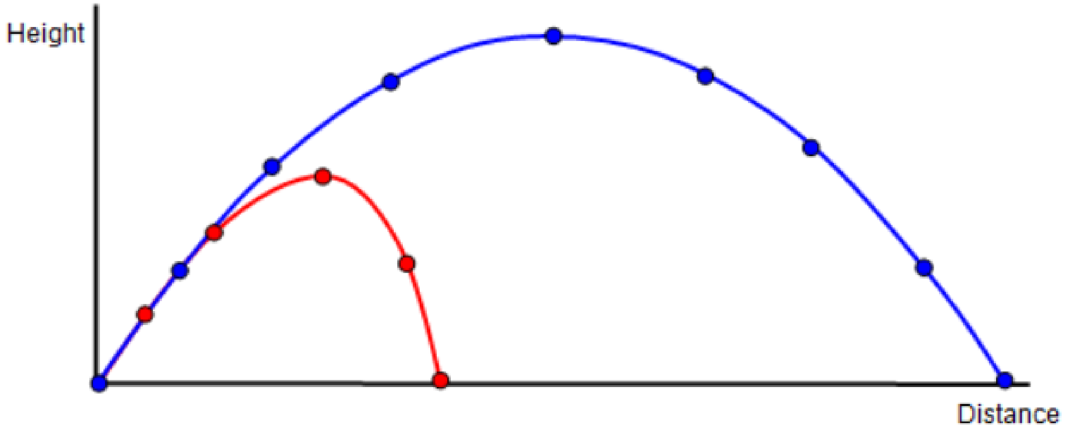
Projectiles and air resistance

Objects moving through air are slowed down due to **air resistance**, sometimes called drag. This air resistance affects a spacecraft when it re-enters the Earth's atmosphere but also the path of a projectile such as a bullet or a ball. When air resistance is taken into account the trajectory of a projectile is changed. The resistance is often taken as being proportional to either the velocity of the object or the square of the velocity of the object.

The medieval scientists believed that a projectile went upwards at an angle along a straight path, then went through a short curved section before falling vertically back to the ground again.

Both the range of a projectile and the maximum height that it reaches are affected by air resistance. The mathematics of the motion is quite complicated (especially if you consider the change in the shape and/or surface of a projectile and the variation of the density of the air with height) but the following diagrams try to simplify things by showing generally how air resistance affects both the trajectory and the velocity of a projectile.

The blue lines show the projectile with no air resistance and the red lines show what happens when air resistance is taken into account. The maximum height, the range and the velocity of the projectile are all reduced.



Project: Trajectories

Consider a batted baseball. Use 6 ft above ground, 176 ft/sec, angle of θ.

https://calculushowto.com/how-to-describe-the-path-of-a-baseball-in-calculus/

Vertical as a function of time: $y(t) = 6 + 176 \sin(\theta)\, t - 16t^2$.

Horizontal as a function of time: $x(t) = 176 \cos(\theta)t$.

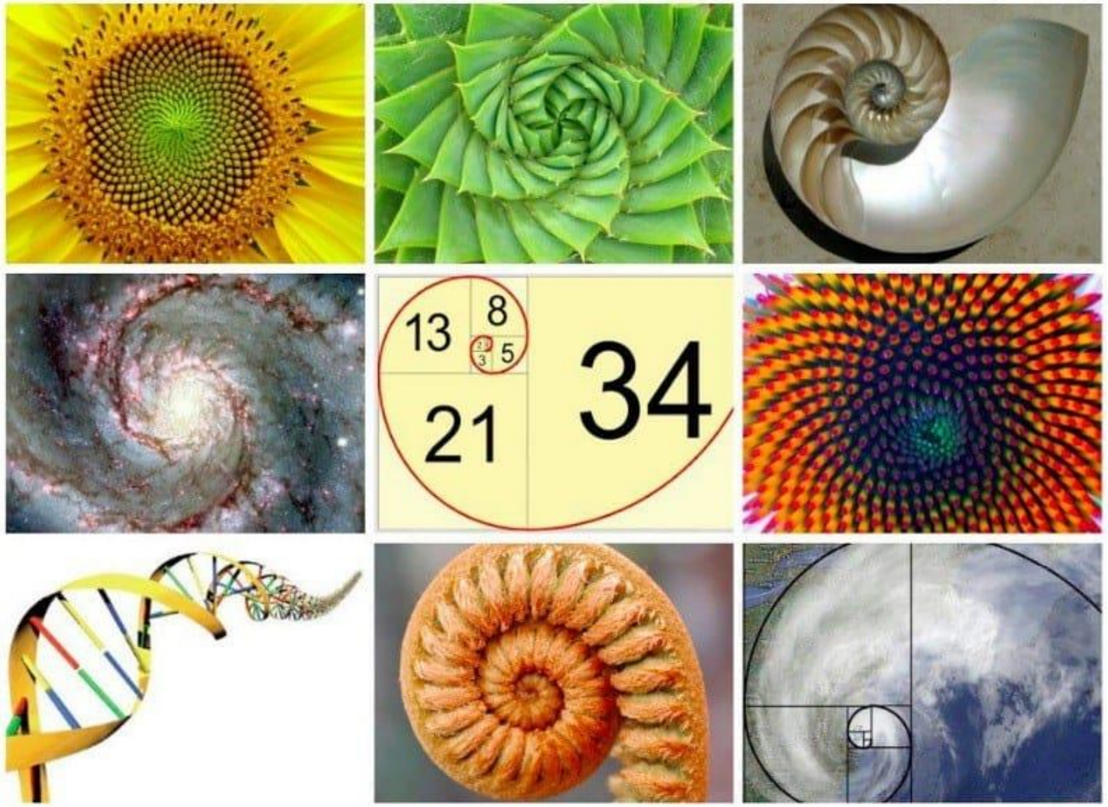
Here $\cos(\theta)$ and $\sin(\theta)$ are the sides of a right triangle, so the sum of their squares is 1.

