COMPUTATIONAL THINKING MODULES:

FROM DATA TO RESULTS (THROUGH CHOCOLATE!)

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http://www.williams.edu/Mathematics/sjmiller/public_html

STANYS (Science Teachers of New York State) Conference: November 7, 2021

Objectives: Discuss computational units.

- Gather data and conjecture.
- Test hypotheses.
- Prove claims.
- Have fun!

DIMACS

The Value of Computational Thinking across Grade Levels 9-12 (VCTAL)

What is Computational Thinking?

Computational thinking is a high level thought process that considers the world in computational terms. It begins with learning to see opportunities to compute something, and it develops to include such considerations as computational efficiency, selecting appropriate ways to represent data, and making approximations. A person skilled in computational thinking is able to harness the power of computing to gain insights. Computational thinking is not programming. It is a thought process that can be brought to bear not only in programming but also in a wide array of other contexts. It relates to mathematical thinking in its use of abstraction, decomposition, measurement and modeling, but is more directly cognizant of the need to compute and the potential benefits of doing so.

The International Society for Technology in Education (ISTE) and the Computer Science Teachers Association (CSTA) put forth an operational definition of computational thinking that captures much of our thinking in VCTAL. The ISTE/CSTA definition for Computational Thinking is quoted below[1].

Leadership:

Midge Cozzens (PI), *DIMACS*
Rebecca Wright (co-PI), *DIMACS*
Tamra Carpenter, *DIMACS*
Sol Garfunkel, *COMAP*
Paul Kehle, *Hobart and William Smith Colleges*

Activities & Events

Writers Meetings – held roughly twice each year to bring module writers together so that writing partners have the opportunity to work on their modules and so that the entire group can discuss common elements and provide feedback to each other.

Student Prototyping Workshops – held for two weeks each summer so that materials under development can be presented to and explored with students for the first time. A primary purpose of the workshops is to provide authors with early feedback based on student reactions to the module topics.

Partner Schools Meetings – held for two days each summer overlapping the Prototyping Workshop so that teachers at partner schools, who will be field-testing the completed modules, can observe the teaching of the modules.

Advisory Board Meetings – held annually to provide VCTAL leadership with overall guidance on all aspects of the project.

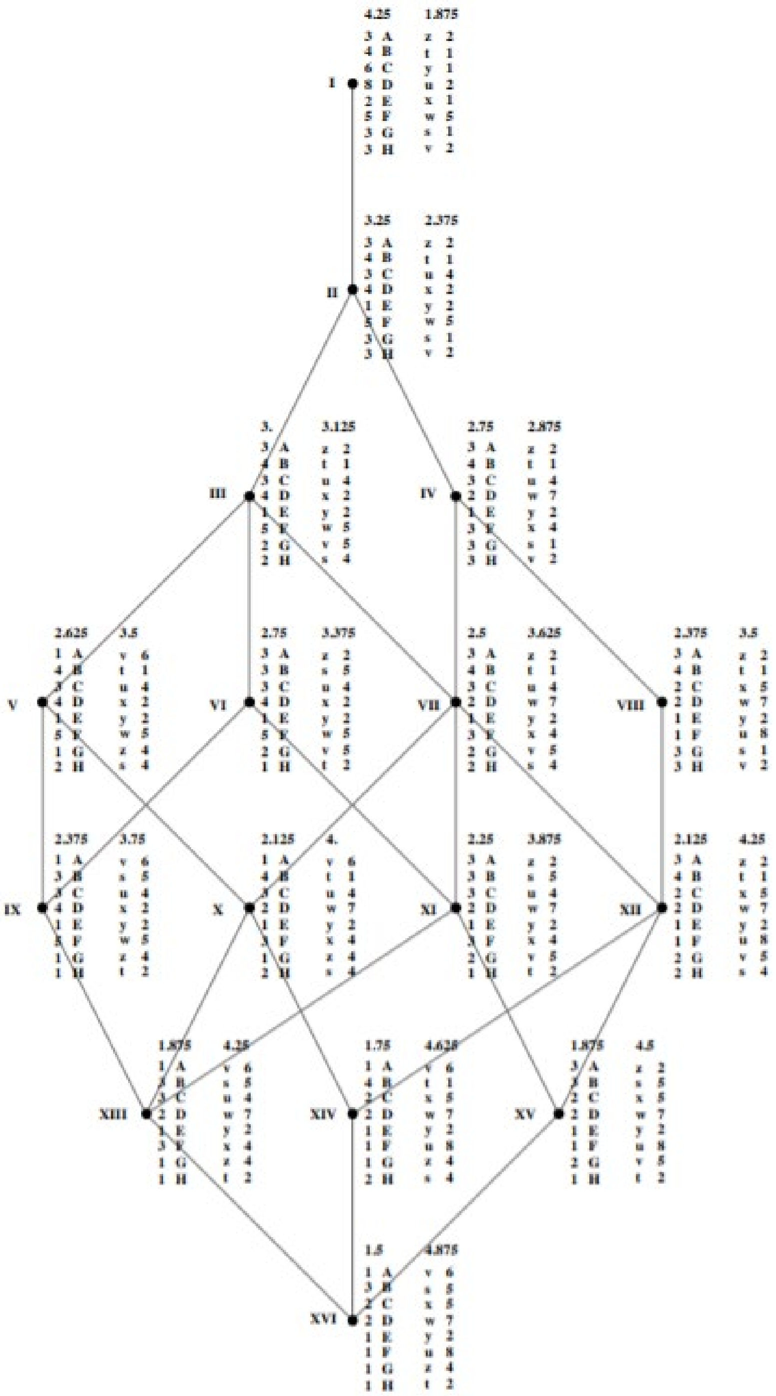
http://dimacs.rutgers.edu/archive/VCTAL/computational.html

Computational thinking (CT) is a problem-solving process that includes (but is not limited to) the following characteristics:

- Formulating problems in a way that enables us to use a computer and other tools to help solve them
- Logically organizing and analyzing data
- Representing data through abstractions such as models and simulations
- Automating solutions through algorithmic thinking (a series of ordered steps)
- Identifying, analyzing, and implementing possible solutions with the goal of achieving the most efficient and effective combination of steps and resources
- Generalizing and transferring this problem solving process to a wide variety of problems

These skills are supported and enhanced by a number of dispositions or attitudes that are essential dimensions of CT. These dispositions or attitudes include:

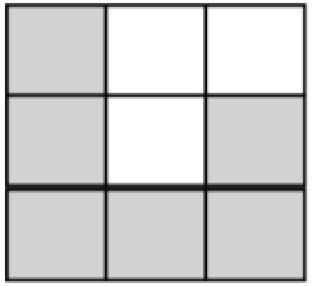
- Confidence in dealing with complexity
- Persistence in working with difficult problems
- Tolerance for ambiguity
- The ability to deal with open ended problems
- The ability to communicate and work with others to achieve a common goal or solution



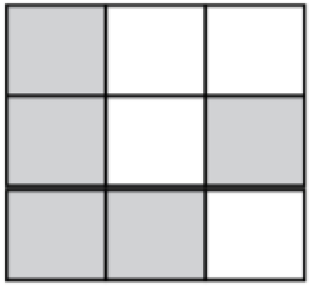
Tomography: A Geometric and Computational Approach

Tomography is the science of examining internal structure with external measurements. Most people think of tomography in the context of medical testing, such as CT scans, but tomography can be used any time it is impossible to directly look inside something. Students tackle activities in which they are challenged to determine what is inside some object. They study how CT scan images of an object are created using 3-D reconstruction of 2-D slices of the object using shadows, pin prints, graphs, and more. The main questions of the module are: How can 3-D images be created from 2-D images (i.e. slices) of it?; How much computational power and skill are required to do these reconstructions; and what do they depend on? This module is appropriate for high school classes in computer science, mathematics, biology, environmental science, and physics.

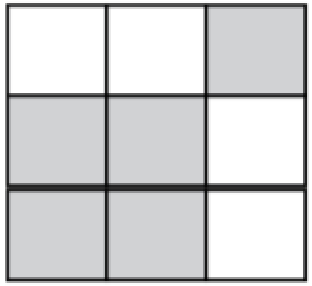
Consider the example below, which shows the top, front, and right views. Each shaded square indicates a Styrofoam ball.



top

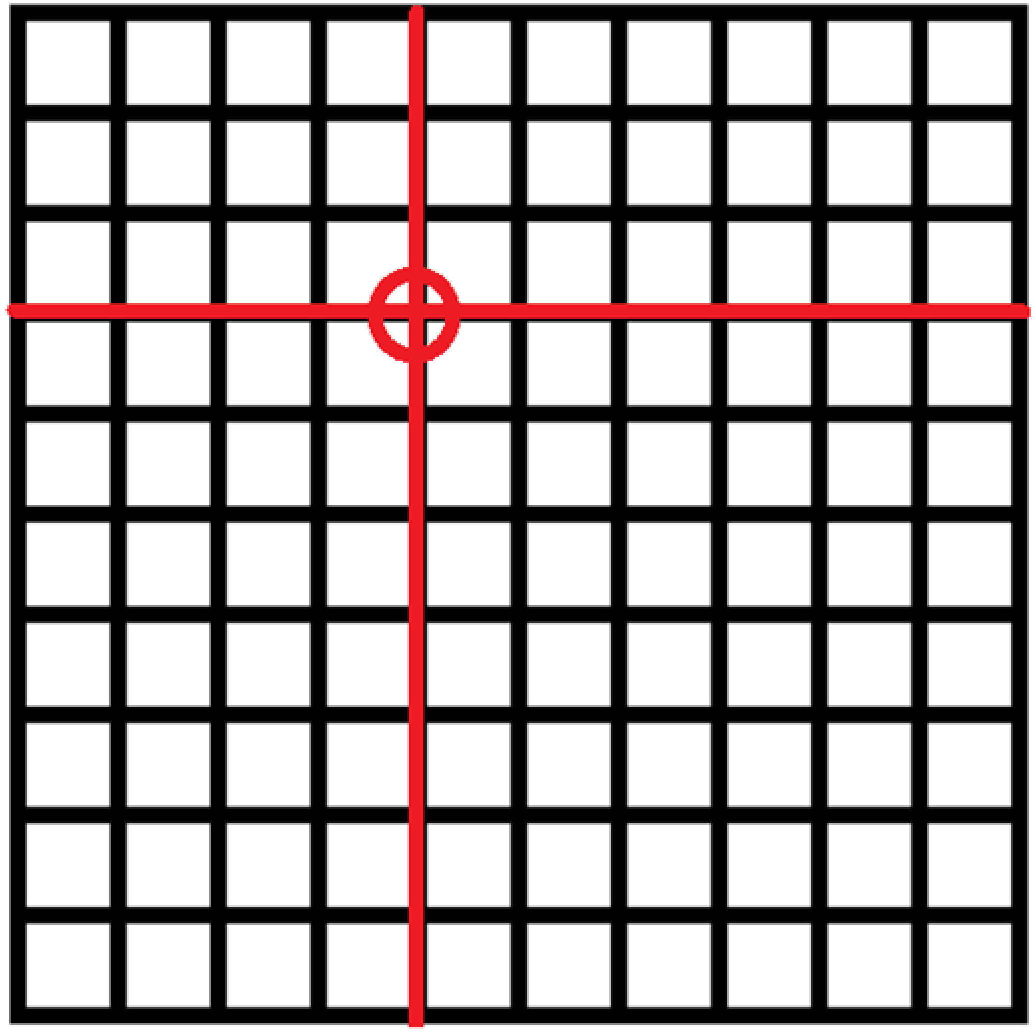


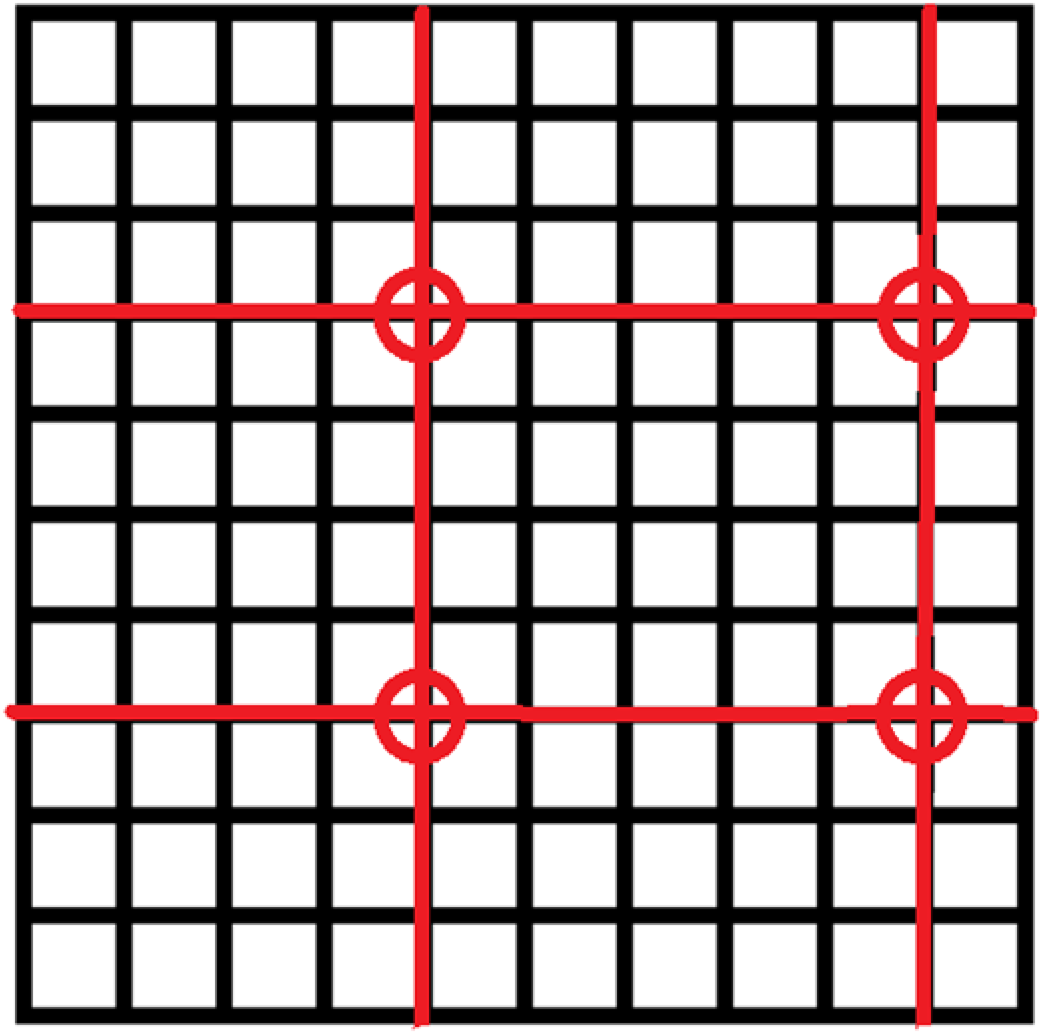
front



right

Pooled Testing: Reduces from order n to order $\sqrt{n}$ tests.





