Distribution of Summands in Zeckendorf Decompositions

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and Combinatorial Number Theory
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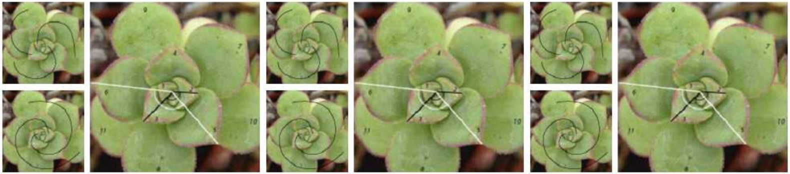
Introduction

Goals of the Talk

- Overview of previous results.
- Highlight techniques: combinatorics, generating fns.
- Interesting generalizations.
- Open problems.



Joint with Olivia Beckwith, Amanda Bower, Louis Gaudet, Rachel Insoft, Shiyu Li, Philip Tosteson.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$.

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Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example:
$2012 = 1597 + 377 + 34 + 3 + 1 = F_{16} + F_{13} + F_8 + F_3 + F_1$.

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Old Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.

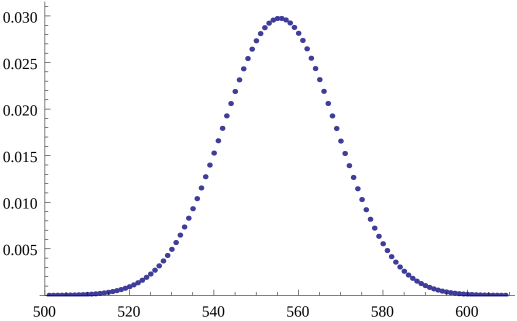

Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

New Results: Bulk Gaps: $m \in [F_n, F_{n+1})$ and $\phi = \frac{1+\sqrt{5}}{2}$

$$m \;=\; \sum_{j=1}^{k(m)=n} F_{i_j}, \quad \nu_{m;n}(x) \;=\; \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Theorem (Zeckendorf Gap Distribution)

Gap measures $\nu_{m;n}$ converge almost surely to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.

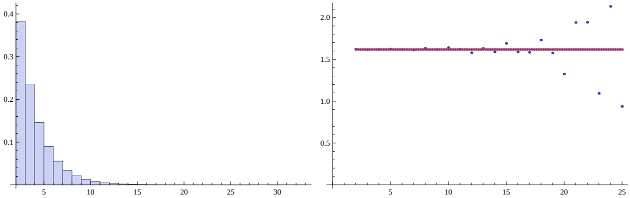

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{2010} \approx 10^{208}$.

New Results: Longest Gap

Theorem (Longest Gap)

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\mathrm{Prob}\left(L_n(m) \leq f(n)\right) \ \approx \ e^{-e^{\log n - f(n)/\log \phi}}$$

Immediate Corollary: If $f(n)$ grows ***slower*** or ***faster*** than $\log n / \log \phi$, then $\mathrm{Prob}(L_n(m) \leq f(n))$ goes to **0** or ***1***, respectively.

Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

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Let $p_{n,k} = \#\left\{N \in [F_n, F_{n+1})\text{: the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\right\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

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$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \; i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1, \; d_j := i_j - i_{j-1} - 2 \; (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1, \; d_j \geq 0.$$

Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1 \; + \; k-1}{k-1} = \binom{n-k}{k-1}$.

Gaps in the Bulk

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

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Can ask similar questions about binary or other expansions: $2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2$.