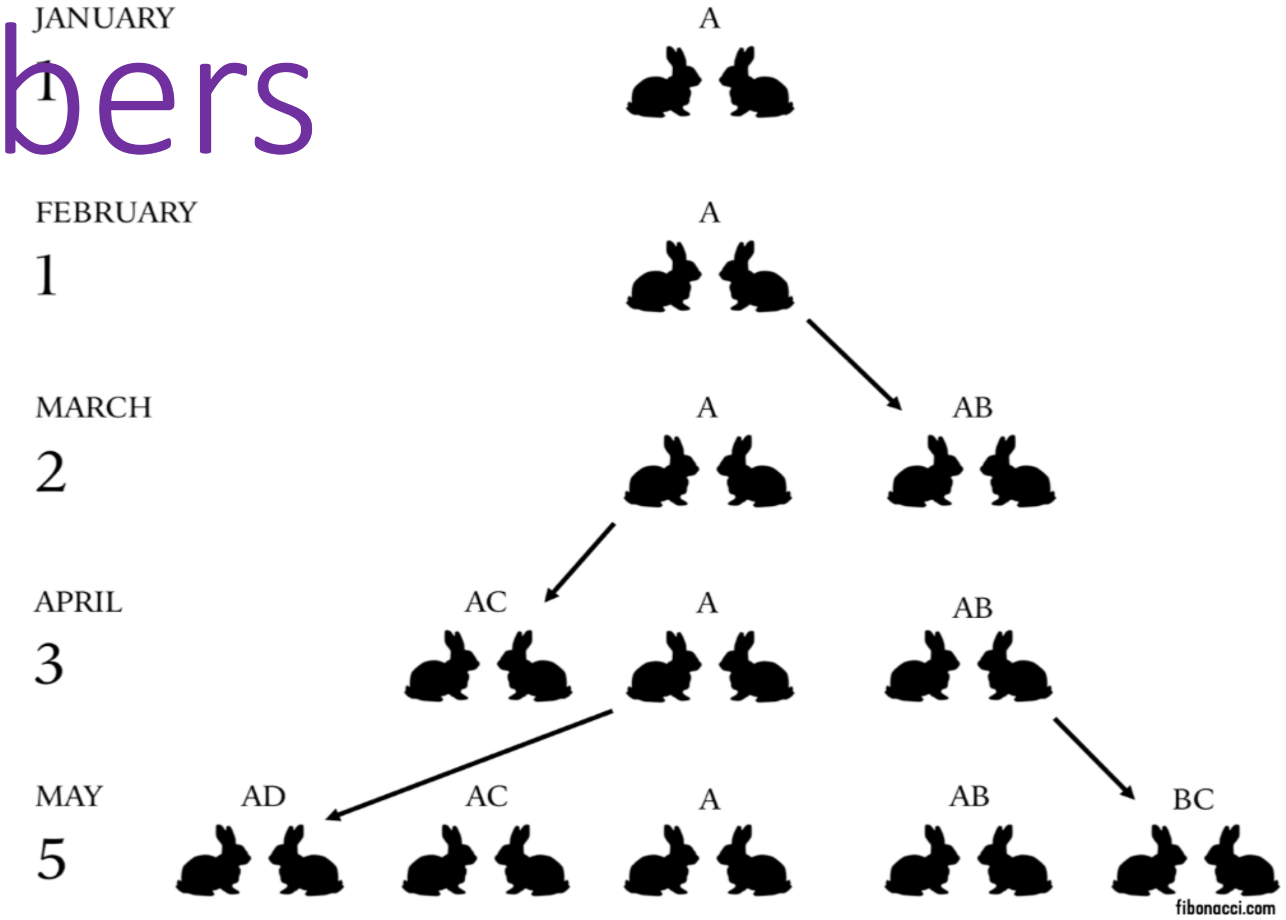
We have $\cos(\theta)$ goes from 0 to 1, and $\sin(\theta) = \sqrt{1 - \cos(\theta)^{2}}$.

Project: Use the quadratic formula to find the flight time and horizontal distance as a function of θ. What θ gives the furthest horizontal travel?

Part III: Applications of Quadratic Equations:The Fibonacci Numbers

JOURNAL OF NUMBER THEORY
Teachers As Scholars
henry.bolter@gmail.com
These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.





https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

The Fibonacci Numbers

There are many ways to define the Fibonacci numbers.

We met them in:

I Love Rectangles Game: https://youtu.be/JHtrzARHwHU (powerpoint here, pdf here) (3/24/2020): Aimed for K, should be good for all ages. 13 minutes.

Standard definition:

F_n is the n^{th} Fibonacci number, $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

Thus the sequence is 0, 1, ...?

The Fibonacci Numbers

There are many ways to define the Fibonacci numbers.

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Thus the sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ….

The Fibonacci Numbers

Look at the Sequence, create some questions: https://oeis.org/A000045

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THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A000045 Fibonacci numbers: F(n) = F(n-1) + F(n-2) with F(0) = 0 and F(1) = 1. 4919

(Formerly M0692 N0256)

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169, 63245986, 102334155 (list; graph; refs; listen; history; text; internal format)

Ask some questions about them!

STOP STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION. STOP

The Fibonacci Numbers

Look at the Sequence, create some questions: https://oeis.org/A000045

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Are there infinitely many that are prime? That are perfect squares? Perfect cubes? How fast do they grow? Could you have a right triangle with all sides Fibonacci? Where do they arise? Is there a formula for each?

Get in the habit of asking questions!

The Fibonacci Numbers

F_n is the n^{th} Fibonacci number, $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

Thus the sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ….

Most of the questions are too hard, but we can get their size….

After the double 1's, we see the Fibonaccis are strictly increasing. Can you bound their growth rate? Upper bound? Lower bound?

This means find functions U(n) and L(n) such that $L(n) \leq F_n \leq U(n)$.

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Thus the sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ….

Want functions U(n) and L(n) such that $L(n) \leq F_n \leq U(n)$.

As $F_{n+1} = F_n + F_{n-1}$ we have $F_{n+1} \leq$ **??? (bound it in terms of F_n)**

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Want functions U(n) and L(n) such that $L(n) \leq F_n \leq U(n)$.

As $F_{n+1} = F_n + F_{n-1}$ we have $F_{n+1} \leq F_n + F_n \leq 2\ F_n$.

What upper bound does this give?

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Want functions U(n) and L(n) such that $L(n) \leq F_n \leq U(n)$.

As $F_{n+1} = F_n + F_{n-1}$ we have $F_{n+1} \leq F_n + F_n \leq 2 F_n$.

$F_4 \leq 2 F_3 \leq 2(2 F_2) = 2^2 F_2 \leq 2^2 (2F_1) = 2^3 F_1$.