MODULES

1. It's an Electrifying Idea!
2. Heart Transplants and the NFL Draft
3. Network Capacity
4. Your Data and Your Privacy: Do you know what "they" can tell about you?
5. Tomography: A Geometric and Computational Approach
6. Lqwurgxfwlrq wr Fubswrjudskb: Introduction to Cryptography
7. Fair and Stable Matching
8. Polynomiography: Visual Displays of Solutions to Polynomial Equations
9. The Analysis of Games
10. Competition or Collusion? Game Theory in Sports, Business, and Life
11. Gently Down the Stream: The Mathematics of Streaming Information
12. Recursion - Problem Solving and Efficiency: How to Define an Infinite Set Finitely

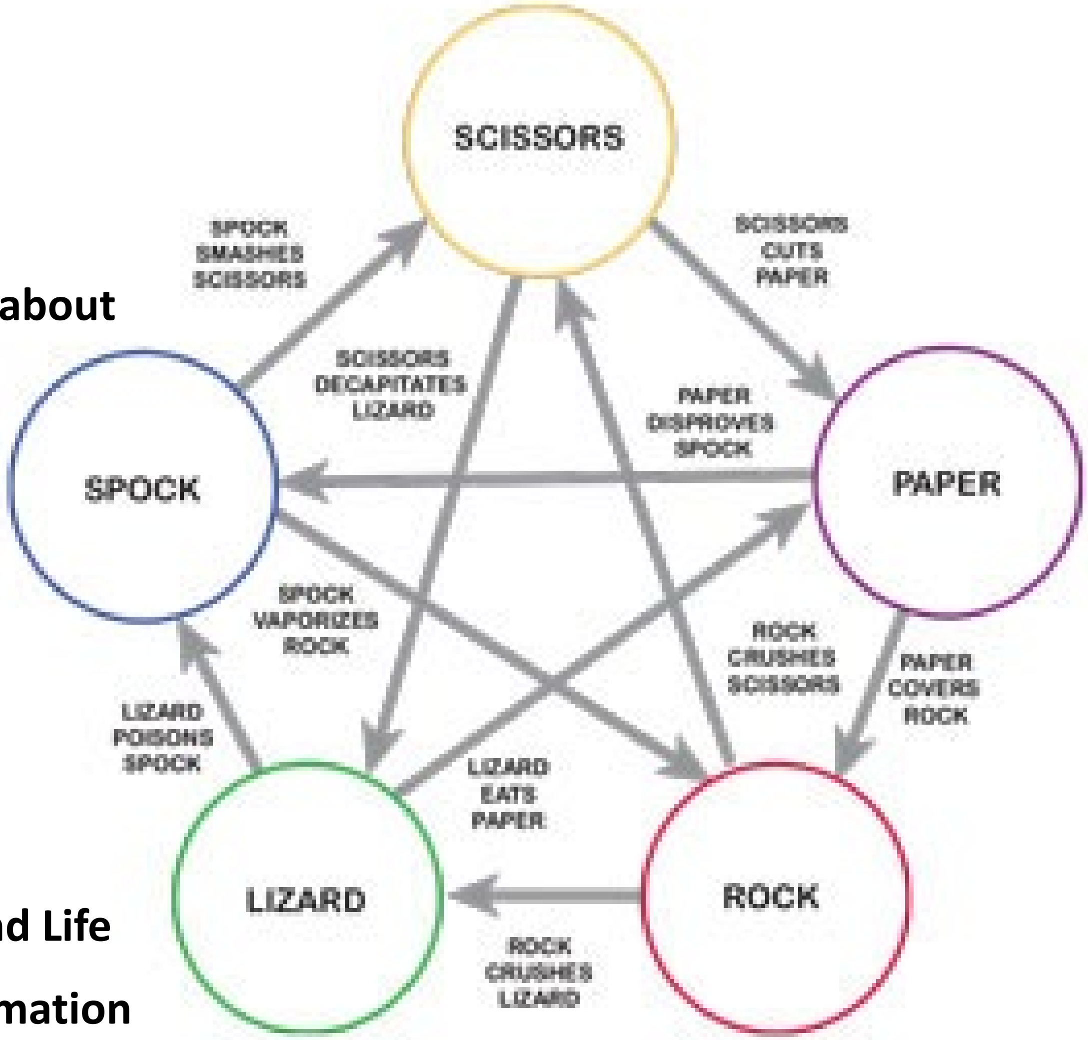


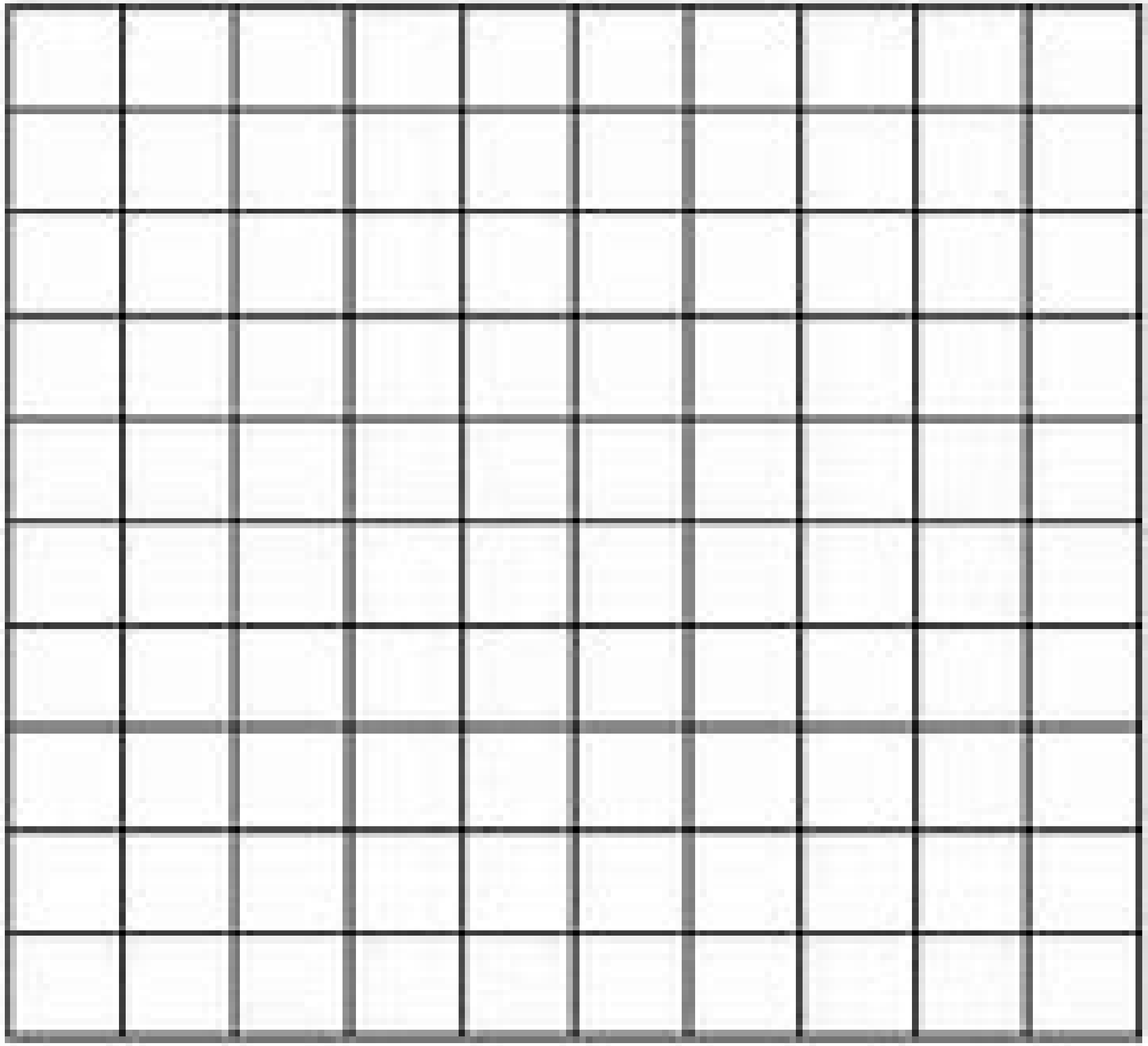
Illustration by COMAP, Inc

THE I LOVE RECTANGLES GAME

Our goal is to explore **tilings**. What is a tiling?

We have a collection of objects and we want to place them down to cover a space.

For example, imagine you want to cover the floor and the floor is a giant square, say 10 feet by 10 feet. What would be a good shape to use to cover it? We want the shape to be smaller that the floor, and we want all the pieces to fit together with no gaps.



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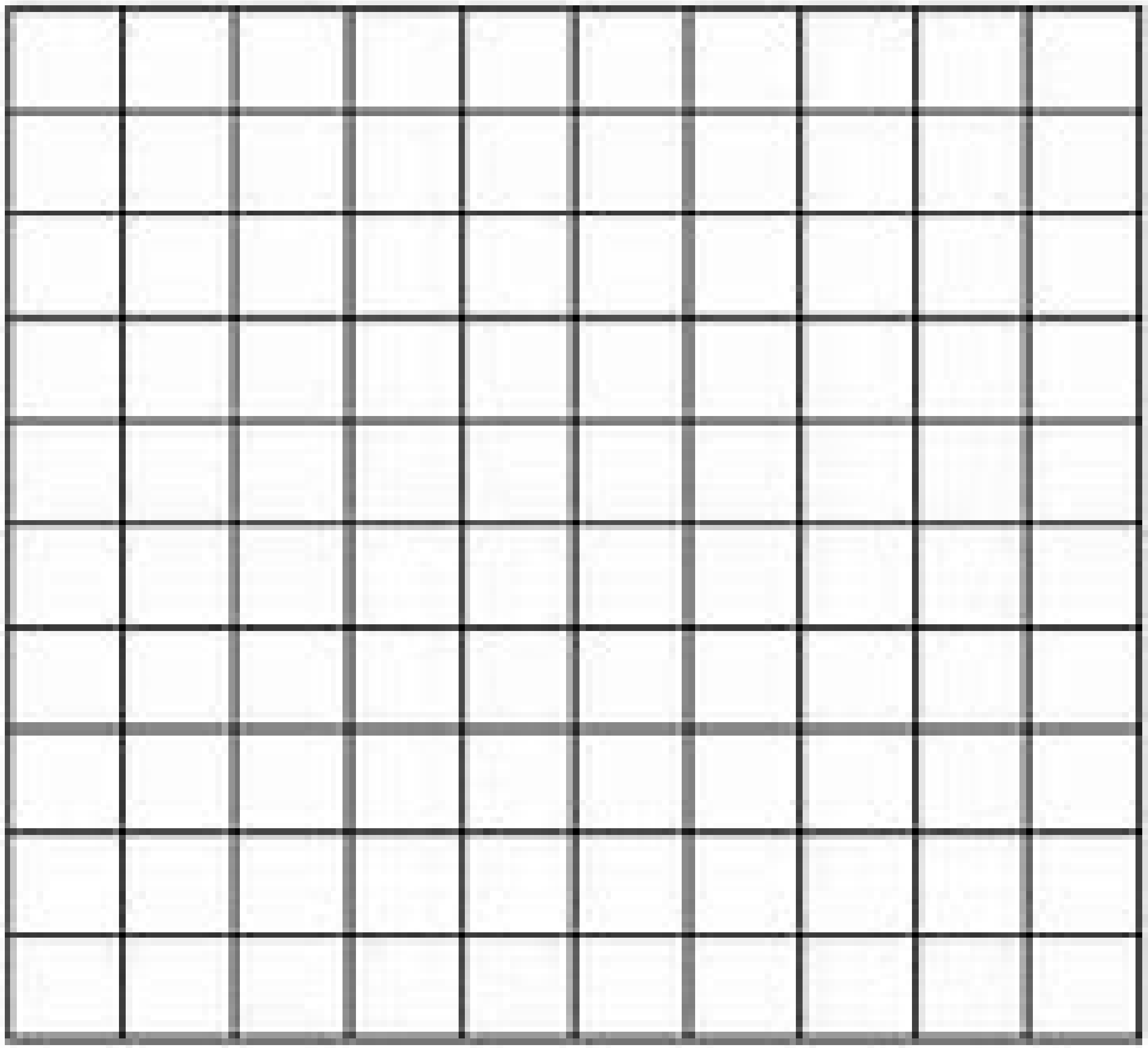

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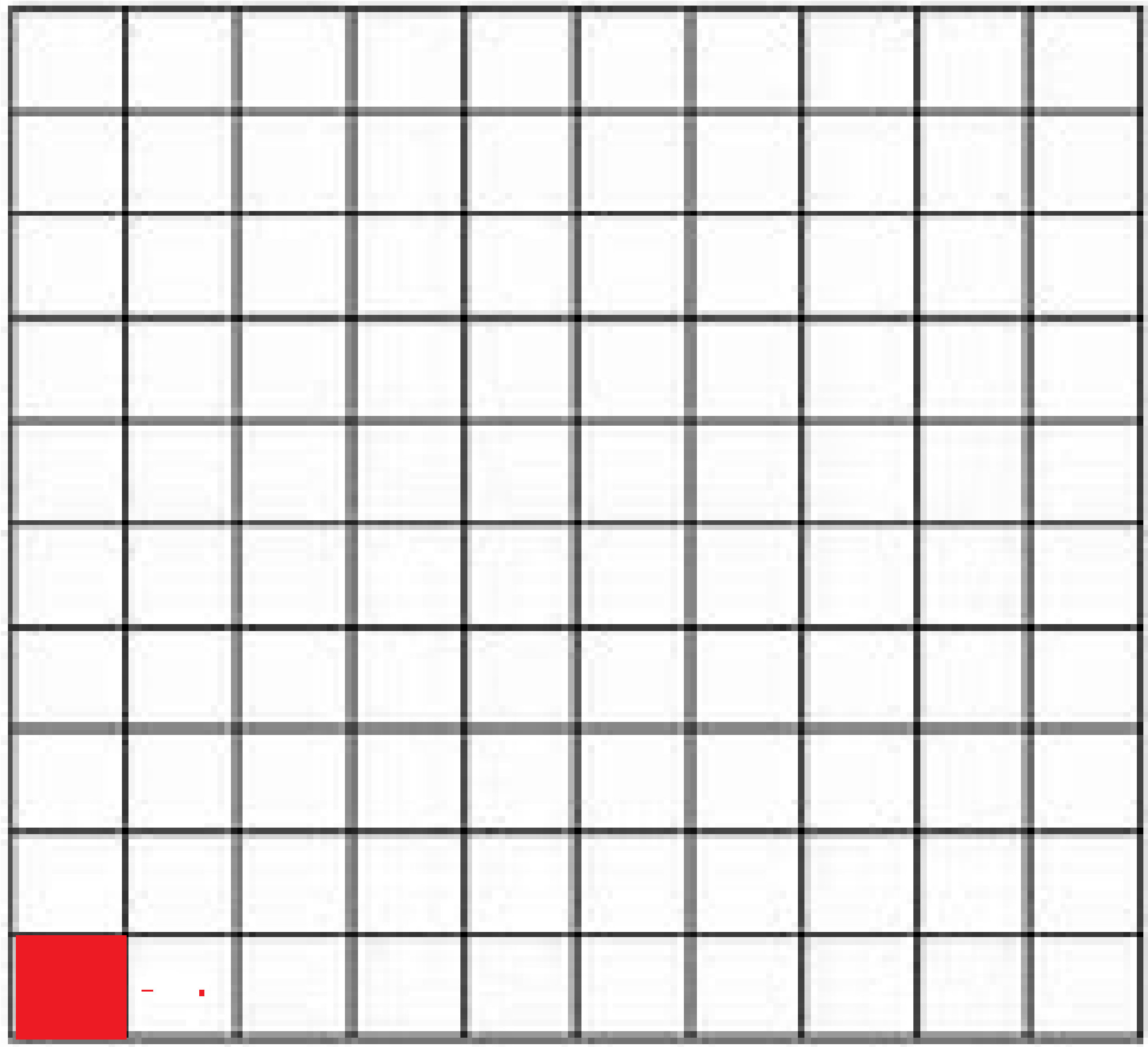