Main Results

Theorem (Base B Gap Distribution)

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, *and for* $k \geq 1$, $P(k) = c_B B^{-k}$, *with* $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution)

For Zeckendorf decompositions, $P(k) = \frac{1}{\phi^k}$ *for* $k \geq 2$, *with* $\phi = \frac{1+\sqrt{5}}{2}$ *the golden mean.*

Main Results

Theorem

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ *be a positive linear recurrence of length* L *where* $c_i \geq 1$ *for all* $1 \leq i \leq L$*. Then* $P(j) =$

$$\begin{cases} 1 - (\frac{a_1}{C_{Lek}})(\lambda_1^{-n+2} - \lambda_1^{-n+1} + 2\lambda_1^{-1} + a_1^{-1} - 3) & j = 0 \\ \lambda_1^{-1}(\frac{1}{C_{Lek}})(\lambda_1(1 - 2a_1) + a_1) & j = 1 \\ (\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-j} & j \geq 2 \end{cases}$$

Proof of Fibonacci Result

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$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

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For the indices greater than $i + k$: $F_{n-k-i-2}$ choices. Why? Have F_n, don't have F_{i+k+1}. Like Zeckendorf with potential summands $F_{i+k+2}, \dots, F_n$. Shifting, like summands $F_1, \dots, F_{n-k-i-1}$, giving $F_{n-k-i-2}$.

Determining $P(k)$ for $k \geq 2$

$$\sum_{i=1}^{n-k} X_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2}$$

- $\sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$ is the x^{n-k-3} coefficient of $(g(x))^2$, where $g(x)$ is the generating function of the Fibonaccis.

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- Alternatively, use Binet's formula and get sums of geometric series.

$P(k) = C/\phi^k$ for a constant C, so $P(k) = 1/\phi^k$.

Proof sketch of almost sure convergence

- $m = \sum_{j=1}^{k(m)} F_{i_j}$,
  $\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right)$.
- $\mu_{m,n}(t) \ = \ \int x^t \mathrm{d}\nu_{m;n}(x)$.
- Show $\mathbb{E}_m[\mu_{m;n}(t)]$ equals average gap moments, $\mu(t)$.
- Show $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^2]$ and $\mathbb{E}_m[(\mu_{m;n}(t) - \mu(t))^4]$ tend to zero.

Key ideas: (1) Replace $k(m)$ with average (Gaussianity); (2) use $X_{i,i+g_1,j,j+g_2}$.

Longest Gap

Fibonacci Case Generating Function

$G_{n,k,f}$ be the number of $m \in [F_n, F_{n+1})$ with k nonzero summands and all gaps less than $f(n)$.

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Let $m = F_n + F_{n-g_1} + F_{n-g_1-g_2} + \cdots + F_{n-g_1-\cdots-g_{n-1}}$, then

- Each gap is ≥ 2.
- Each gap is $< f(n)$.
- The sum of the gaps of x is $\leq n$.

Gaps uniquely identify m by Zeckendorf's Theorem.

The Combinatorics

$G_{n,k,f}$ is the n^{th} coefficient of

$$\frac{1}{1-x}\left[x^2+\cdots+x^{f(n)-2}\right]^{k-1} = \frac{x^{2(k-1)}}{1-x}\left(\frac{1-x^{f(n)-3}}{1-x}\right)^{k-1}.$$

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For fixed k hard to analyze, but only care about sum over k.

The Generating Function

Sum over k gives number of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$, call it $G_{n,f}$.

It's the n^{th} coefficient (up to potentially small algebra errors!) of

$$F(x) = \frac{1}{1-x}\sum_{k=1}^{\infty}\left(\frac{x^2 - x^{f-2}}{1-x}\right)^{k-1} = \frac{x}{1-x-x^2+x^{f(n)}}.$$

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Use **partial fractions** and **Rouché** to find the CDF.

Partial Fractions

Write the roots of $x^f - x^2 - x - 1$ as $\{\alpha_i\}_{i=1}^{f}$, generating function is

$$F(x) = \frac{x}{1 - x - x^2 + x^{f(n)}} = \sum_{i=1}^{f(n)} \frac{-\alpha_i}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i} \sum_{j=1}^{\infty} \left(\frac{x}{\alpha_i}\right)^j.$$

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Take the n^{th} coefficient to find the number of m with gaps less than $f(n)$.

Partial Fractions

Divide the number of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$ by the number of m, which is
$F_{n+1} - F_n = F_{n-1} = 5^{-1/2}\left(\phi^{n-1} - (1/\phi)^{n-1}\right)$.