Similarly $F_5 \leq 2 F_4 \leq 2 (2^3 F_1) = 2^4 F_1$.

What upper bound does this give?

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Similarly $F_5 \leq 2 F_4 \leq 2 (2^3 F_1) = 2^4 F_1$.

What upper bound does this give? $F_n \leq 2^{n-1} F_n$. So grows at most exponentially (doubling). Lower bound?

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The Fibonacci Numbers

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Want functions U(n) and L(n) such that $L(n) \leq F_n \leq U(n)$.

As $F_{n+1} = F_n + F_{n-1}$ we have $F_{n+1} \geq F_n + F_{n-1} \geq 2\, F_{n-1}$ (note index decreases by 2).

Thus $F_8 \geq 2\, F_6 \geq 2\,(2F_4) = 2^2\, F_4 \geq 2^2\,(2\, F_2) = 2^3\, F_2$.

Then $F_{10} \geq 2\, F_8 \geq 2(2^3\, F_2) = 2^4\, F_2$, and similarly $F_{12} \geq 2^5\, F_2$.

So basically get F_n is at least $2^{(n-1)/2} = 2^{n/2}\, 2^{-1/2} = 2^{-1/2}\,(2^{1/2})^n = 2^{-1/2}\,(\sqrt{2}\,)^n$

The Fibonacci Numbers

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Want functions U(n) and L(n) such that $L(n) \leq F_n \leq U(n)$.

Basically have Constant $* (\sqrt{2})^n \leq F_n \leq 2^n$.

So maybe there is some constant r such that F_n grows like r^n? We can try that….

Note such an r must be between **???** and **???**.

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Note such an r must be between $\sqrt{2}$ and 2.

The Fibonacci Numbers: Finding a Formula

F_n is the n^{th} Fibonacci number, $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

Thus the sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ….

Try $F_n = r^n$ and substitute in.

Thus $F_{n+1} = r^{n+1}$, $F_n = r^n$, $F_{n-1} = r^{n-1}$.

Substituting: $r^{n+1} = r^n + r^{n-1}$.

Algebra: $r^{n+1} - r^n - r^{n-1} = 0$, so $r^{n-1}(r^2 - r - 1) = 0$.

We know r=0 is absurd, so r must satisfy $r^2 - r - 1 = 0$; this is a quadratic!

What can we use to solve it?

The Fibonacci Numbers: Finding a Formula

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Quadratic Formula: If $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$, the solutions to $f(x) = 0$ or $y = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For us a = **???**, b = **???**, c = **???**.

The Fibonacci Numbers: Finding a Formula

F_n is the n^{th} Fibonacci number, $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.
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STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

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$f(x) = 0$ or $y = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For us a = **1**, b = **-1**, c = **-1**. **Thus roots are…**

$$\boldsymbol{r = \frac{1 \pm \sqrt{(-1)^2 - 4 * 1 * (-1)}}{2} = \frac{1 + \sqrt{5}}{2} \ or \ \frac{1 - \sqrt{5}}{2}}$$

The Fibonacci Numbers: Finding a Formula

F_n is the n^{th} Fibonacci number, $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

Thus the sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ….

Try $F_n = r^n$ and substitute, r must satisfy $r^2 - r - 1 = 0$.

Found $\boldsymbol{r_1 = \frac{1+\sqrt{5}}{2}}$ ***and*** $\boldsymbol{r_2 = \frac{1-\sqrt{5}}{2}}$.

Advanced: if r^n works, so does $a * r^n$ for any constant a.

Also if r_1^n and r_2^n work, so too do $a_1 r_1^n + a_2 r_2^n$ for any constants a_1, a_2.

Thus the question is: can we find constants a_1, a_2 such that $F_n = a_1 r_1^n + a_2 r_2^n$?

How would we do this?

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The Fibonacci Numbers: Finding a Formula

F_n is the n^{th} Fibonacci number, $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$.

Thus the sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ….

Try $F_n = r^n$ and substitute, r must satisfy $r^2 – r – 1 = 0$.

Found $\boldsymbol{r_1 = \frac{1+\sqrt{5}}{2}}$ ***and*** $\boldsymbol{r_2 = \frac{1-\sqrt{5}}{2}}$.

Want $F_n = a_1 r_1^n + a_2 r_2^n$, know $F_0 = 0$ and $F_1 = 1$, two equations in two unknowns!

Solve: $0 = a_1 + a_2$ and $1 = a_1 \frac{1+\sqrt{5}}{2} + a_2 \frac{1-\sqrt{5}}{2}$. After algebra (see next page)

get $a_1 = -a_2 = -1/\sqrt{5}$

Yields Binet's Formula: $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$. Amazing!

Solving system of Equations

Details to solve: $0 = a_1 + a_2$ and $1 = a_1 \frac{1+\sqrt{5}}{2} + a_2 \frac{1-\sqrt{5}}{2}$

From the first we get $a_2 = -a_1$, substitute that into the second:

$$1 = a_1 \frac{1+\sqrt{5}}{2} - a_1 \frac{1-\sqrt{5}}{2} = a_1 \frac{1}{2} + a_1 \frac{\sqrt{5}}{2} - a_1 \frac{1}{2} + a_1 \frac{\sqrt{5}}{2} = a_1 \frac{2\sqrt{5}}{2}.$$

Thus $a_1 \sqrt{5} = 1$ so $a_1 = 1 / \sqrt{5}$.

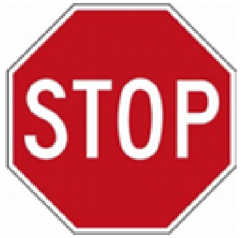
Solved this by the substitution method: used the first equation to replace one unknown (a_2) with an expression involving just a_1. Could also subtract first from second.

The Fibonacci Numbers and the Quadratic Formula

Using the Quadratic Formula we found

Binet's Formula: $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$

Absolutely amazing – it allows us to jump to any Fibonacci number without computing earlier ones! How hard is it to compute $\left(\frac{1+\sqrt{5}}{2}\right)^n$? How do you do that?



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Absolutely amazing – it allows us to jump to any Fibonacci number without computing earlier ones! How hard is it to compute $\left(\frac{1+\sqrt{5}}{2}\right)^n$? How do you do that?