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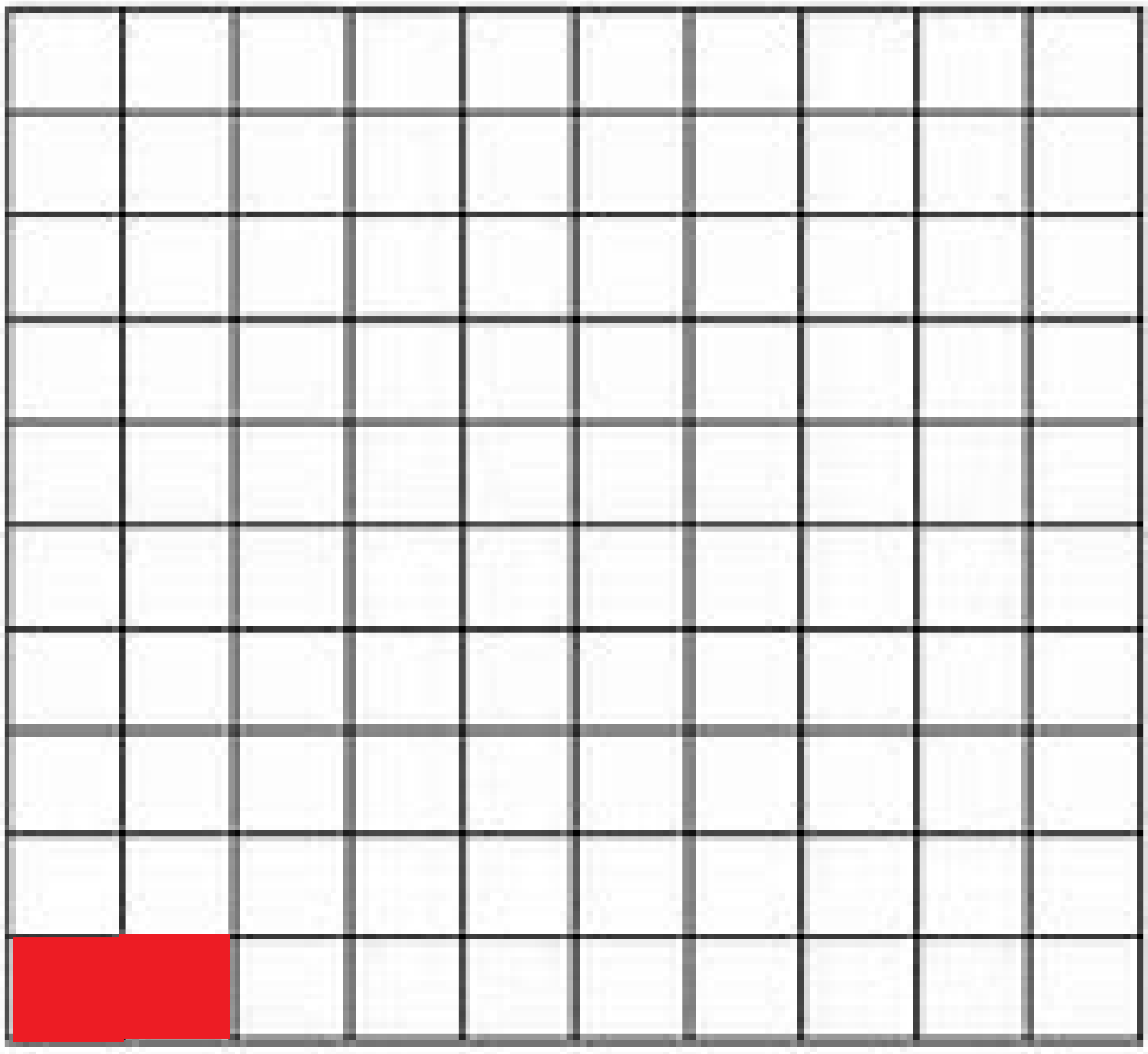

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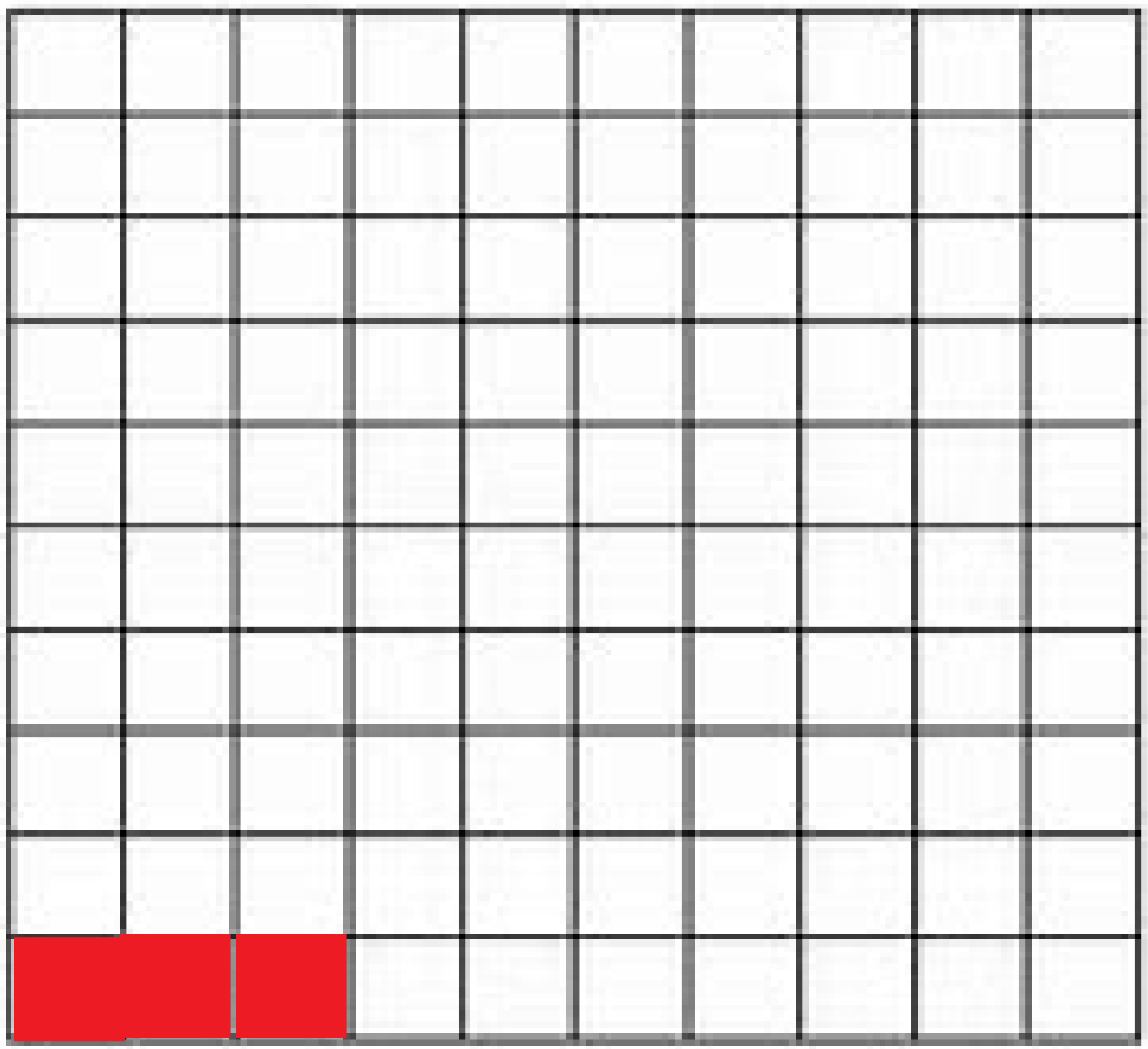
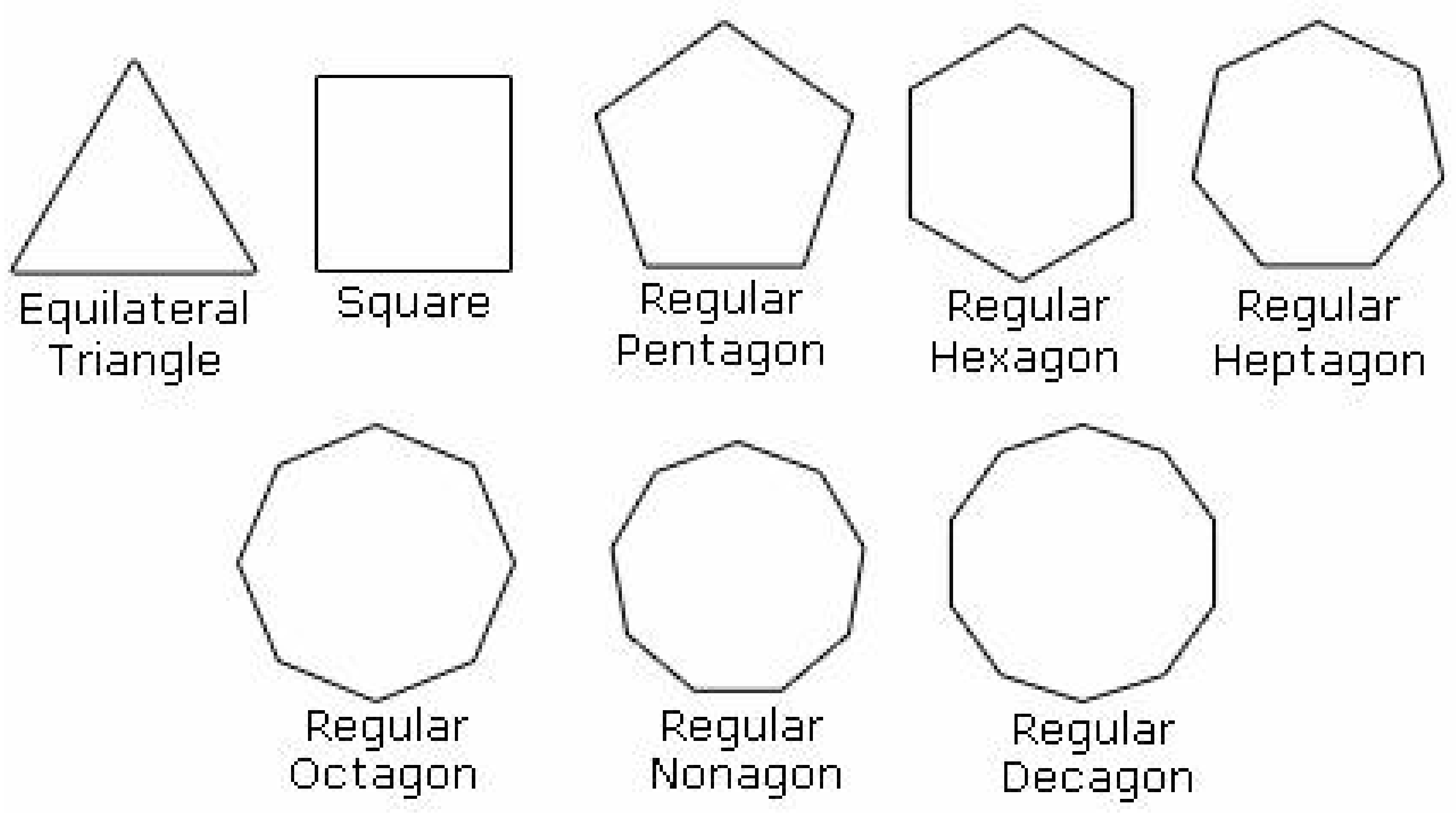


We just continue adding the smaller squares…..

Building on our success, as a fun problem see if you can tile larger and larger regions, with no gaps, with the following shapes.



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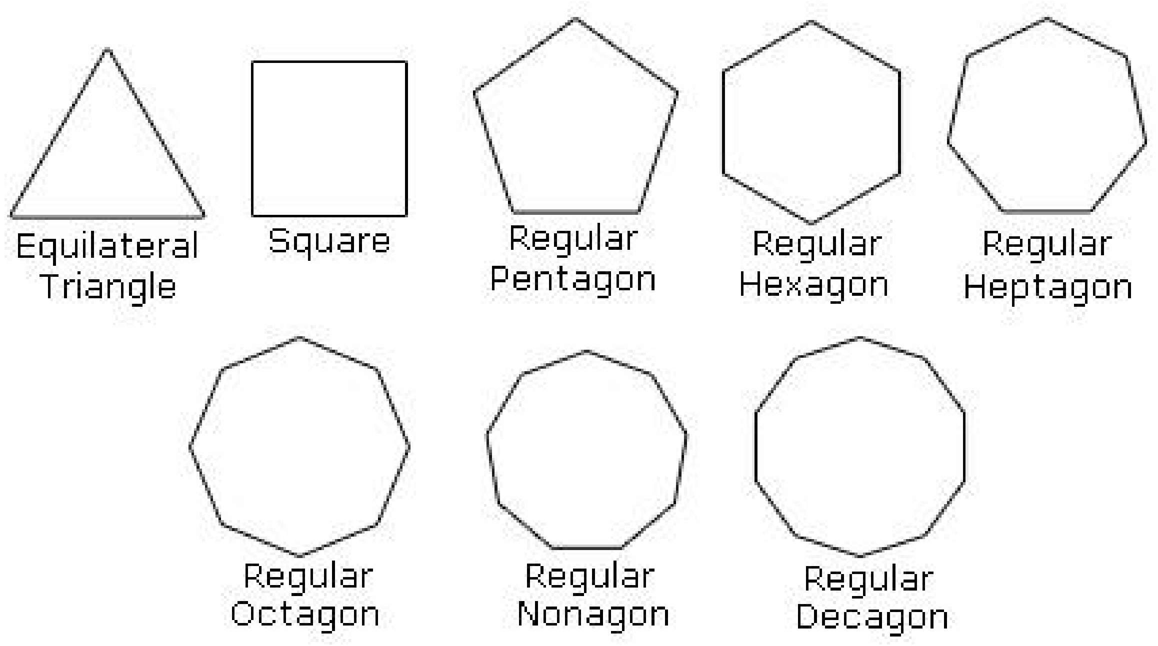
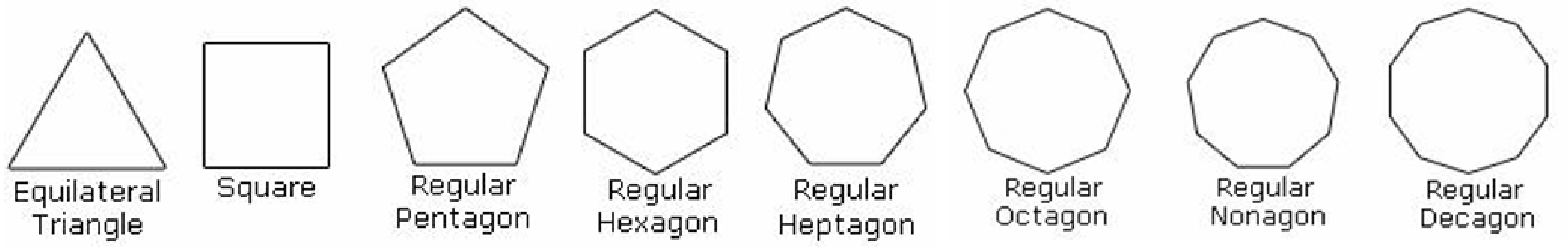


Note each shape above has all sides of the same length. We saw we can do it with the square. What about the triangle? What about the pentagon? What if we mix and match? GOOD LUCK!