Theorem

The proportion of $m \in [F_n, F_{n+1})$ with $L(x) < f(n)$ is exactly

$$\sum_{i=1}^{f(n)} \frac{-\sqrt{5}(\alpha_i)}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i} \left(\frac{1}{\alpha_i}\right)^{n+1} \frac{1}{(\phi^n - (-1/\phi)^n)}$$

Now, we find out about the roots of $x^f - x^2 - x + 1$.

Rouché and Roots

When $f(n)$ is large $z^{f(n)}$ is very small, for $|z| < 1$. Thus, by Rouché's theorem from complex analysis:

Lemma

For $f \in \mathbb{N}$ and $f \geq 4$, the polynomial $p_f(z) = z^f - z^2 - z + 1$ has exactly one root z_f with $|z_f| < .9$. Further, $z_f \in \mathbb{R}$ and $z_f = \frac{1}{\phi} + \left|\frac{z_f^f}{z_f + \phi}\right|$, so as $f \to \infty$, z_f converges to $\frac{1}{\phi}$.

We only care about the **smallest root**.

Getting the CDF

As f grows, only one root goes to $1/\phi$. The other roots don't matter. So,

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Theorem

If $\lim_{n\to\infty} f(n) = \infty$, the proportion of m with $L(m) < f(n)$ is, as $n \to \infty$

$$\lim_{n\to\infty} (\phi z_f)^{-n} = \lim_{n\to\infty} \left(1 + \left|\frac{\phi z_f^{f(n)}}{\phi + z_f}\right|\right)^{-n}.$$

If $f(n)$ is bounded, then $P_f = 0$.

Take logarithms, Taylor expand, result follows from algebra.

References

References

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Appendix:
Gaussian Behavior

Generalizing Lekkerkerker: Erdos-Kac type result

Theorem (KKMW 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian.

Sketch of proof: Use Stirling's formula,

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

to approximates binomial coefficients, after a few pages of algebra find the probabilities are approximately Gaussian.

(Sketch of the) Proof of Gaussianity

The probability density for the number of Fibonacci numbers that add up to an integer in $[F_n, F_{n+1})$ is $f_n(k) = \binom{n-1-k}{k}/F_{n-1}$. Consider the density for the $n+1$ case. Then we have, by Stirling

$$\begin{aligned} f_{n+1}(k) &= \binom{n-k}{k}\frac{1}{F_n} \\ &= \frac{(n-k)!}{(n-2k)!k!}\frac{1}{F_n} = \frac{1}{\sqrt{2\pi}}\frac{(n-k)^{n-k+\frac{1}{2}}}{k^{(k+\frac{1}{2})}(n-2k)^{n-2k+\frac{1}{2}}}\frac{1}{F_n} \end{aligned}$$

plus a lower order correction term.

Also we can write $F_n = \frac{1}{\sqrt{5}}\phi^{n+1} = \frac{\phi}{\sqrt{5}}\phi^n$ for large n, where ϕ is the golden ratio (we are using relabeled Fibonacci numbers where $1 = F_1$ occurs once to help dealing with uniqueness and $F_2 = 2$). We can now split the terms that exponentially depend on n.

$$f_{n+1}(k) = \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}\right)\left(\phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}\right).$$

Define

$$N_n = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}, \quad S_n = \phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}.$$

Thus, write the density function as

$$f_{n+1}(k) = N_n S_n$$

where N_n is the first term that is of order $n^{-1/2}$ and S_n is the second term with exponential dependence on n.

(Sketch of the) Proof of Gaussianity

Model the distribution as centered around the mean by the change of variable $k = \mu + x\sigma$ where μ and σ are the mean and the standard deviation, and depend on n. The discrete weights of $f_n(k)$ will become continuous. This requires us to use the change of variable formula to compensate for the change of scales:

$$f_n(k)dk = f_n(\mu + \sigma x)\sigma dx.$$

Using the change of variable, we can write N_n as

$$\begin{aligned} N_n &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n-k}{k(n-2k)}}\frac{\phi}{\sqrt{5}} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-k/n}{(k/n)(1-2k/n)}}\frac{\sqrt{5}}{\phi} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-(\mu+\sigma x)/n}{((\mu+\sigma x)/n)(1-2(\mu+\sigma x)/n)}}\frac{\sqrt{5}}{\phi} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C-y}{(C+y)(1-2C-2y)}}\frac{\sqrt{5}}{\phi} \end{aligned}$$