Use Pascal's Triangle to expand $(1 + \sqrt{5})^n$; see From Pascal to Calculus: Part I: https://youtu.be/dv15VTyEWyQ (powerpoint here, pdf here) (3/25/2020): For those knowing Algebra I (equations of lines): 52 minutes. Note even powers of $\sqrt{5}$ are integers, odd powers are integers times $\sqrt{5}$.

The Fibonacci Numbers and the Quadratic Formula

Using the Quadratic Formula we found

Binet's Formula: $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$

Absolutely amazing – it allows us to jump to any Fibonacci number without computing earlier ones!

It gives us the growth rate! $\frac{1+\sqrt{5}}{2}$ is about 1.618, $\frac{1-\sqrt{5}}{2}$ is about -.618.

As $\frac{1+\sqrt{5}}{2}$ is greater than 1 in absolute value and $\frac{1-\sqrt{5}}{2}$ is less than 1, what happens as n gets large?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Fibonacci Numbers and the Quadratic Formula

Using the Quadratic Formula we found

Binet's Formula: $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$

Absolutely amazing – it allows us to jump to any Fibonacci number without computing earlier ones! It gives us the growth rate!

As $\frac{1+\sqrt{5}}{2}$ is greater than 1 in absolute value and $\frac{1-\sqrt{5}}{2}$ is less than 1, what happens as n gets large? The second term in Binet's formula doesn't matter much, the n^{th} Fibonacci number is the closest integer to $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$.

We will use this in the next lecture to solve gambling problems.

Computing Fibonacci Numbers

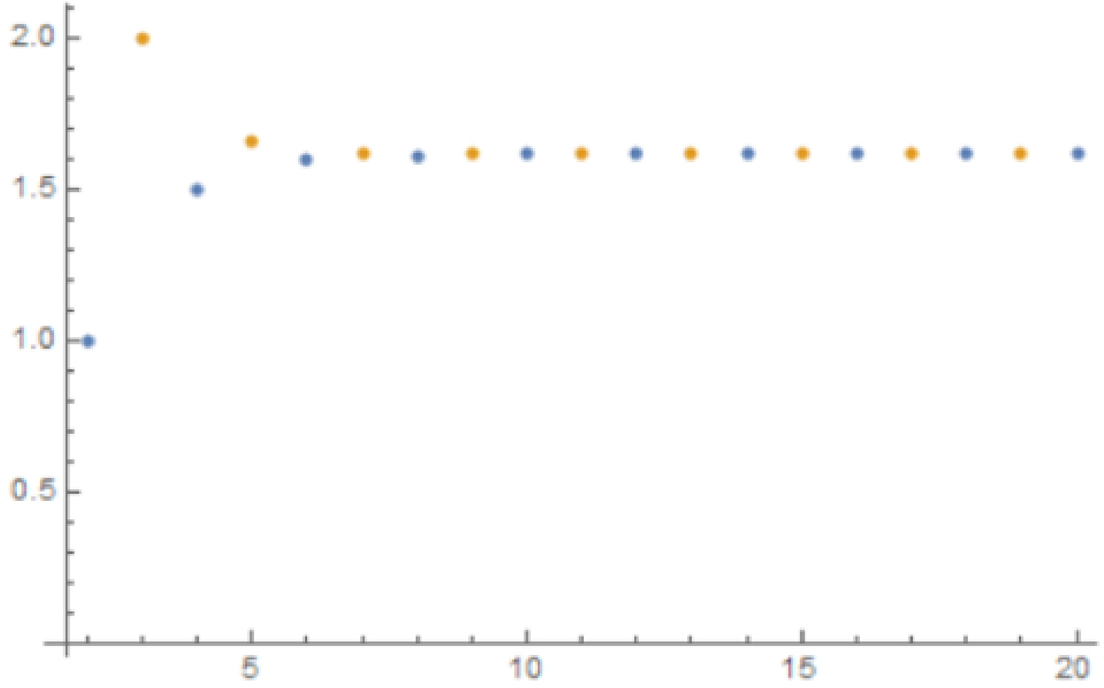
F_{100} = 354224848179261915075

F_{1000} =
434665576869374564356885276750406258025646605173717804024817290895365554179490518904038798400792551692959225930803226347752096896232398733224711616429964409065331879382989696499285160037044761377951668492288 75

F_{10000} =
3364476487643178326662161200510754331030214846068006390656476997468008144216666236815559551363373402558206533268083615937373479048386526826304089246305643188735454436955982749160660209988418393386465273130008883026923567361313511757929743785441375213052050434770160226475831890652789085515436615958298727968298751063120057542878345321551510387081829896979161312785626503319548714021428753269818796204693609787990035096230229102636813149319527563022783762844154036058440257211433496118002309120828704608892396232883546150577658327125254609359112820392528539343462090424524892940390170623388899108584106518317336043747073790855263176432573399371287193758774689747992630583706574283016163740896917842637862421283525811282051637029808933209990570792006436742620238978311147005407499845925036063356093388383192338678305613643535189213327973290813373264265263398976392272340788292817795358057099369104917547080893184105614632233821746563732124822638309210329770164805472624384237486241145309381220656491403275108664339451751216152654536133311131404243685480510676584349352383695965342807176877532834823434555736671973139274627362910821067928078471803532913117677892465908993863545932789452377767440619224033763867400402133034329749690202832814593341882681768389307200363479562311710310129195316979460763273758925353077255237594378843450406771555577905645044301664011946258097221672975861502696844314695203461493229110597067624326851599283470989128470674086200858713501626031207190317208609408129832158107728207635318662461127824553720853236530577595643007251774431505153960090516860322034916322264088524885243315805153484962243484829938090507048348244932745373262456775587908918719080366205800959474315005240253270974699531877072437682590741993963226598414749819360928522394503970716544315642132815768890805878318340491743455627052022356484649519611246026831397097506938264870661326450766507461151267752274862159864253071129844118262266105716351506926002986170494542504749137811515413994155067125627119713325276363193960690289565028826860836224108205056243070179497617112123306607331005994736687 5

Ratio of Fibonacci Numbers and the Golden Mean

n	F(n)	F(n+1)/F(n)	Golden Mean
1	1		(1+sqrt(5)) / 2
2	2	2.000000000	1.618033989
3	3	1.500000000	
4	5	1.666666667	(1-sqrt(5))/1
5	8	1.600000000	-0.618033989
6	13	1.625000000	
7	21	1.615384615	
8	34	1.619047619	
9	55	1.617647059	
10	89	1.618181818	
11	144	1.617977528	
12	233	1.618055556	
13	377	1.618025751	
14	610	1.618037135	
15	987	1.618032787	
16	1597	1.618034448	
17	2584	1.618033813	
18	4181	1.618034056	
19	6765	1.618033963	
20	10946	1.618033999	



Ratios of Fibonacci Numbers and the Golden Mean

Why does F_{n+1}/F_n approach the Golden Mean?