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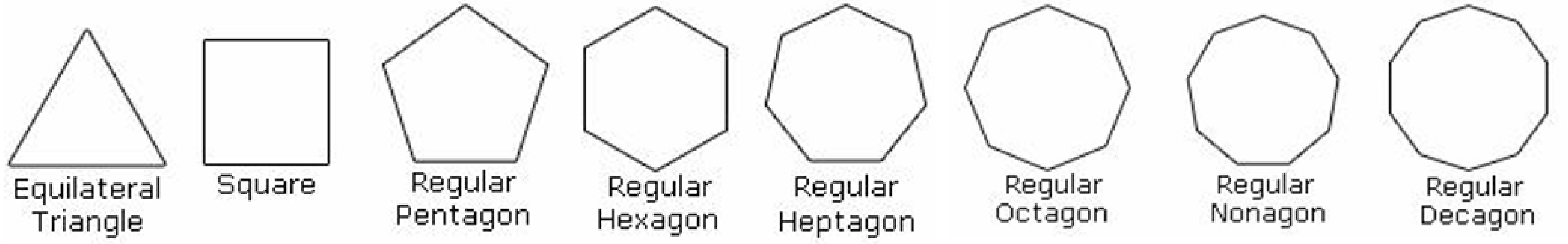
Notice we have implicitly made an assumption about what we are studying? What is that assumption?



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We are working IN THE PLANE – What happens in higher dimensions? Think of the five platonic solids – how do those GENERALIZE?

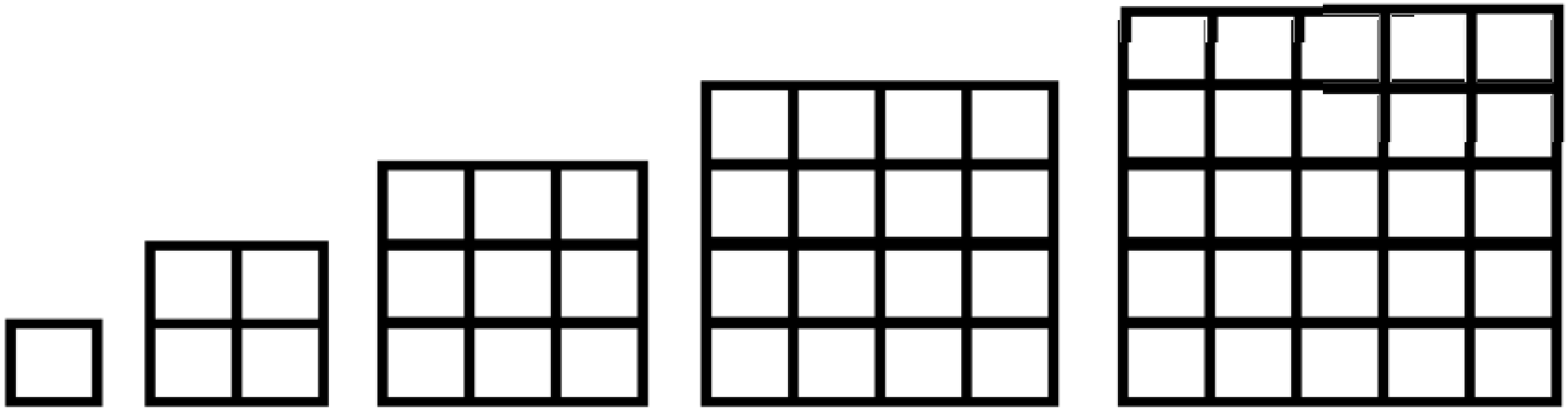
Objectives: Discuss computational units.

- Gather data and conjecture.
- Test hypotheses.
- Prove claims.
- **GENERALIZE**!
- Have fun!

The I LOVE RECTANGLES Game

If we have an unlimited supply of 1 foot by 1 foot squares, we can cover larger and larger rectangles.

Let's make it more interesting. Imagine now we have EXACTLY ONE of each size square. We have one 1 by 1 rectangle, one 2 by 2 rectangle, one 3 by 3 rectangle, one 4 by 4 rectangle, and so on.



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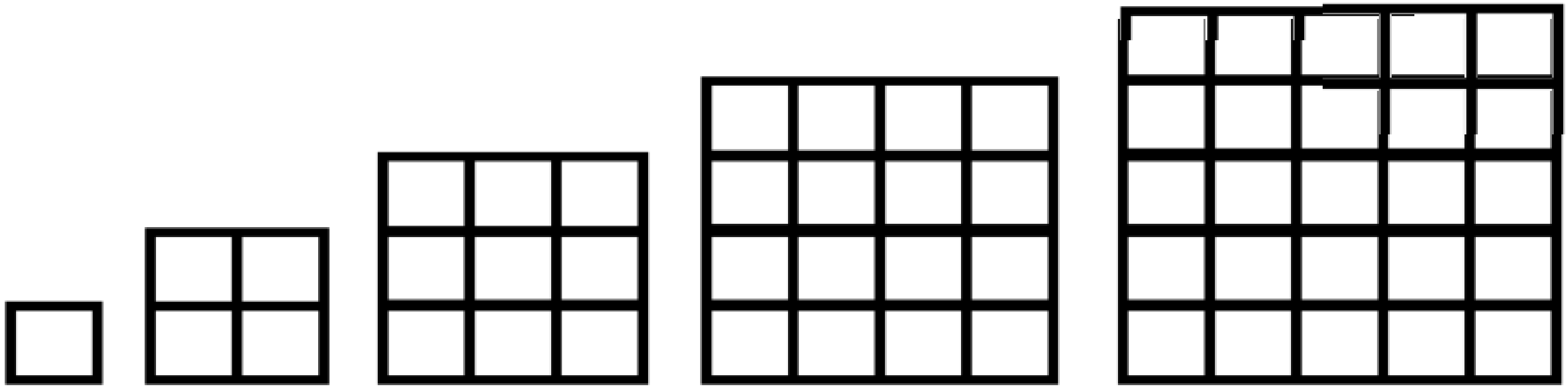
Here's the rule: we put these squares down ONE AT A TIME, and at EVERY MOMENT IN TIME our shape MUST be a rectangle. Can it be done? Note a square IS a rectangle.



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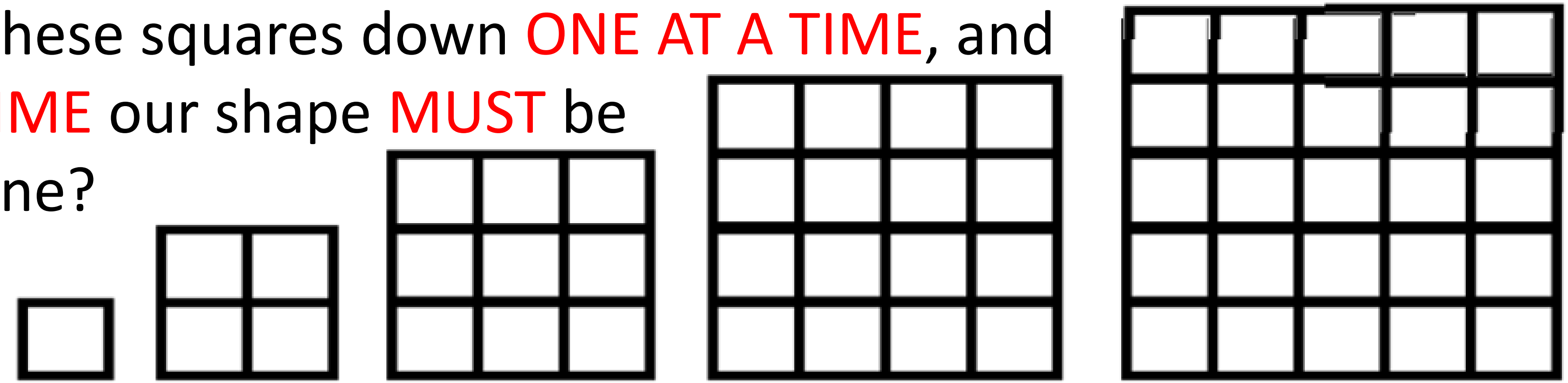
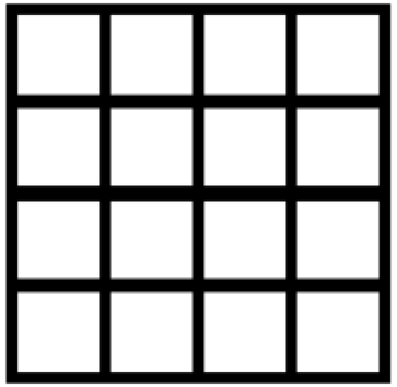


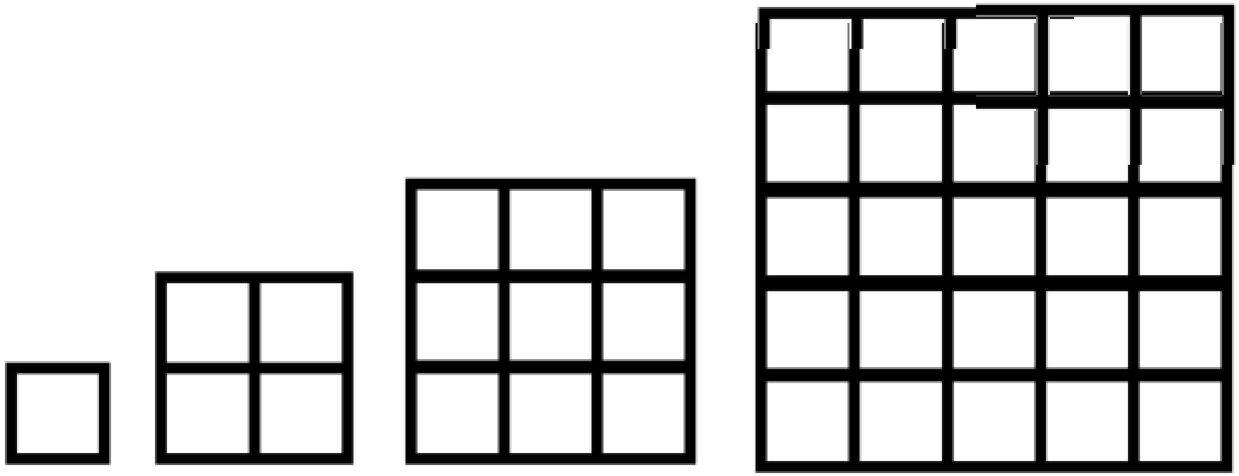
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

Imagine we put the 4 by 4 square down. That gives us a rectangle, so far so good. Can we put down anything else next to it and still have a rectangle?



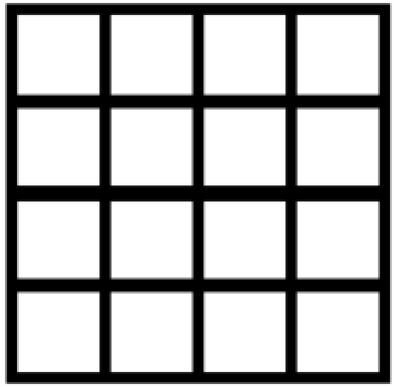
We have placed a 4 by 4 square. This is a rectangle!



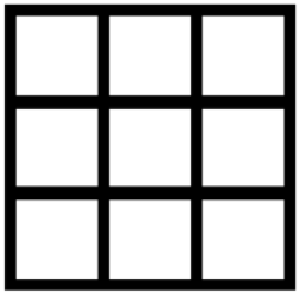
These are the squares we have left. We have a 1 by 1, a 2 by 2, a 3 by 3, a 5 by 5, a 6 by 6 (not drawn) and so on. Can we place anything next to the 4 by 4 and still have a rectangle?

The I LOVE RECTANGLES Game

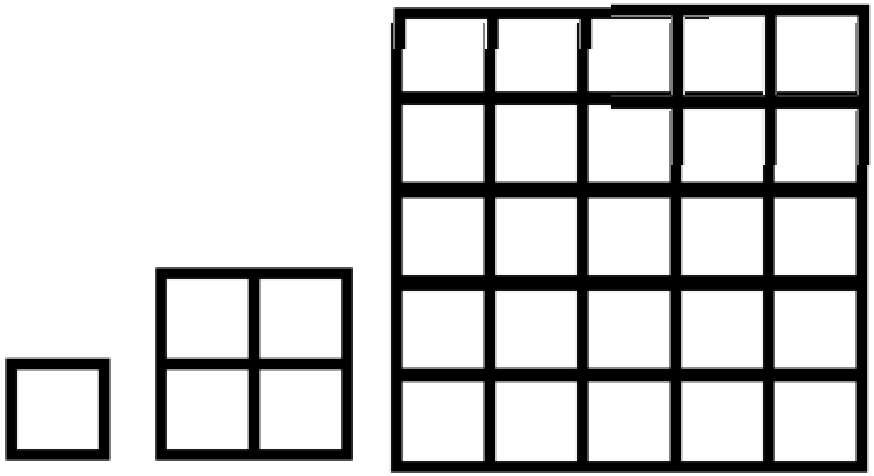
Imagine we put the 4 by 4 square down. That gives us a rectangle, so far so good. Can we put down anything else? Let's try putting down the 3 by 3.



We have placed a 4 by 4 square. This is a rectangle!



We see the 3 by 3 will not fit next to the 4 by 4 and still give a rectangle!

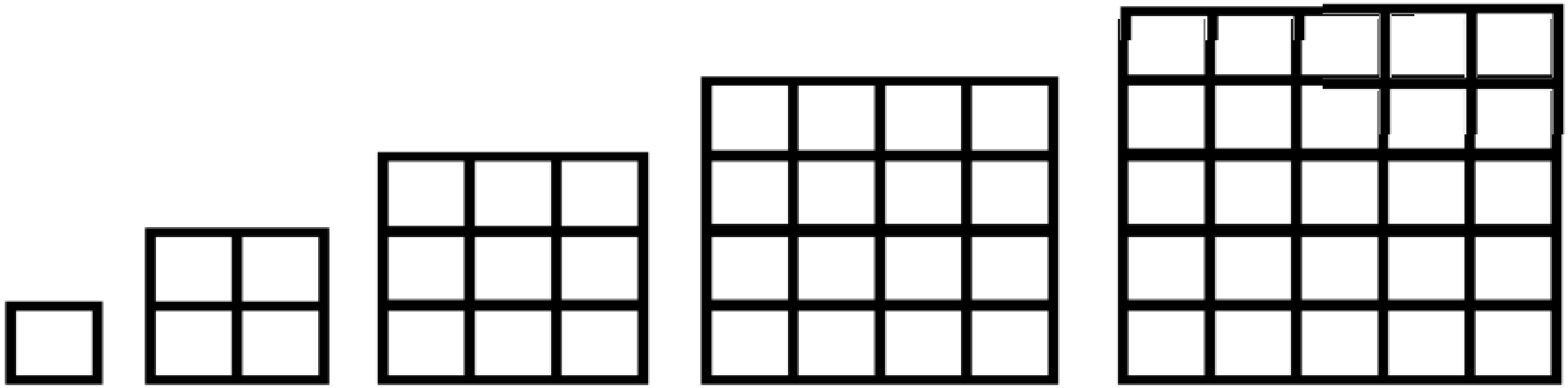


These are the squares we would have left if we try to use a 3 by 3. We would have a 1 by 1, a 2 by 2, a 5 by 5, a 6 by 6 (not drawn) and so on.