where $C = \mu/n \approx 1/(\phi+2)$ (note that $\phi^2 = \phi + 1$) and $y = \sigma x/n$. But for large n, the y term vanishes since $\sigma \sim \sqrt{n}$ and thus $y \sim n^{-1/2}$. Thus

$$N_n \approx \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C}{C(1-2C)}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{(\phi+1)(\phi+2)}{\phi}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{5(\phi+2)}{\phi}} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

since $\sigma^2 = n\frac{\phi}{5(\phi+2)}$.

(Sketch of the) Proof of Gaussianity

For the second term S_n, take the logarithm and once again change variables by $k = \mu + x\sigma$,

$$\begin{aligned}
\log(S_n) &= \log\left(\phi^{-n}\frac{(n-k)^{(n-k)}}{k^k(n-2k)^{(n-2k)}}\right) \\
&= -n\log(\phi) + (n-k)\log(n-k) - (k)\log(k) \\
&\quad - (n-2k)\log(n-2k) \\
&= -n\log(\phi) + (n-(\mu+x\sigma))\log(n-(\mu+x\sigma)) \\
&\quad - (\mu+x\sigma)\log(\mu+x\sigma) \\
&\quad - (n-2(\mu+x\sigma))\log(n-2(\mu+x\sigma)) \\
&= -n\log(\phi) \\
&\quad + (n-(\mu+x\sigma))\left(\log(n-\mu) + \log\left(1-\frac{x\sigma}{n-\mu}\right)\right) \\
&\quad - (\mu+x\sigma)\left(\log(\mu) + \log\left(1+\frac{x\sigma}{\mu}\right)\right) \\
&\quad - (n-2(\mu+x\sigma))\left(\log(n-2\mu) + \log\left(1-\frac{x\sigma}{n-2\mu}\right)\right) \\
&= -n\log(\phi) \\
&\quad + (n-(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-1\right) + \log\left(1-\frac{x\sigma}{n-\mu}\right)\right) \\
&\quad - (\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right) \\
&\quad - (n-2(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-2\right) + \log\left(1-\frac{x\sigma}{n-2\mu}\right)\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

Note that, since $n/\mu = \phi + 2$ for large n, the constant terms vanish. We have $\log(S_n)$

$$\begin{aligned}
= \quad & -n\log(\phi) + (n-k)\log\left(\frac{n}{\mu}-1\right) - (n-2k)\log\left(\frac{n}{\mu}-2\right) + (n-(\mu+x\sigma))\log\left(1-\frac{x\sigma}{n-\mu}\right) \\
& -(\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right) - (n-2(\mu+x\sigma))\log\left(1-\frac{x\sigma}{n-2\mu}\right) \\
= \quad & -n\log(\phi) + (n-k)\log(\phi+1) - (n-2k)\log(\phi) + (n-(\mu+x\sigma))\log\left(1-\frac{x\sigma}{n-\mu}\right) \\
& -(\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right) - (n-2(\mu+x\sigma))\log\left(1-\frac{x\sigma}{n-2\mu}\right) \\
= \quad & n(-\log(\phi) + \log\left(\phi^2\right) - \log(\phi)) + k(\log(\phi^2) + 2\log(\phi)) + (n-(\mu+x\sigma))\log\left(1-\frac{x\sigma}{n-\mu}\right) \\
& -(\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right) - (n-2(\mu+x\sigma))\log\left(1-2\frac{x\sigma}{n-2\mu}\right) \\
= \quad & (n-(\mu+x\sigma))\log\left(1-\frac{x\sigma}{n-\mu}\right) - (\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right) \\
& -(n-2(\mu+x\sigma))\log\left(1-2\frac{x\sigma}{n-2\mu}\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

Finally, we expand the logarithms and collect powers of $x\sigma/n$.