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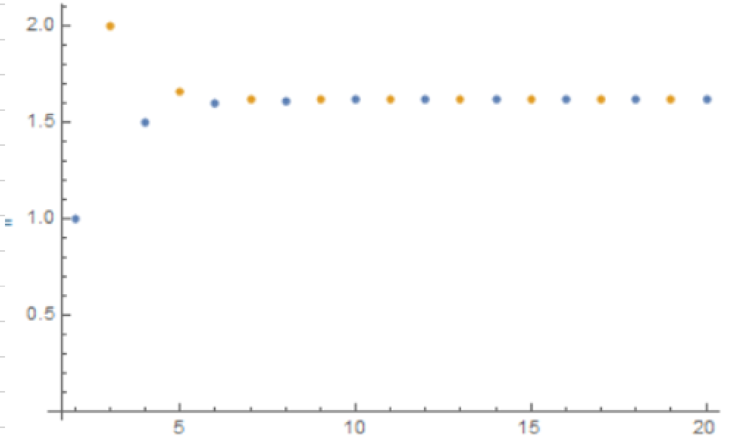

Binet's Formula:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n .$$

The Golden mean $\frac{1+\sqrt{5}}{2}$ is about 1.618, while the other factor is about -.618.

For n large, F_n is approximately $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$, F_{n+1} is approximately $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}$, so the ratio is approximately $\frac{1+\sqrt{5}}{2}$. Can you tell why it alternates above/below?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Ratios of Fibonacci Numbers and the Golden Mean

Why does F_{n+1}/F_n approach the Golden Mean?

Binet's Formula:

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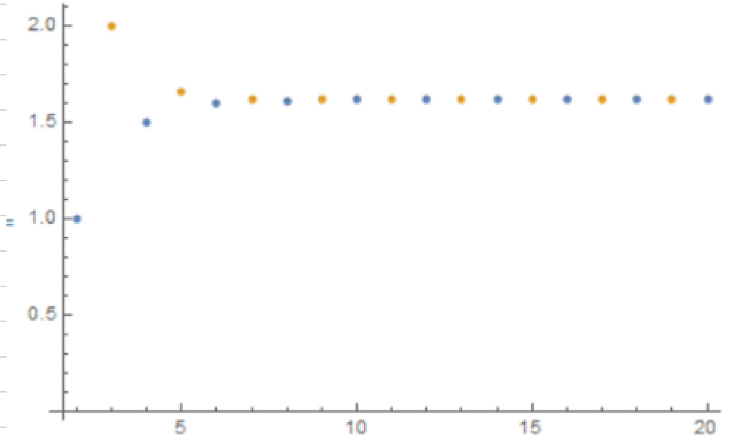

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so the ratio is approximately $\frac{1+\sqrt{5}}{2}$. Can you tell why it alternates above/below?

The term tending to zero is negative, so alternates b/w being positive/negative.

Part IV: Applications of Quadratic and Higher: Recurrences and Roulette

Watch this video: https://youtu.be/Esa2TYwDmwA (7 minutes)

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

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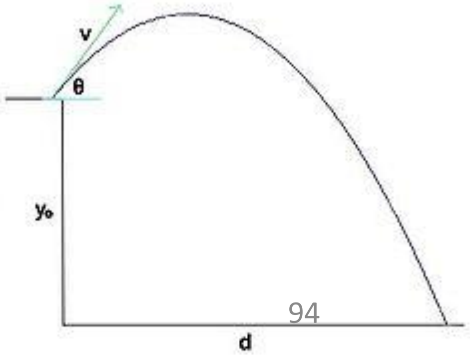
Part V: Applications of Quadratic Equations: Trajectories