The I LOVE RECTANGLES Game

In fact, no matter WHAT square we put down first, we cannot put any more down! If we put down a 5 by 5, to keep it a rectangle we would need something that has a side of length 5, but we only have ONE of each square!

We have to modify the game. We need to give at least ONE more square. What is the smallest square we can give?



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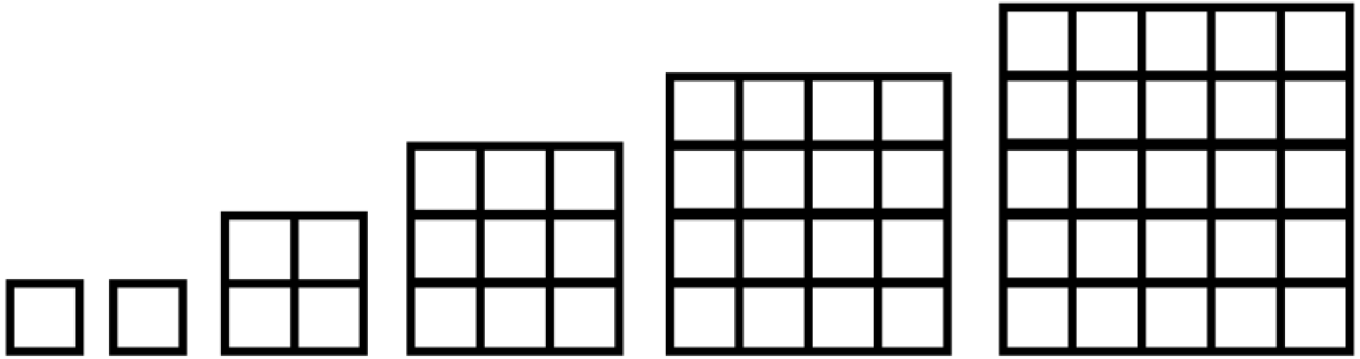

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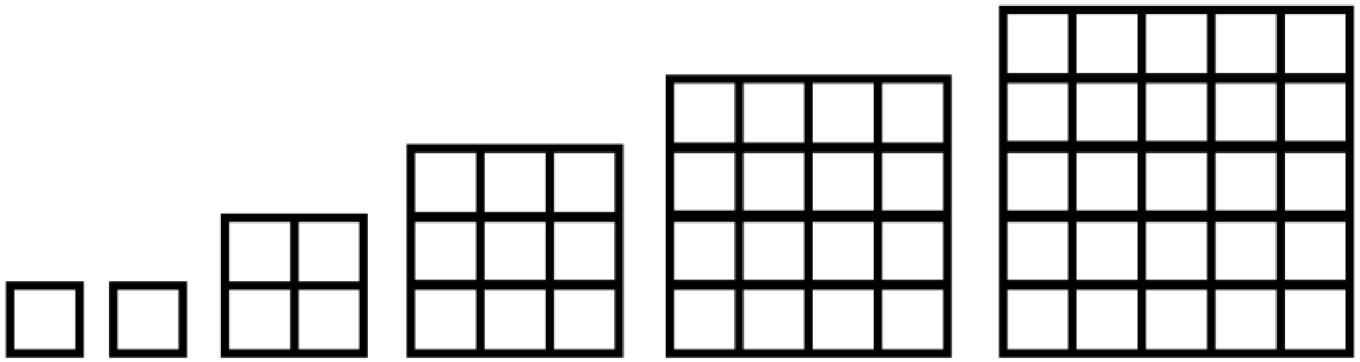
OK, we want to put the squares down one at a time so that we always have a rectangle. We cannot put a square on top of a square. Which should we put down first? Which should we put down second?



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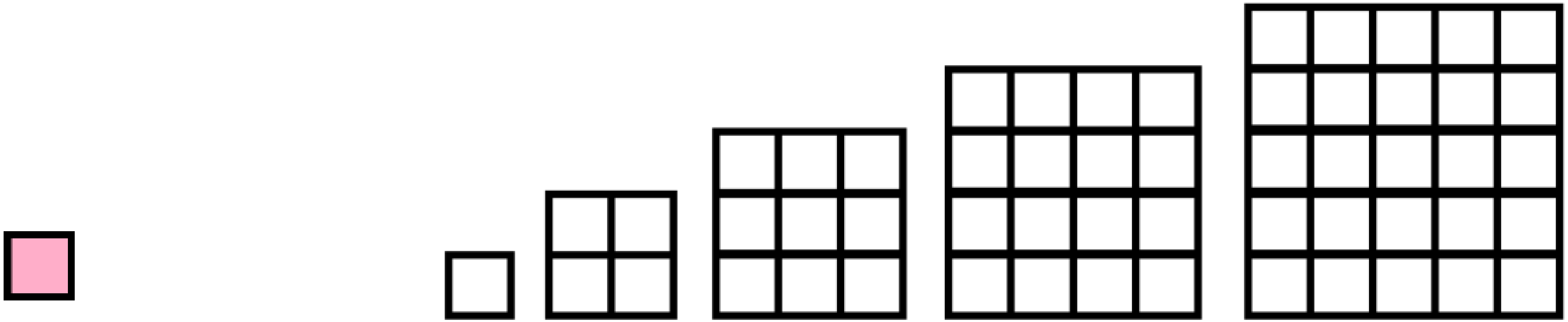
OK, we want to put the squares down one at a time so that we always have a rectangle. We cannot put a square on top of a square. Which should we put down first? Which should we put down second?

Makes sense to start with the two 1 by 1 squares, as they fit! Here is placing the first 1 by 1 square. Now we have one 1 by 1, one 2 by 2, one 3 by 3, and so on.



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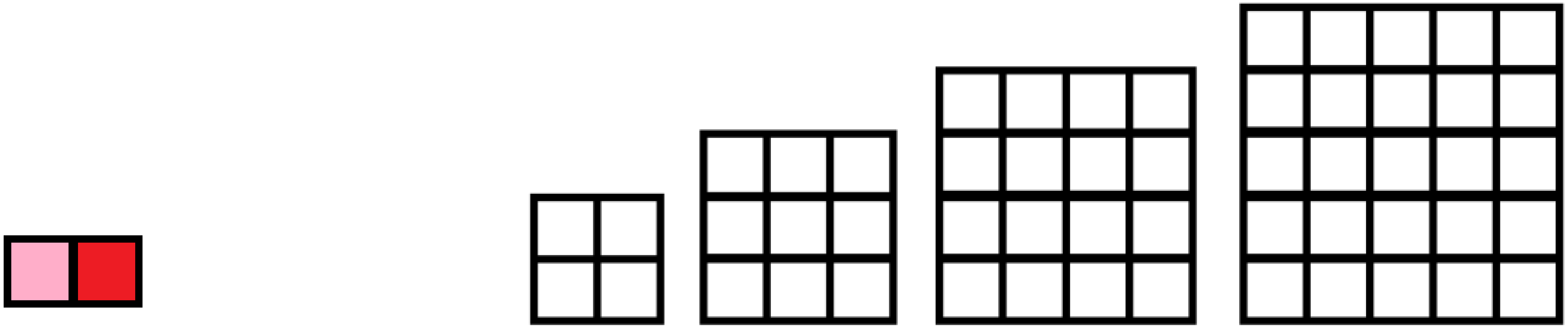
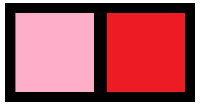
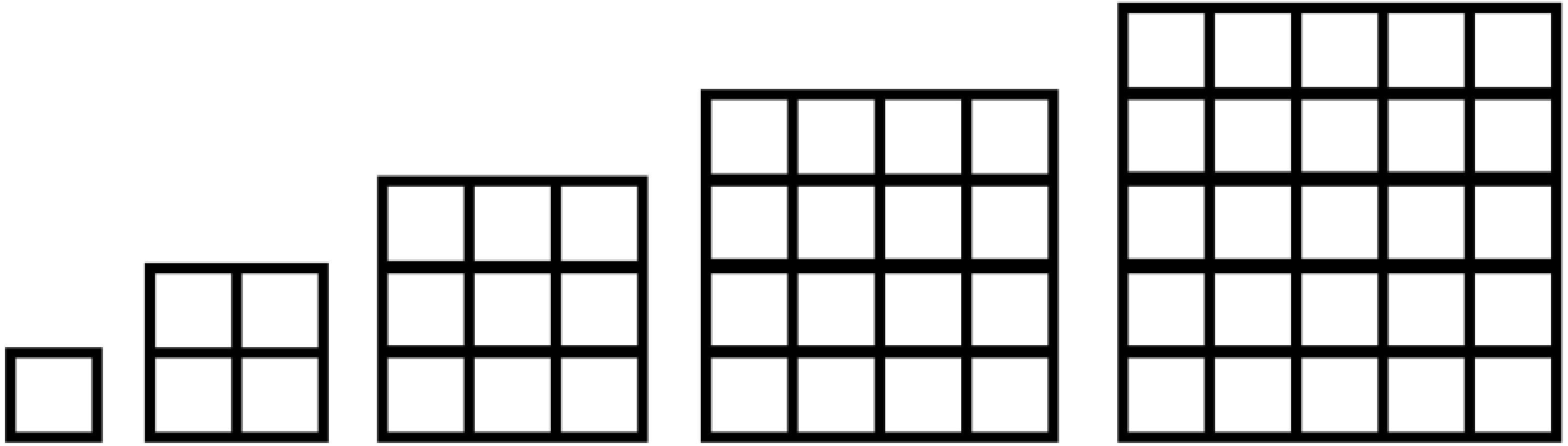


The I LOVE RECTANGLES Game

We have placed the two 1 by 1 squares, we have a 2 by 2, a 3 by 3, a 4 by 4, a 5 by 5 and so on. What should we place next to the two 1 by 1 squares so that we still have a rectangle? Note the two 1 by 1 squares have formed a 1 by 2 rectangle…..



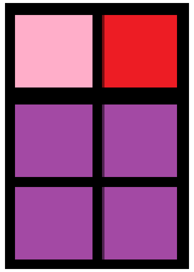


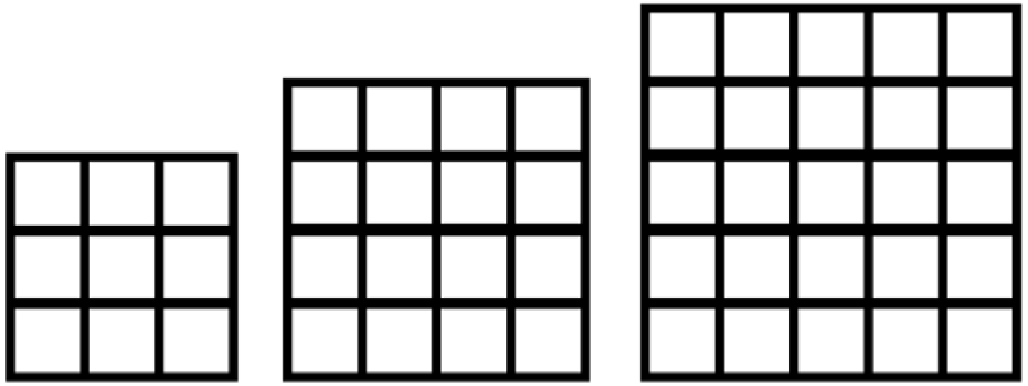
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

We had a 1 by 2 rectangle, so we need a square that has a side of length 1 or a side of length 2. Looking at our squares, we see we can use the 2 by 2 square!

Building on this success, what should we put down next? Note we now have a rectangle that is 2 by 3….



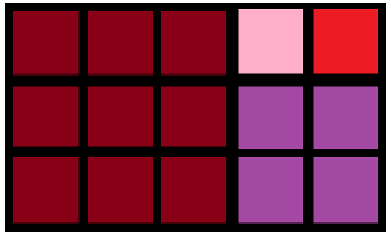


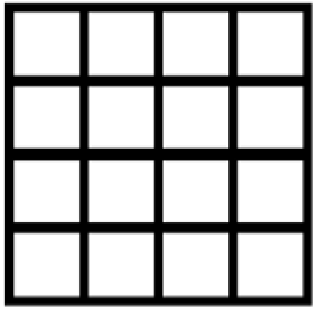
SPEND A MOMENT AND SEE IF YOU
NSWER THIS!

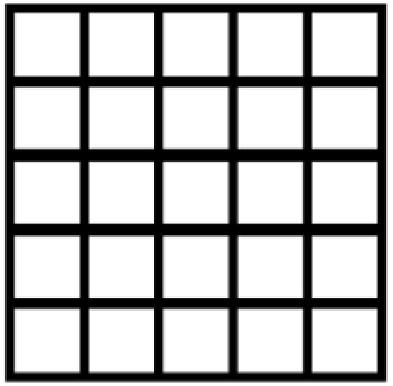
The I LOVE RECTANGLES Game

We had a 2 by 3 rectangle, so we need a square that has a side of length 2 or a side of length 3. Looking at our squares, we see we can use the 3 by 3 square!

Building on this success, what should we put down next? Note we now have a 3 by 5 rectangle.





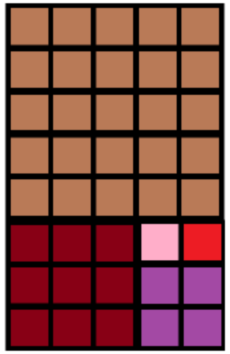


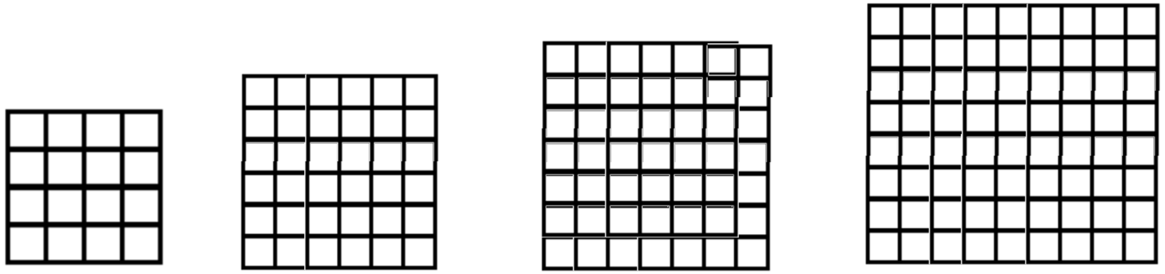
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

We had a 3 by 5 rectangle. Looking at our squares, we see we can use the 5 by 5 square!

Building on this success, what should we put down next? Note we now have a 5 by 8 rectangle. The 4 by 4 is too small, we still have a 6 by 6, …..