$$\begin{aligned}
\log(S_n) &= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n-\mu}-\frac{1}{2}\left(\frac{x\sigma}{n-\mu}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\mu}-\frac{1}{2}\left(\frac{x\sigma}{\mu}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-2\frac{x\sigma}{n-2\mu}-\frac{1}{2}\left(2\frac{x\sigma}{n-2\mu}\right)^2+\dots\right)\\
&= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}-\frac{1}{2}\left(\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}\right)^2+\dots\right)\\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\frac{n}{\phi+2}}-\frac{1}{2}\left(\frac{x\sigma}{\frac{n}{\phi+2}}\right)^2+\dots\right)\\
&\quad -(n-2(\mu+x\sigma))\left(-\frac{2x\sigma}{n\frac{\phi}{\phi+2}}-\frac{1}{2}\left(\frac{2x\sigma}{n\frac{\phi}{\phi+2}}\right)^2+\dots\right)\\
&= \frac{x\sigma}{n}n\left(-\left(1-\frac{1}{\phi+2}\right)\frac{(\phi+2)}{(\phi+1)}-1+2\left(1-\frac{2}{\phi+2}\right)\frac{\phi+2}{\phi}\right)\\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n\left(-2\frac{\phi+2}{\phi+1}+\frac{\phi+2}{\phi+1}+2(\phi+2)-(\phi+2)+4\frac{\phi+2}{\phi}\right)\\
&\quad +O\left(n\,(x\sigma/n)^3\right)
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

$$\begin{aligned}
\log(S_n) &= \frac{x\sigma}{n} n \left(-\frac{\phi+1}{\phi+2}\frac{\phi+2}{\phi+1} - 1 + 2\frac{\phi}{\phi+2}\frac{\phi+2}{\phi} \right) \\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n(\phi+2)\left(-\frac{1}{\phi+1} + 1 + \frac{4}{\phi}\right) \\
&\quad + O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4}{\phi(\phi+1)} + 1\right) + O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4+2\phi+1}{\phi(\phi+1)}\right) + O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}x^2\sigma^2\left(\frac{5(\phi+2)}{\phi n}\right) + O\left(n\left(x\sigma/n\right)^3\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

But recall that

$$\sigma^2 = \frac{\phi n}{5(\phi + 2)}.$$

Also, since $\sigma \sim n^{-1/2}$, $n\left(\frac{x\sigma}{n}\right)^3 \sim n^{-1/2}$. So for large n, the $O\left(n\left(\frac{x\sigma}{n}\right)^3\right)$ term vanishes. Thus we are left with

$$\begin{aligned} \log S_n &= -\frac{1}{2}x^2 \\ S_n &= e^{-\frac{1}{2}x^2}. \end{aligned}$$

Hence, as n gets large, the density converges to the normal distribution:

$$\begin{aligned} f_n(k)dk &= N_n S_n dk \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}x^2} \sigma dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \end{aligned}$$

□

Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \ n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf: Every positive integer can be written uniquely as $\sum a_i H_i$ with natural constraints on the a_i's (e.g. cannot use the recurrence relation to remove any summand).
- Lekkerkerker
- Central Limit Type Theorem

Generalizing Lekkerkerker

Generalized Lekkerkerker's Theorem

The average number of summands in the generalized Zeckendorf decomposition for integers in $[H_n, H_{n+1})$ tends to $Cn + d$ as $n \to \infty$, where $C > 0$ and d are computable constants determined by the c_i's.

$$C = -\frac{y'(1)}{y(1)} = \frac{\sum_{m=0}^{L-1}(s_m + s_{m+1} - 1)(s_{m+1} - s_m)y^m(1)}{2\sum_{m=0}^{L-1}(m+1)(s_{m+1} - s_m)y^m(1)}.$$

$$s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m.$$

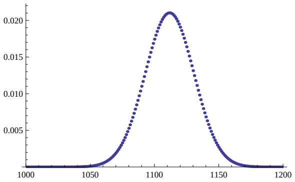
$y(x)$ is the root of $1 - \sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}$.