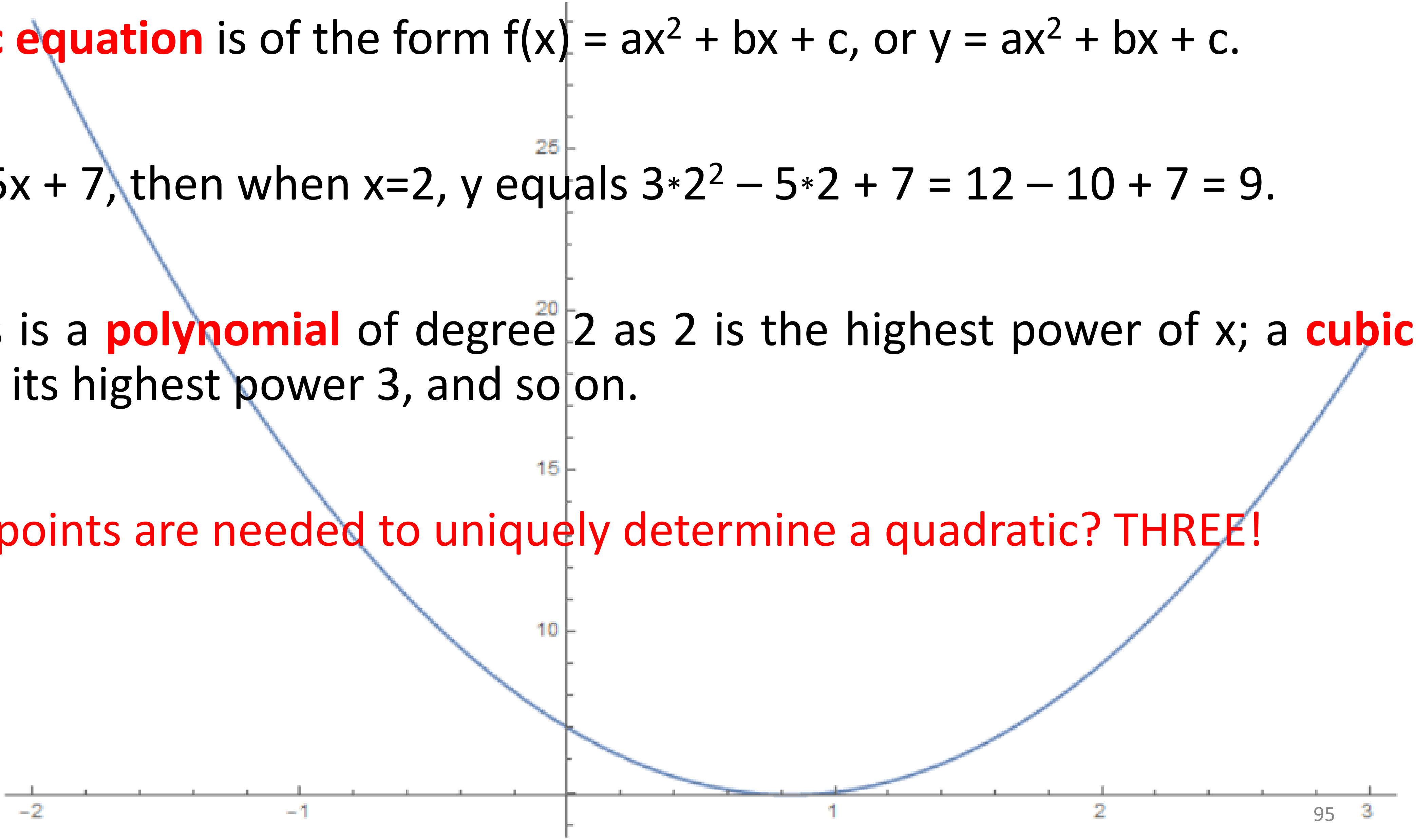
Review: Quadratic Equations

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

If $y = 3x^2 - 5x + 7$, then when $x=2$, y equals $3*2^2 - 5*2 + 7 = 12 - 10 + 7 = 9$.

We say this is a **polynomial** of degree 2 as 2 is the highest power of x; a **cubic** would have its highest power 3, and so on.

How many points are needed to uniquely determine a quadratic? THREE!



Review: Quadratic Equations

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Imagine know (0, 7), (1, 5) and (2,9) are on the quadratic. Then

- $7 = a * 0^2 + b * 0 + c$, so $7 = c$.
- $5 = a * 1^2 + b * 1 + c$, so $5 = a + b + c$, but since $c=7$ we get $-2 = a+b$.
- $9 = a * 2^2 + b * 2 + c$, so $9 = 4a + 2b + c$, but since $c=7$ we get $2 = 4a+2b$.

We have two equations in two unknowns: we can solve! If we subtract two copies of the second from the first, the b-terms vanish:

$2 = 4a + 2b$

$-4 = 2a + 2b$

So $6 = 2a$ or $a = 3$; since $-2 = a + b$ we find $b = -2-a = -5$.

We could also have used $-2 = a+b$ to say $b = -2-a$, and substitute that into $2 = 4a+2b$, which would give $2 = 4a + 2(-2-a)$, or $2 = 4a – 4 – 2a$. Thus $6 = 2a$ and again $a = 3$.

Application: Finding the Cannon!

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

Imagine we are being attacked by a cannon. We don't know where it is, but we observe the trajectory as it nears us. Say we observe 10 points on the curve as the cannon ball approaches us.

If we assume it is a perfect parabola, can we determine where the cannon is? If yes, how?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Application: Finding the Cannon!

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If we assume it is a perfect parabola, can we determine where the cannon is? If yes, how?

- Knowing three points on the parabola, can find a, b and c. Use method of last page, system of three equations and three unknowns.
- Find where the polynomial is zero (Quadratic Formula), that gives the launch spot!

Application: Trajectories are Parabolas

A **quadratic equation** is of the form f(x) = $ax^2 + bx + c$, or $y = ax^2 + bx + c$.

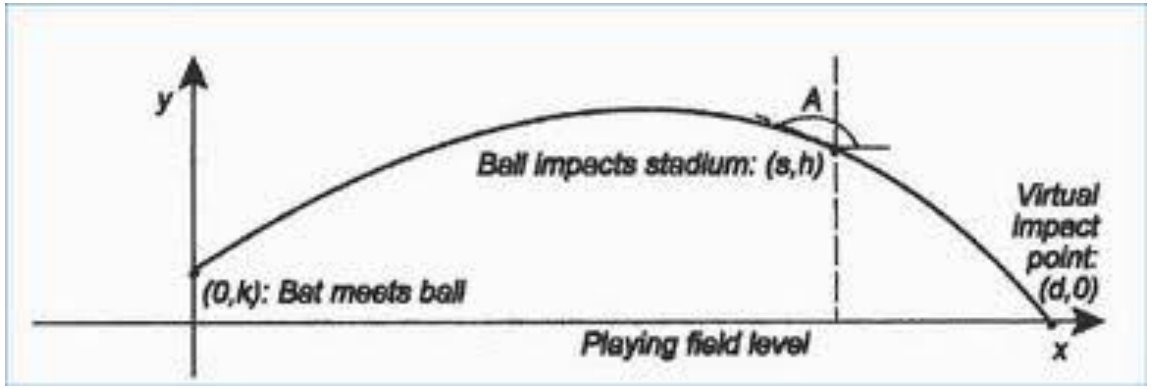
We assumed the cannon ball's path (no air resistance) is a parabola. Is it?

Consider a batted baseball. https://calculushowto.com/how-to-describe-the-path-of-a-baseball-in-calculus/

Path of a baseball: Steps

Step 1: Define the variables used in both the parametric equations.

- Represent the height in feet by 'h',
- The angle in degrees by 'a',
- The initial velocity in feet per second by 'v'
- The time in seconds by 't'.



Step 2: Write an equation for the horizontal motion of the baseball as a function of time:

- $x(t) = v * \text{Cos}(a) * t$.

Step 3: Write an equation to describe the vertical motion of the baseball as a function of time:

- $y(t) = h + v * \text{Sin}(a) * t - 16 * t^2$.

Two steps. We figure out how long it is airborn until the y-coordinate is zero, and that gives the time to plug in to the formula for x. What do you think is the best angle?