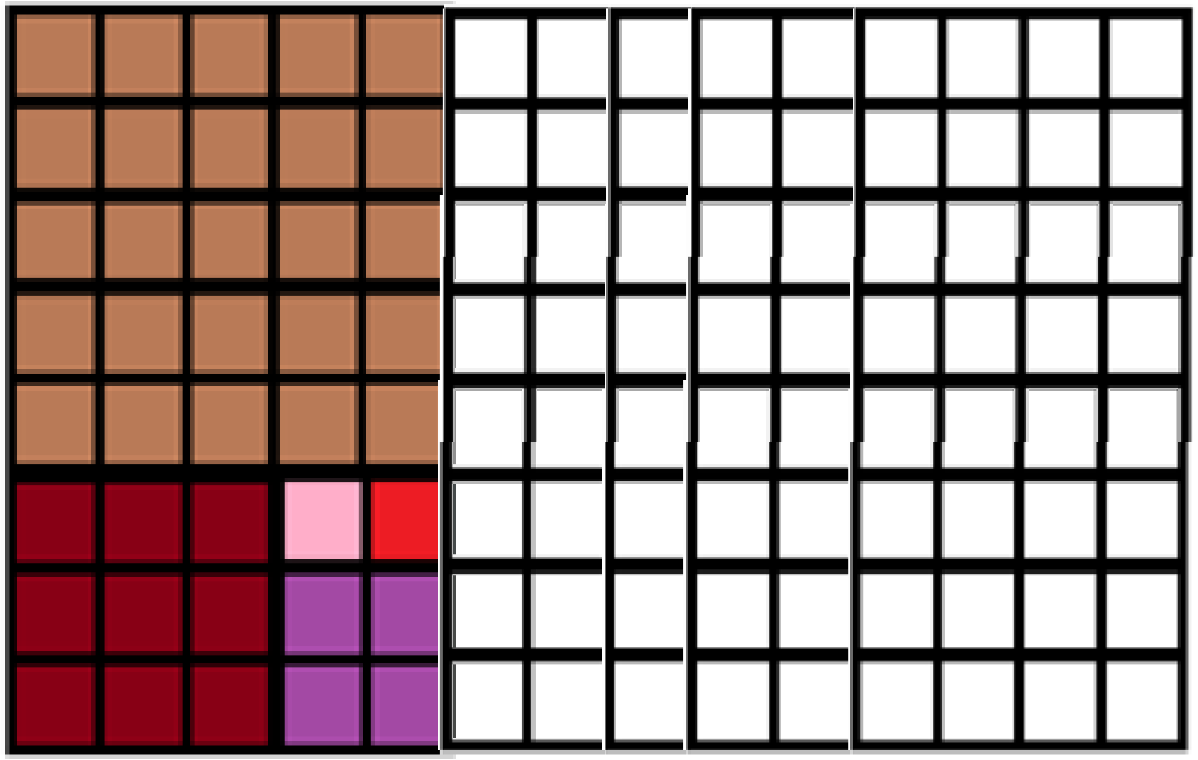


We still have a 6 by 6, a 7 by 7, an 8 by 8, a 9 by 9 (not drawn), a 10 by 10 (not drawn), and so on…..

SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

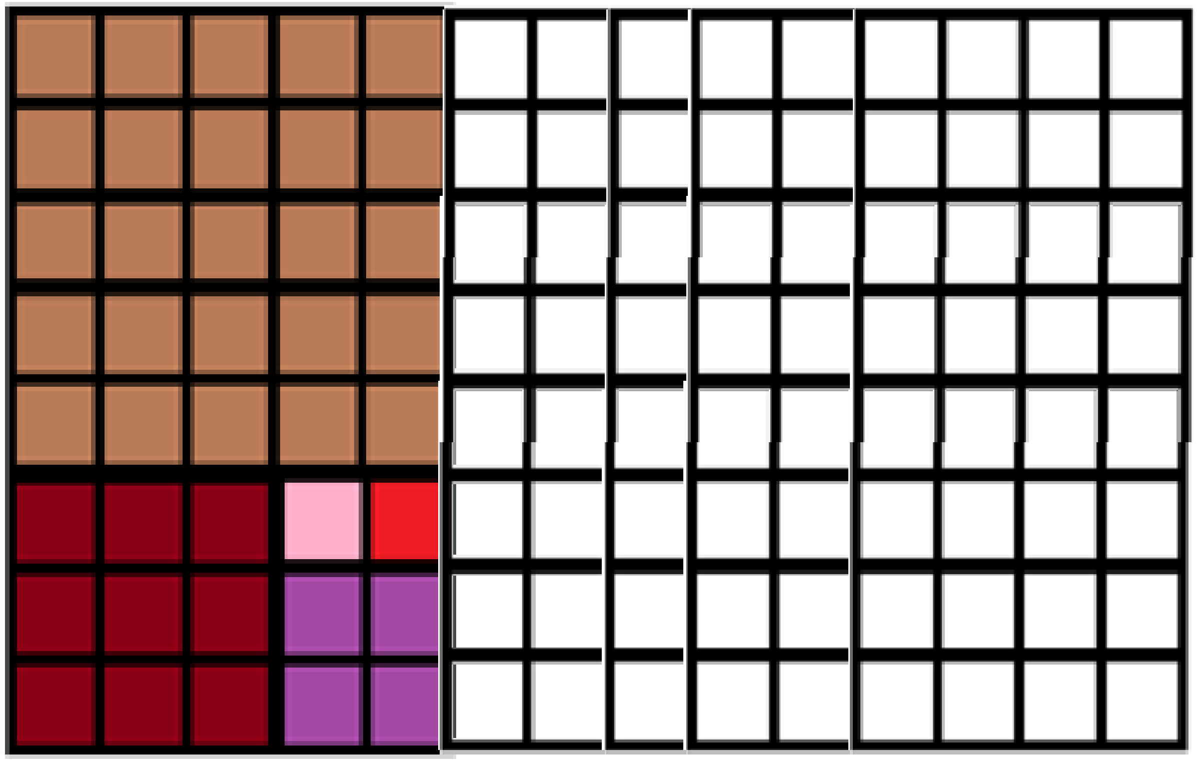
We had a 5 by 8 rectangle. We need to add something with a side of length 5 or 8. Thus we won't use the 4 by 4, the 6 by 6 or the 7 by 7, but we will use the 8 by 8……



The I LOVE RECTANGLES Game

We write down the squares used in the order used:

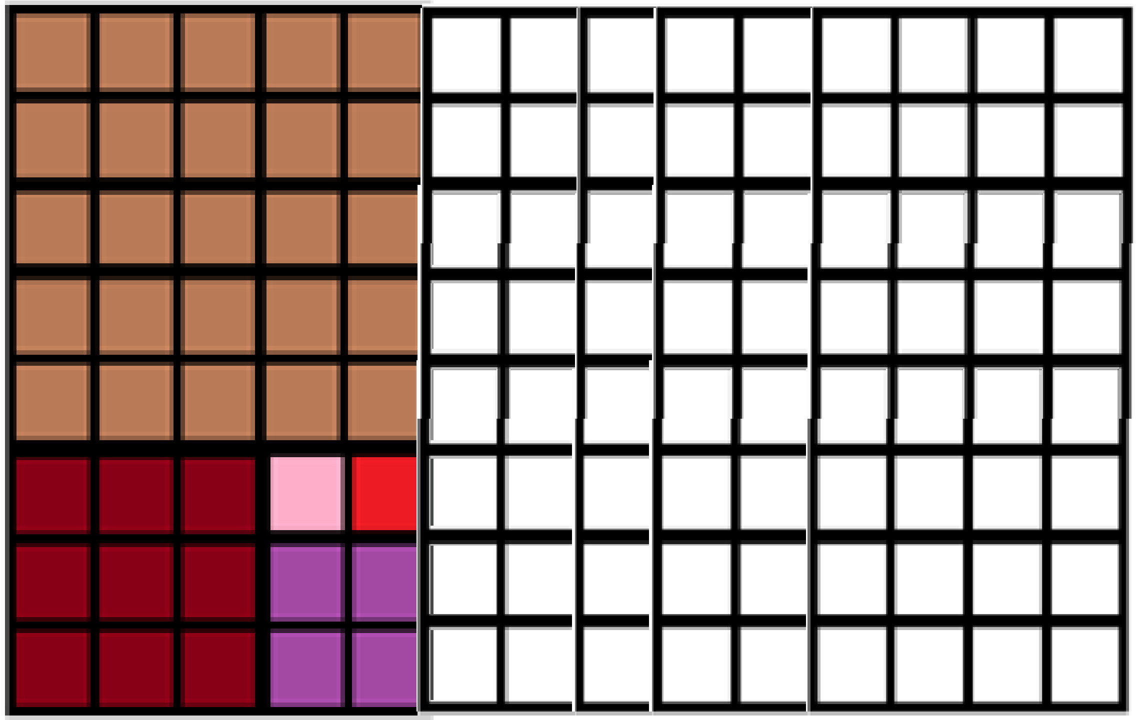
1 by 1, 1 by 1, 2 by 2, 3 by 3, 5 by 5, 8 by 8, …..



The I LOVE RECTANGLES Game

Let’s just write down the side lengths of the squares in the order used:

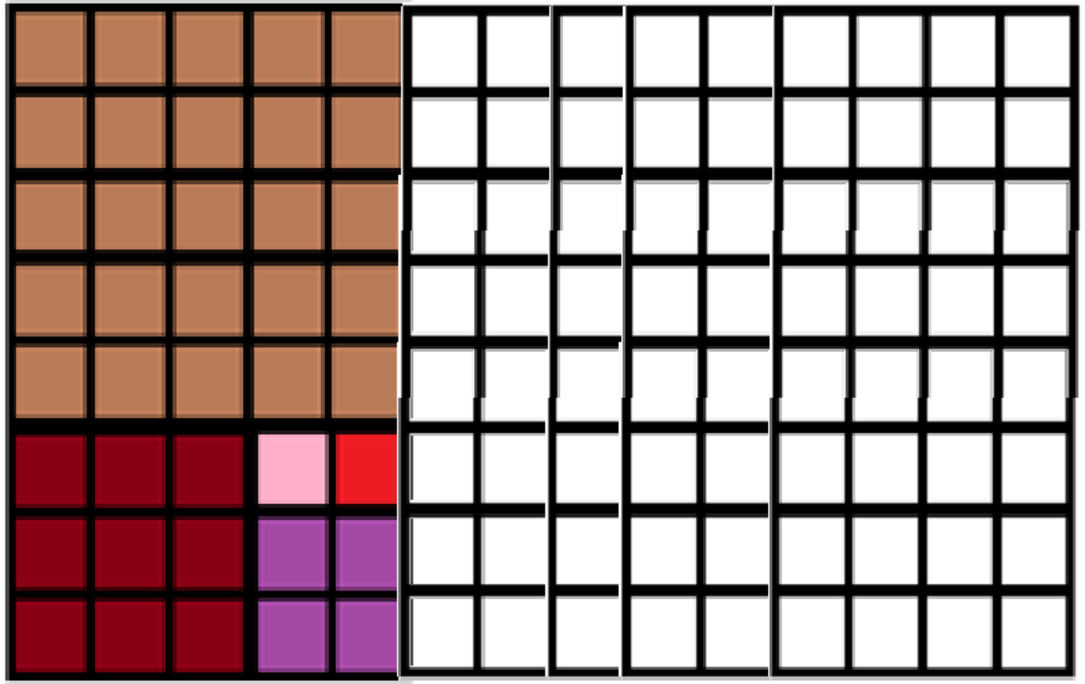
1, 1, 2, 3, 5, 8, …. DO YOU NOTICE A PATTERN?



The I LOVE RECTANGLES Game

Let's just write down the side lengths of the squares in the order used (we'll add a few more terms to the sequence):

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …. DO YOU NOTICE A PATTERN?



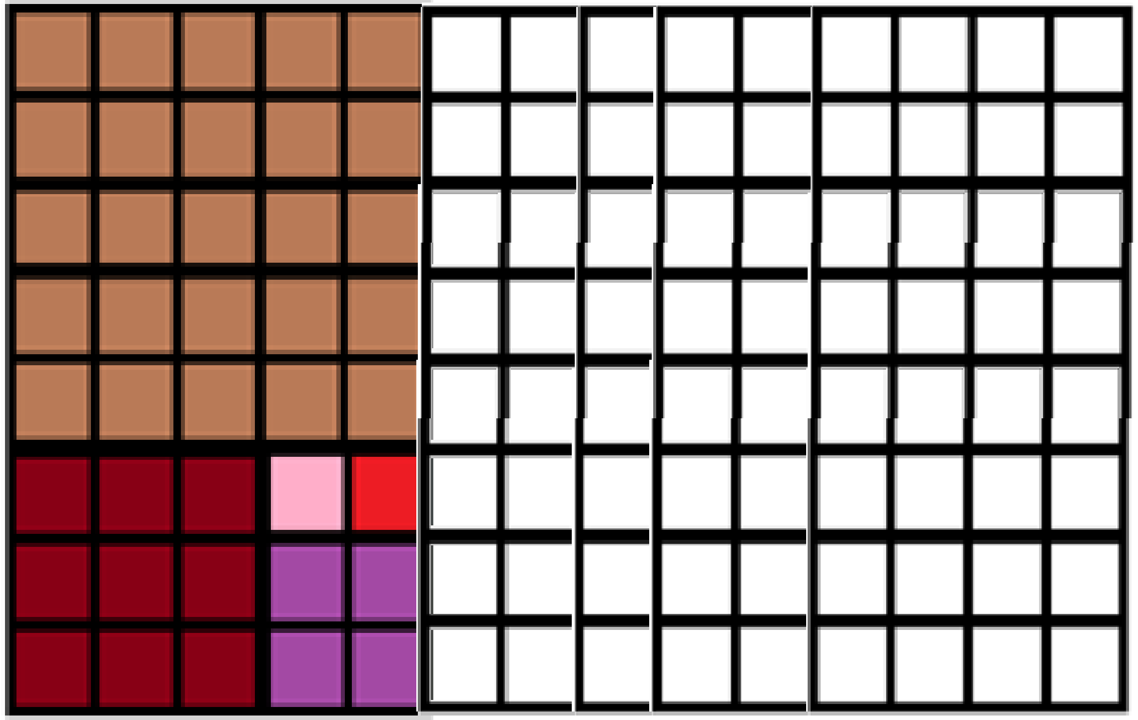
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

Let's just write down the side lengths of the squares in the order used:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ….

We start 1, 1, and then after that each term is the sum of the previous two terms! 2 = 1 + 1, 3 = 2 + 1, 5 = 3 + 2, 8 = 5 + 3, and so on. Can you continue the pattern?

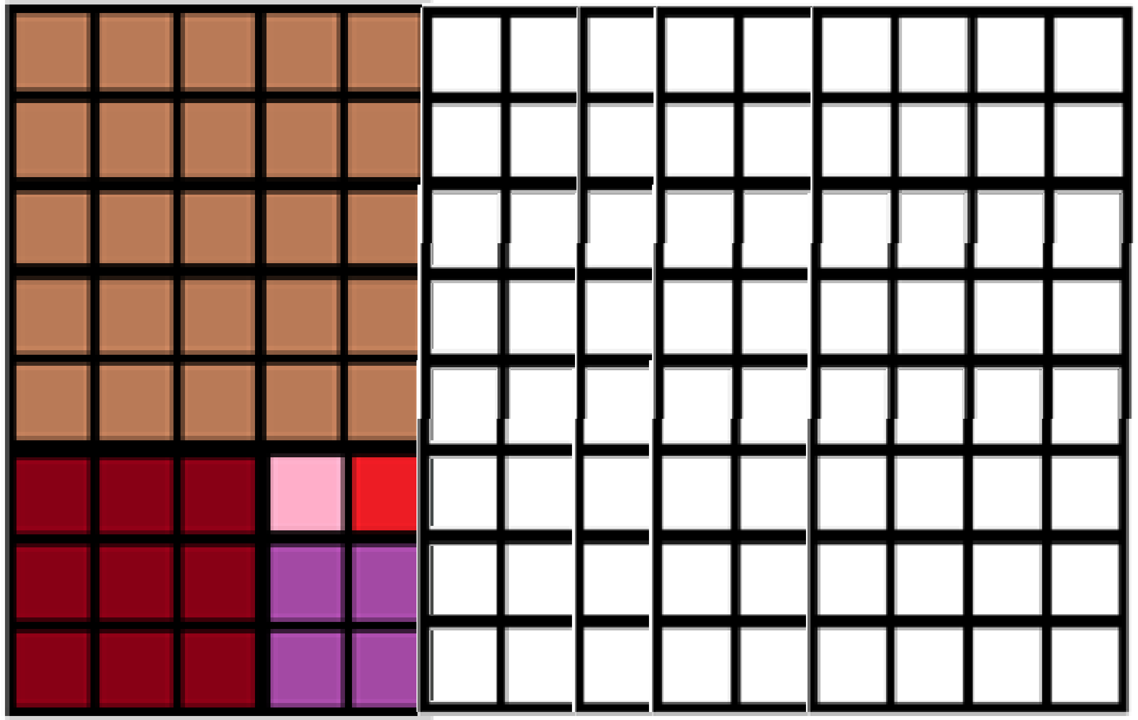


The Fibonacci Sequence

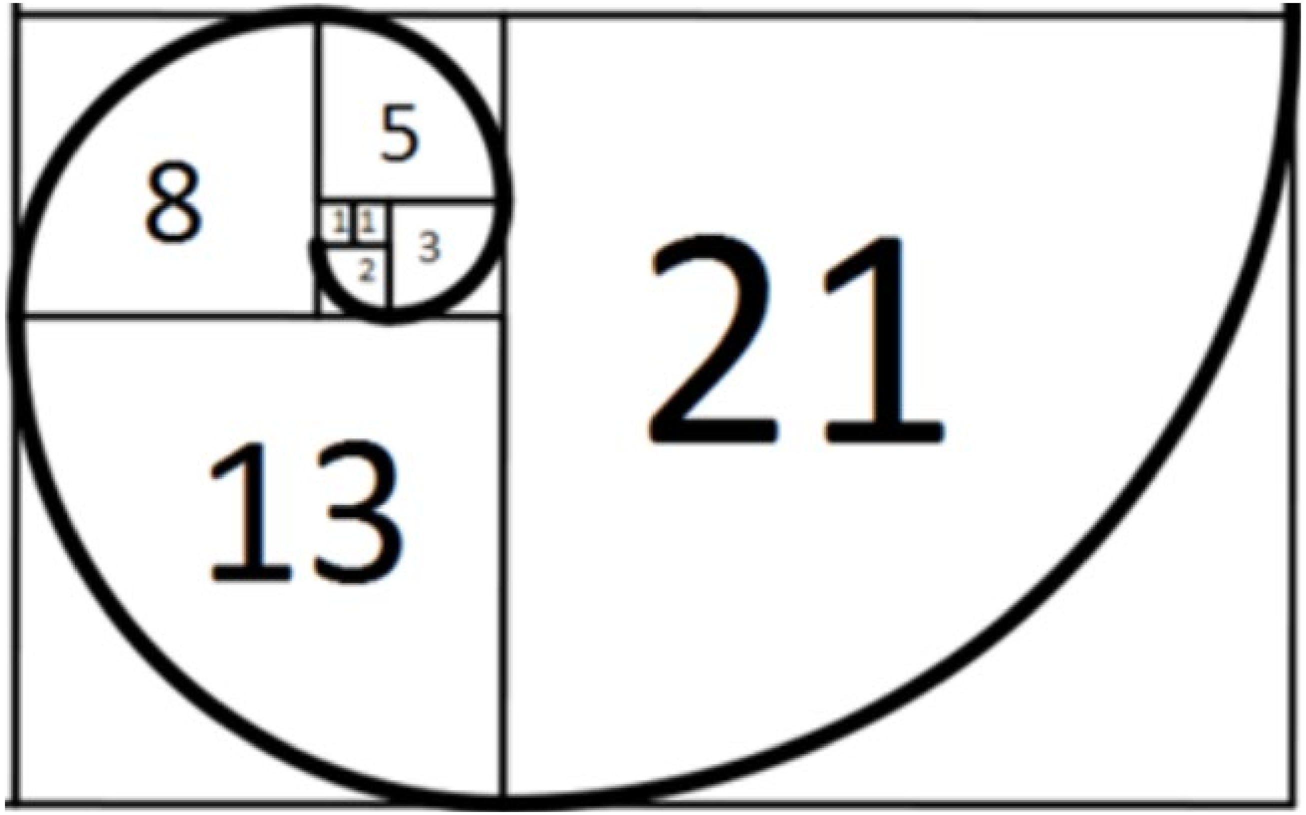
The numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ….

are called the Fibonacci numbers, and have many wondrous properties. See for example https://www.youtube.com/watch?v=me6Dnl2DOtM .



ADVANCED TOPIC!



Advanced: you can calculate area two ways. It is length times width, which here is 21 by 34. It is also the sum of the areas of each square, which is $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 + 21^2$. These are equal! You can thus prove the sum of the squares of the first n Fibonacci numbers is the n^{th} Fibonacci number times the $(n+1)^{st}$ Fibonacci number!

Summary: I Love Rectangles Game.

- Experimentally discovered Fibonaccis!
- Alternative definition: connections!
- Can you **GENERALIZE**?

Chocolate Bar Game

https://tinyurl.com/aca5usrd

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

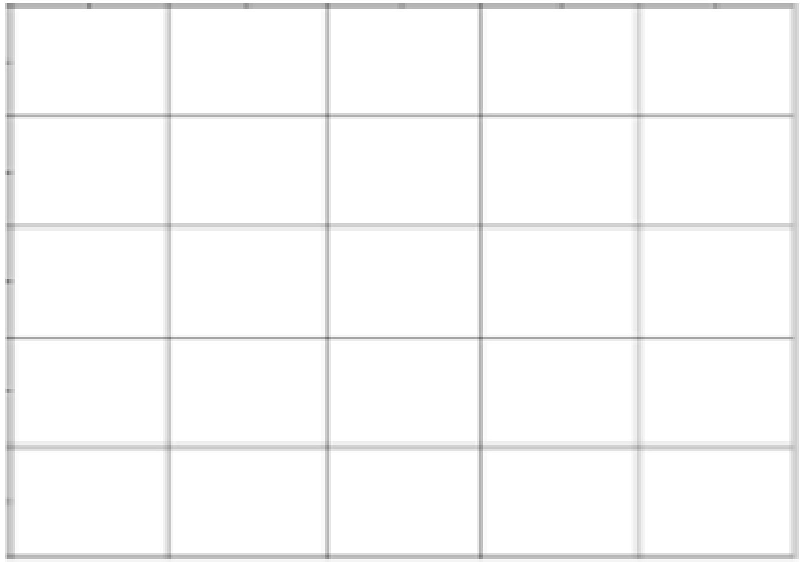


Figure: Winning strategy? Function of board dimension?

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

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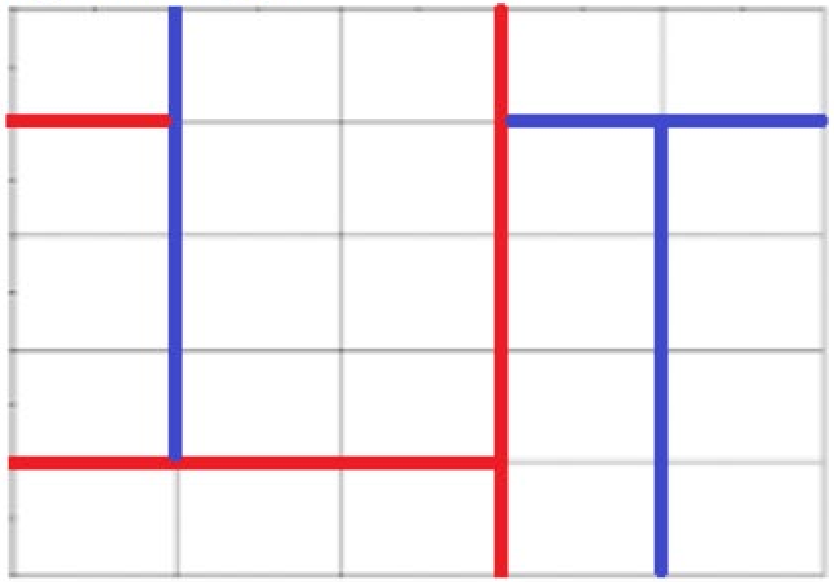

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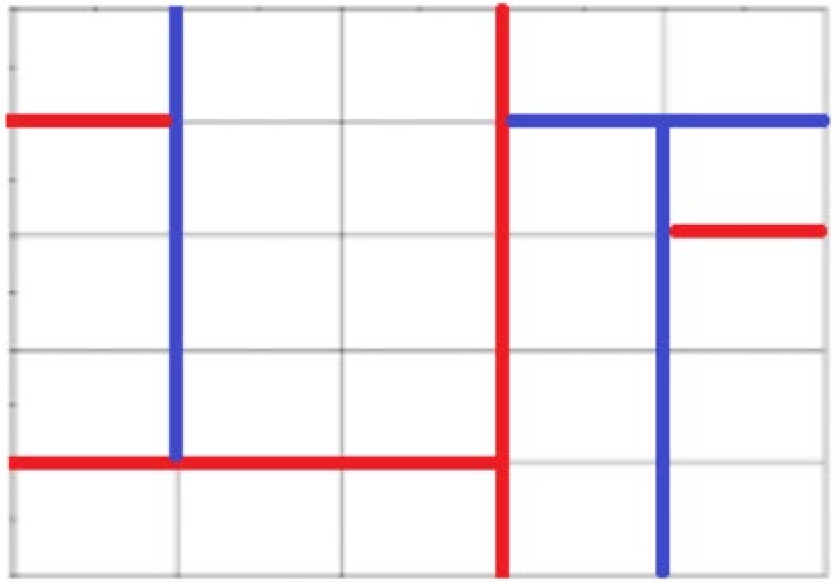

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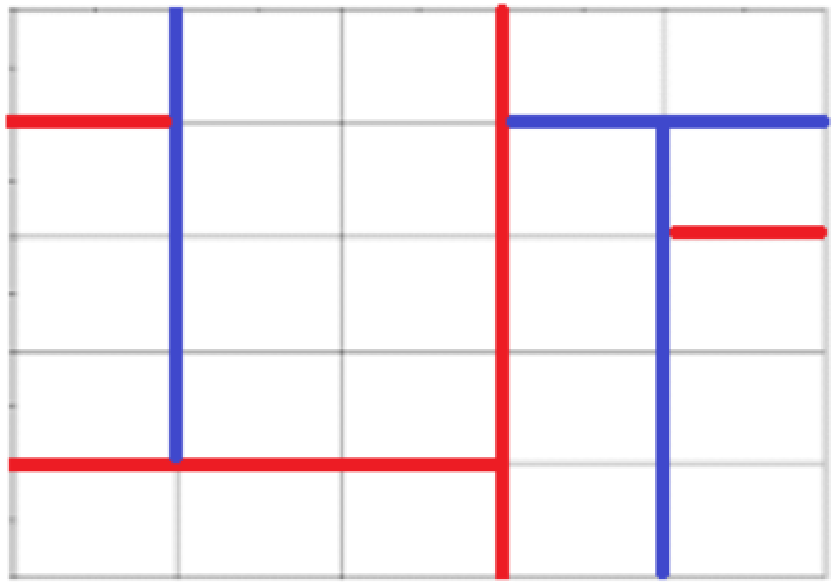
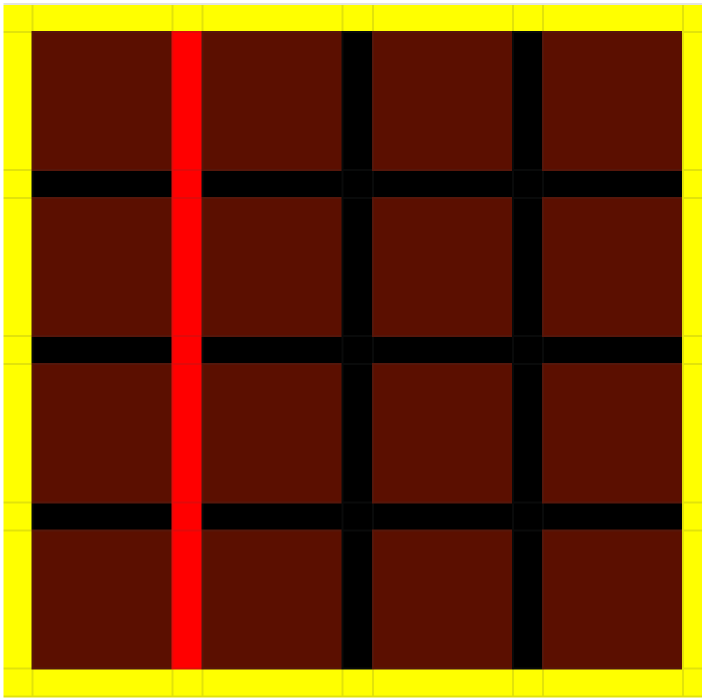


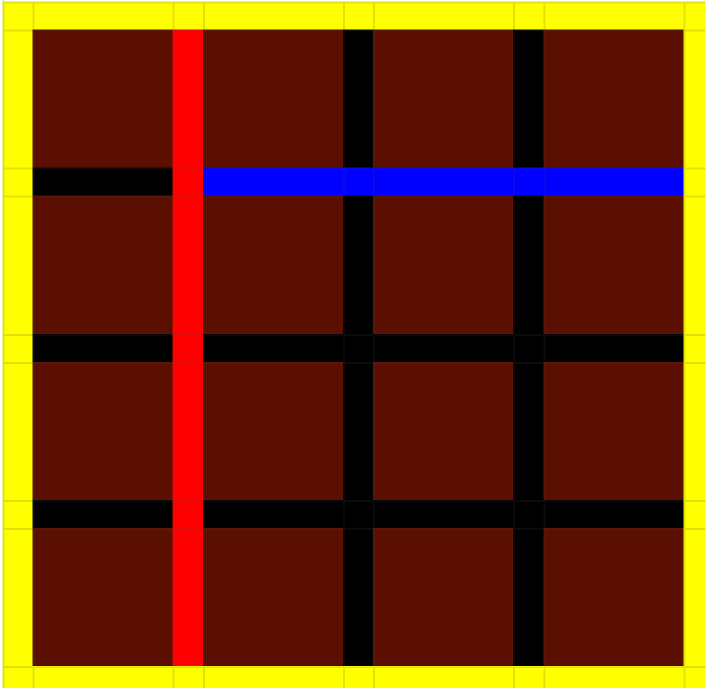
Gather data! Try various sized boards, strategies.