Application: Trajectories are Parabolas

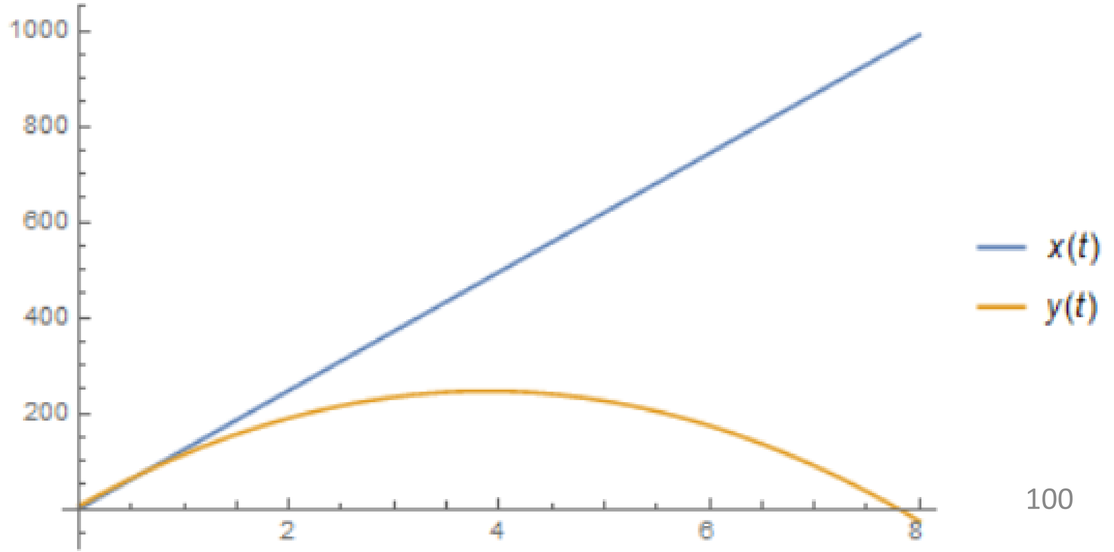
A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

Consider a batted baseball. Use 6 ft above ground, 176 ft/sec, 45 degree angle.

Vertical as a function of time: $y(t) = 6 + 124t - 16t^2$.

Horizontal as a function of time: $x(t) = 124t$.

```
x[t_] := 124 t
y[t_] := 6 + 124 t - 16 t^2
Plot[{x[t], y[t]}, {t, 0, 8}, PlotLegends → "Expressions"]
```



We have x and y as functions of time. This gives us a linear function for x, and a quadratic (parabola) for y as a function of time.

How do we find y as a function of x?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Application: Trajectories are Parabolas

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

Consider a batted baseball. Use 6 ft above ground, 176 ft/sec, 45 degree angle.

Vertical as a function of time: $y(t) = 6 + 124t - 16t^2$.

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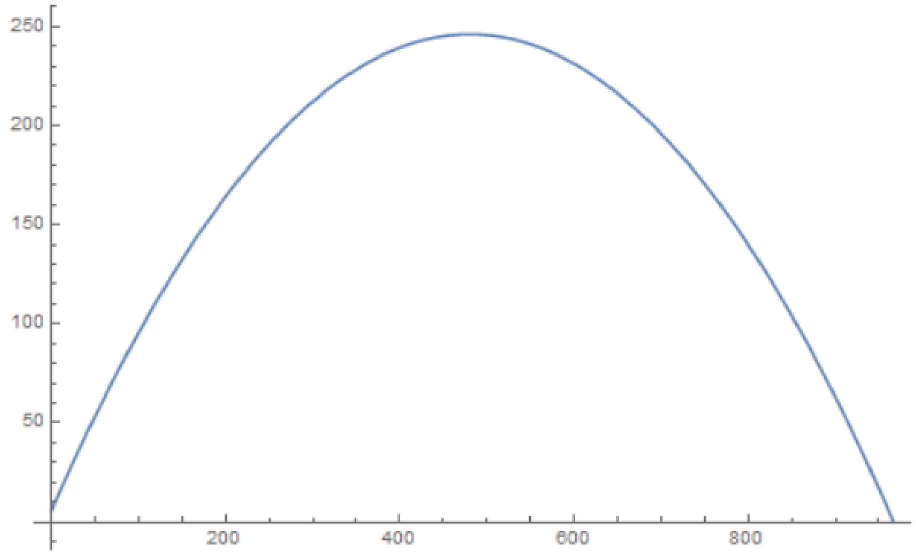


We can write t as a function of x, and then use that to write y as a function of x!