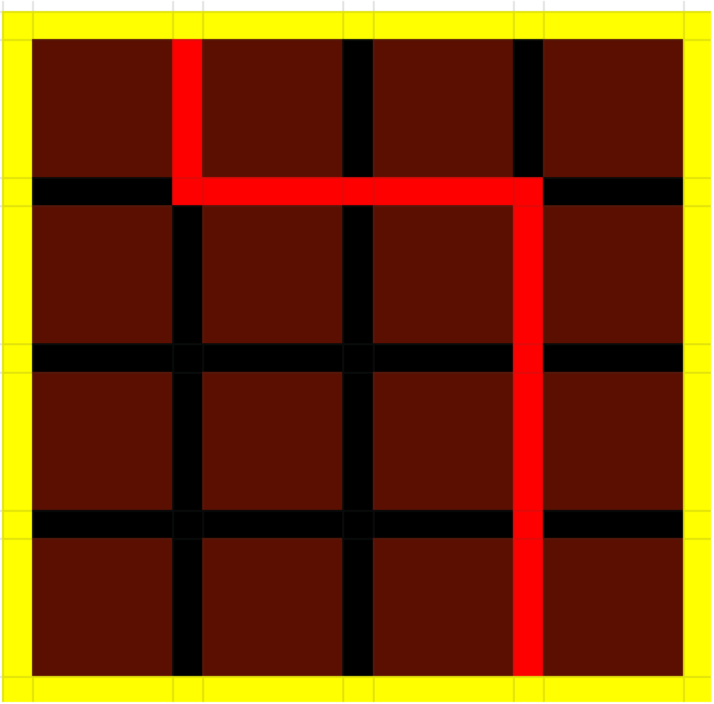
https://tinyurl.com/aca5usrd



This ***is*** a valid move. Now the bar is cut into two pieces.

Given the legal red cut, here is a legal blue cut (it cuts along the entire length of a black line!):

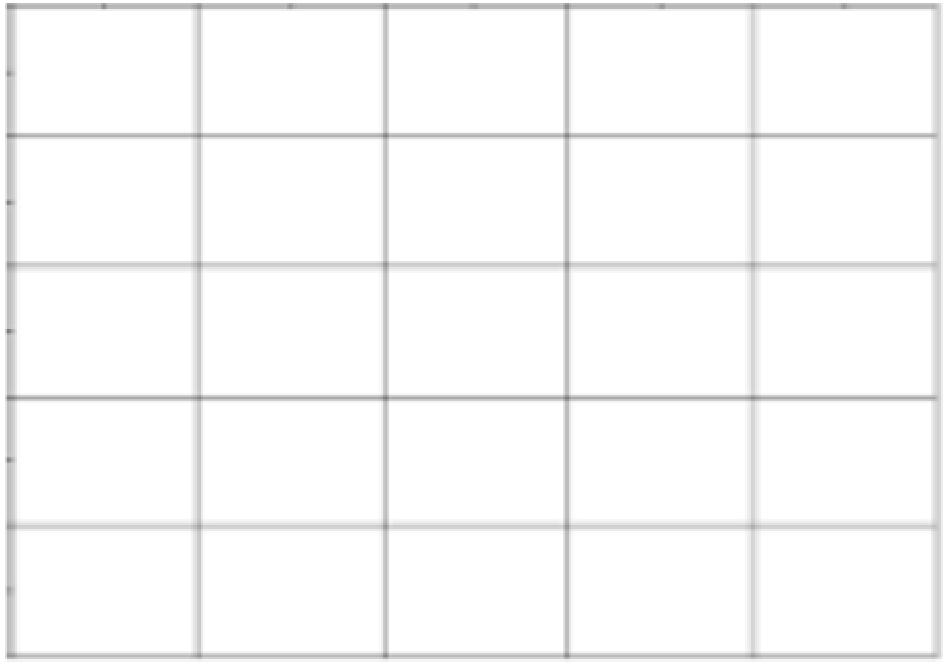




The above cut is illegal!

https://tinyurl.com/aca5usrd

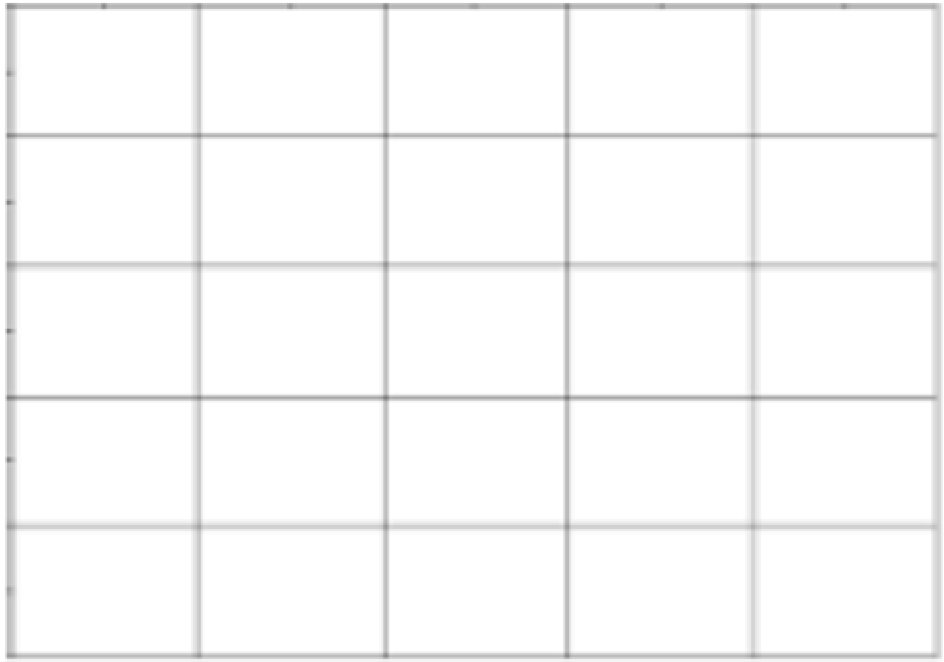
RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Length	Width	Winner
2	2	
2	3	
3	3	
2	4	
3	4	
4	4	
3	5	

Do you see a pattern?

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Length	Width	Winner
2	2	--1--
2	3	--1--
3	3	--2--
2	4	--1--
3	4	--1--
4	4	--1--
3	5	--2--

Figure: Do you see a pattern?

A mono-variant is a quantity that moves on only one direction (either non-decreasing or non-increasing).

Idea: Associate a mono-variant to the

Chocolate Bar Game

Every time move, increase number of pieces by 1!

Move: 0; Pieces: 1.

Every time move, increase number of pieces by 1!

Move: 1; Pieces: 2.

Every time move, increase number of pieces by 1!

Move: 2; Pieces: 3.

Every time move, increase number of pieces by 1!

Move: 3: Pieces: 4.

Every time move, increase number of pieces by 1!

Move: 4; Pieces: 5.

Every time move, increase number of pieces by 1!

Move: 5; Pieces: 6. Player 1 Wins.

The mono-variant is the number of pieces.

If board is m x n, game ends with mn pieces.

Thus takes mn - 1 moves.

If mn is even then Player 1 wins else Player 2 wins.

THE MOST IMPORTANT PART OF THE GAME IS TO EAT THE CHOCOLATE. DON'T FORGET!

Summary: Chocolate Bar Game.

- Gather data! Conjecture!
- New approach: Mono-variants.
- Can you **GENERALIZE**?

Bonus slides: Cookie Problem

The Cookie Problem

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

Proof: Consider $C + P - 1$ cookies in a line.

Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.

Divides the cookies into P sets.

Example: 8 cookies and 5 people ($C = 8$, $P = 5$):

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Example: 8 cookies and 5 people ($C = 8$, $P = 5$):

How to find and conjecture….?

- **Gather data from small values of C and P.**
- **Feed into the Online Encyclopedia of Integer Sequences:**

OEIS: https://oeis.org/

- **P=1:** 1,1,1,1,1,1,1,1,1
- **P=2:** 2,3,4,5,6,7,8,9,10,11
- **P=3:** 3,6,10,15,21,28,36,45,55,66
- **P=4:** 4,10,20,35,56,84,120,165,220,286
- **P=5:** 5,15,35,70,126,210,330,495,715,1001
- **P=6:** 6,21,56,126,252,462,792,1287,2002,3003
- **P=7:** 7,28,84,210,462,924,1716,3003,5005,8008
- **P=8:** 8,36,120,330,792,1716,3432,6435,11440,19448

Search: **seq:5,15,35,70,126,210,330,495,715,1001**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A000332 Binomial coefficient binomial(n,4) = n*(n-1)*(n-2)*(n-3)/24. +30 355

(Formerly M3853 N1578)

0, 0, 0, 0, 1, **5, 15, 35, 70, 126, 210, 330, 495, 715, 1001**, 1365, 1820, 2380, 3060, 3876, 4845, 5985, 7315, 8855, 10626, 12650, 14950, 17550, 20475, 23751, 27405, 31465, 35960, 40920, 46376, 52360, 58905, 66045, 73815, 82251, 91390, 101270, 111930, 123410 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,6

COMMENTS Number of intersection points of diagonals of convex n-gon where no more than two diagonals intersect at any point in the interior.
Also the number of equilateral triangles with vertices in an equilateral triangular array of points with n rows (offset 1), with any orientation. - Ignacio Larrosa Cañestro, Apr 09 2002. [See Les Reid link for proof. - N. J. A. Sloane, Apr 02 2016]
Start from cubane and attach amino acids according to the reaction scheme that describes the reaction between the active sites. See the hyperlink on chemistry. - Robert G. Wilson v, Aug 02 2002
For n>0, a(n) = (-1/8)*(coefficient of x in Zagier's polynomial P_(2n,n)). (Zagier's polynomials are used by PARI/GP for acceleration of alternating or positive series.)
Figurate numbers based on the 4-dimensional regular convex polytope called the regular 4-simplex, pentachoron, 5-cell, pentatope or 4-hypertetrahedron with Schlaefli symbol {3,3,3}. a(n)=((n*(n-1)*(n-2)*(n-3))/4!). - Michael J. Welch (mjw1(AT)ntlworld.com), Apr 01 2004, R. J. Mathar, Jul 07 2009
Maximal number of crossings that can be created by connecting n vertices with straight lines. - Cameron Redsell-Montgomerie (credsell(AT)uoguelph.ca), Jan 30 2007

Bonus slides: using Lego….

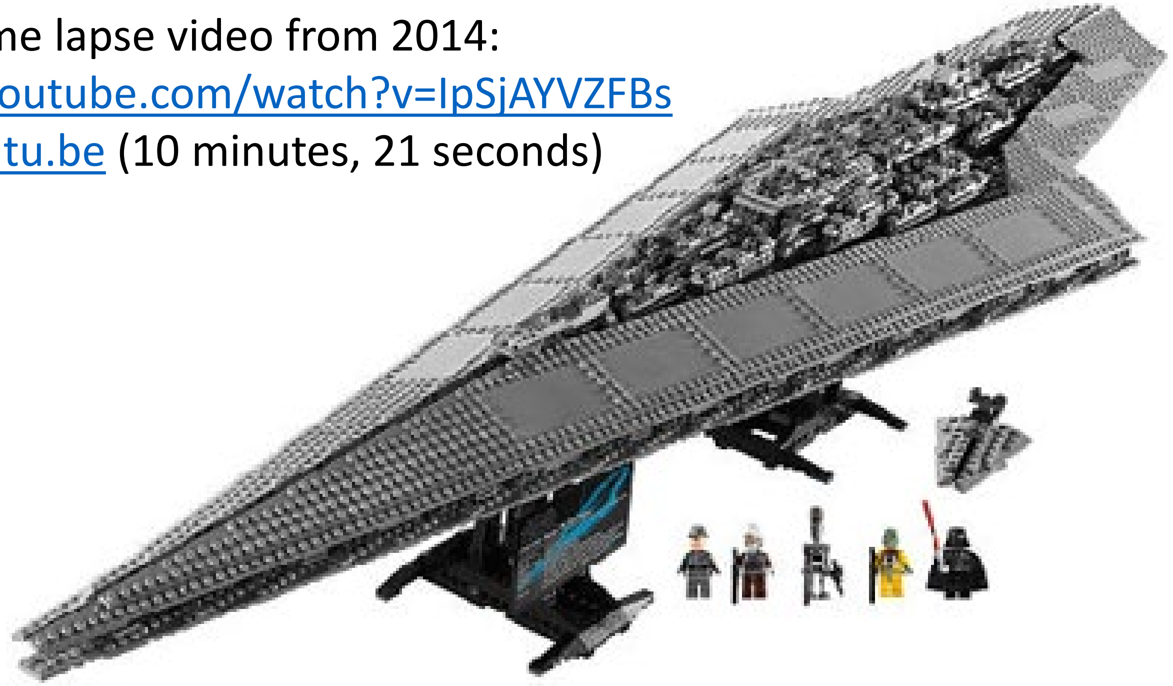
The Mathematics of LEGO Bricks

Professor Steven Miller (Math/Stats); TAs Cameron and Kayla Miller

https://web.williams.edu/Mathematics/sjmiller/public_html/legos

Article about previous courses **(translation generously done by Antony Kim: https://docs.google.com/document/d/1gVixldnb9FPOIumq6y2qBQhb9W--R8OuKUqMHXfD6Vk/edit)**

AWESOME time lapse video from 2014:
http://www.youtube.com/watch?v=IpSjAYVZFBs&feature=youtu.be (10 minutes, 21 seconds)

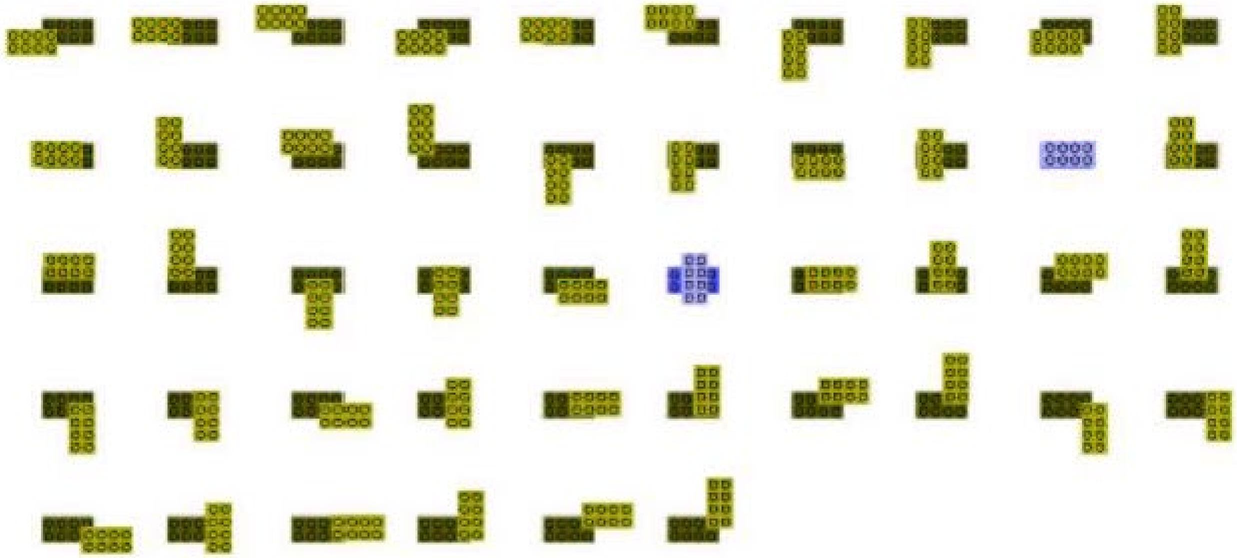


How many ways are there to stack six 2×4 bricks?

Before you count need to decide *what* to count.

http://c-mt.dk/counting/images/modelsintrosmall.png

Require right angles; below is two 2×4:



http://www.math.ku.dk/~eilers/46.jpg

Require right angles; below is two 2×4:

http://www.math.ku.dk/~eilers/46.jpg

Leads to chirality, tic-tac-toe....