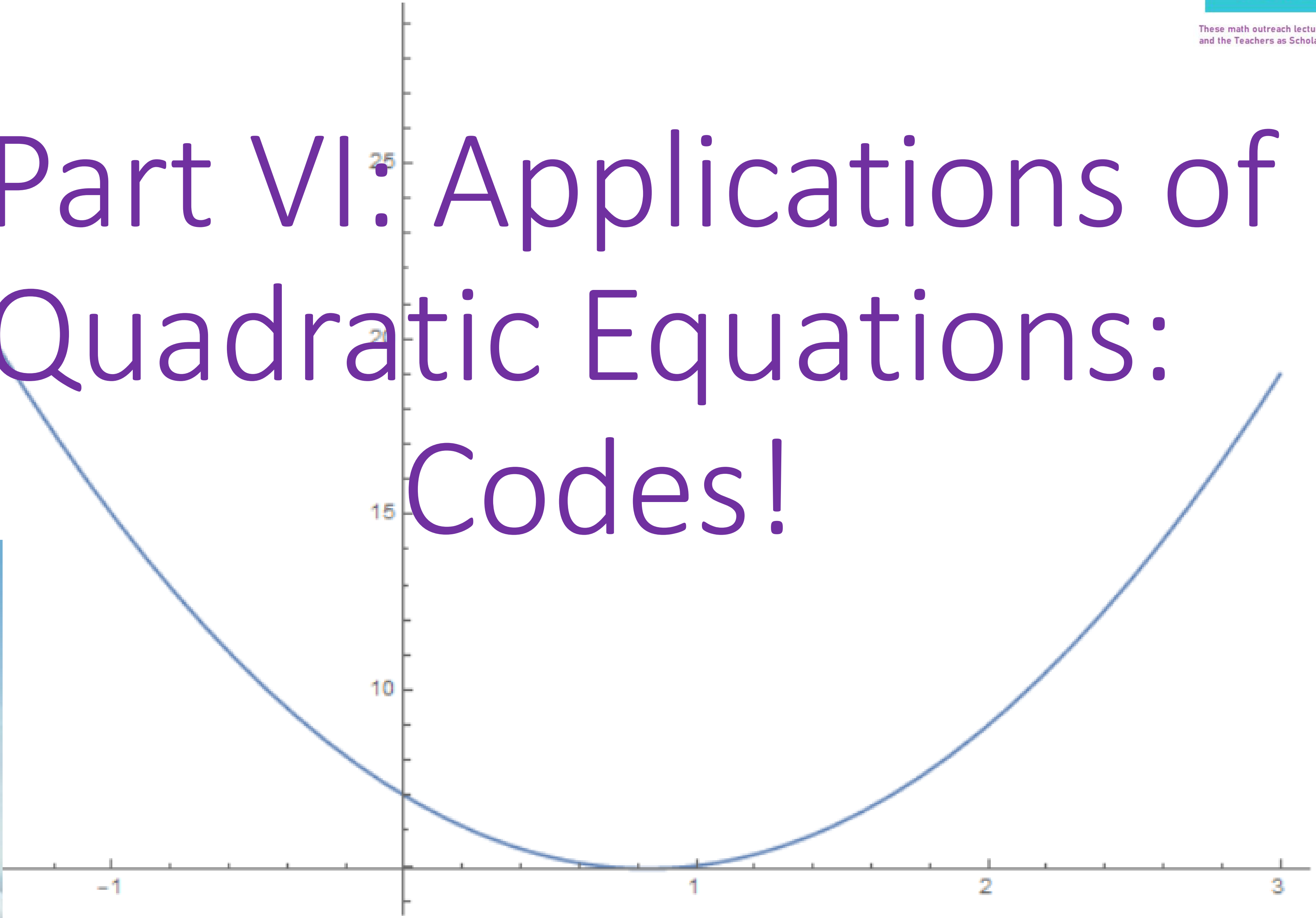
If $x = 124t$ then $t = x/124$. Thus $y = 6 + 124\,\frac{x}{124} - 16\left(\frac{x}{124}\right)^2$, so we find

$y = 6 + x - \frac{1}{961}x^2$, and thus we do see it is a parabola!

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

JOURNAL OF NUMBER THEORY
Teachers As Scholars henry.bolter@gmail.com
These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

Part VI: Applications of Quadratic Equations: Codes!



Application: Codes

This problem is posted on my riddles page: https://mathriddles.williams.edu/

If you are interested in using the student/teacher corner, email me at sjm1@williams.edu for details on how to get the password.

Consider an army with 10 generals. One wants a security system such that any three of them can determine the code to launch nuclear missiles, but no two of them can. It is possible to devise such a system by using a quadratic polynomial, such as $ax^2 + bx + c$; to launch the missiles, one must input (a,b,c).

- **One cannot just tell each general one of a, b, or c (as then it is possible that some subset of three generals won't know a, b and c).**
- **However, if a general knows two of (a,b,c), then a set of two generals can launch the missiles!**

What information should be given to the generals so that any three can find (a,b,c) but no two can? What about the general situation with N generals and any M can launch (but no set of M-1) can?

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Application: Codes

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- **One cannot just tell each general one of a, b, or c (as then it is possible that some subset of three generals won't know a, b and c).**
- **However, if a general knows two of (a,b,c), then a set of two generals can launch the missiles!**

First Thoughts

This seems like a difficult problem to tackle head on, so let's first consider a simpler case. Let's think of a way to create a similar missile system for the case where there are 3 generals and any 2 can launch the missiles.

- A first idea is to use a linear polynomial, $ax + b$. If we give each of our three generals one of the numbers a or b, we run into the same problem as in the original setup: a situation may arise where some subset of two generals can't launch the missiles.
- Similarly, if we give everyone a and b, then each general can launch the missile.

This dilemma suggests that we think of another way to determine the equation of a line. Fortunately, we soon realize that given two points on a line, we can completely solve for the equation of that line. Therefore, if we give each of the generals a point on the line, any subset of two generals can determine the line.

Finding the equation of a line

Imagine we have the line $y = ax + b$ and we have two points: (1,1) and (2,3).

Thus

$3 = a * 2 + b$ or $3 = 2a + b$

$1 = a * 1 + b$ or $1 = a + b$

We have two equations in two unknowns. If we subtract the second from the first we get $2 = a$, and then plugging that in to $1 = a + b$ gives $1 + 2 = b$, or $b = -1$.

Or we could use $1 = a + b$ to get $b = 1-a$, substitute that into the first equation and find $3 = 2a + (1-a)$, or $3 = a + 1$, thus $a = 2$ as before.

Application: Codes

Consider an army with 10 generals. One wants a security system such that any three of them can determine the code to launch nuclear missiles, but no two of them can. It is possible to devise such a system by using a quadratic polynomial, such as $ax^2 + bx + c$; to launch the missiles, one must input (a,b,c).

- **One cannot just tell each general one of a, b, or c (as then it is possible that some subset of three generals won't know a, b and c).**
- **However, if a general knows two of (a,b,c), then a set of two generals can launch the missiles!**

Second Thoughts

We now wish to apply the reasoning to our original problem. Perhaps we can give each general a point and have the same result; any collection of three generals can solve for the equation of the parabola. Let's explore this option. Take any subset of three generals with points (m, n), (r, s) and (u, v). Then we get the following system of equations: The only unknowns are a, b and c.

$am^2 + bm + c = n$

$ar^2 + br + c = s$

$au^2 + bu + c = v$

We observe that this is a system of three equations in three unknowns, (a, b, c), since all of the other parameters are already known by the generals (that is, we know m, n, r, s, u, v). We don't have to worry about there being no solution to this system because we have chosen three points on the parabola we are given, so we know there is at least one parabola through the three points. We must therefore be able to solve this system of equations for a unique (a, b, c).

Quadratic Equations: Finding coefficients from Points

A **quadratic equation** is of the form $f(x) = ax^2 + bx + c$, or $y = ax^2 + bx + c$.

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What information should be given to the generals so that any three can find (a,b,c) but no two can? What about the general situation with N generals and any M can launch (but no set of M-1) can?

The General Solution:

Following a similar line of reasoning, we see that in the general case with N generals where any M can launch the missiles, we need to give each of the generals a unique point on a degree $M-1$ curve. Also "generalizations" where some get more points….