$y(1)$ is the root of $1 - c_1 y - c_2 y^2 - \cdots - c_L y^L$.

Central Limit Type Theorem

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands, i.e., $a_1 + a_2 + \cdots + a_m$ in the generalized Zeckendorf decomposition $\sum_{i=1}^{m} a_i H_i$ for integers in $[H_n, H_{n+1})$ is Gaussian.



Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$: $a_i \in \{0, 1, \dots, 9\}$ $(1 \leq i < m)$, $a_m \in \{1, \dots, 9\}$.
- For $N \in [H_n, H_{n+1})$, $m = n$, i.e., first term is $a_n H_n = a_n 10^{n-1}$.
- A_i: the corresponding random variable of a_i. The A_i's are independent.
- For large n, the contribution of A_n is immaterial. A_i $(1 \leq i < n)$ are identically distributed random variables with mean 4.5 and variance 8.25.
- Central Limit Theorem: $A_2 + A_3 + \cdots + A_n \to$ Gaussian with mean $4.5n + O(1)$ and variance $8.25n + O(1)$.

Far-difference Representation

Theorem (Alpert, 2009) (Analogue to Zeckendorf)

Every integer can be written uniquely as a sum of the $\pm F_n$'s, such that every two terms of the same (opposite) sign differ in index by at least 4 (3).

Example: $1900 = F_{17} - F_{14} - F_{10} + F_6 + F_2$.

K: # of positive terms, L: # of negative terms.

Generalized Lekkerkerker's Theorem

As $n \to \infty$, $E[K]$ and $E[L] \to n/10$.
$E[K] - E[L] = \varphi/2 \approx .809$.

Central Limit Type Theorem

As $n \to \infty$, K and L converges to a bivariate Gaussian.

- $\text{corr}(K, L) = -(21 - 2\varphi)/(29 + 2\varphi) \approx -.551$, $\varphi = \frac{\sqrt{5}+1}{2}$.

Generating Function (Example: Binet's Formula)

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$ (1)
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$\begin{aligned}
(1) \quad &\Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2\sum_{n\geq 1} F_n x^n \\
&\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x) \\
&\Rightarrow g(x) = x/(1 - x - x^2).
\end{aligned}$$

Partial Fraction Expansion (Example: Binet's Formula)

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow \; g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}}\left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x}\right).$$

Coefficient of x^n (power series expansion):

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).

Differentiating Identities and Method of Moments

- Differentiating identities
  Example: Given a random variable X such that
  $\Pr(X = 1) = \frac{1}{2}$, $\Pr(X = 2) = \frac{1}{4}$, $\Pr(X = 3) = \frac{1}{8}$, ...
  then what's the mean of X (i.e., $E[X]$)?
  Solution: Let $f(x) = \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \cdots = \frac{1}{1-x/2} - 1.$
  $$f'(x) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4}x + 3 \cdot \frac{1}{8}x^2 + \cdots.$$
  $$f'(1) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \cdots = E[X].$$
- Method of moments: Random variables $X_1, X_2, \dots$.
  If the ℓ^{th} moment $E[X_n^\ell]$ converges to that of the standard normal distribution $(\forall \ell)$, then X_n converges to a Gaussian.

  Standard normal distribution:
  $2m^{\text{th}}$ moment: $(2m-1)!! = (2m-1)(2m-3)\cdots 1$,
  $(2m-1)^{\text{th}}$ moment: 0.

New Approach: Case of Fibonacci Numbers

$p_{n,k} = \#\left\{N \in [F_n, F_{n+1})\text{: the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\right\}$.

- Recurrence relation:
  $N \in [F_{n+1}, F_{n+2})$: $N = F_{n+1} + F_t + \cdots$, $t \le n-1$.

$$\begin{aligned} p_{n+1,k+1} &= p_{n-1,k} + p_{n-2,k} + \cdots \\ p_{n,k+1} &= p_{n-2,k} + p_{n-3,k} + \cdots \\ \Rightarrow p_{n+1,k+1} &= p_{n,k+1} + p_{n-1,k}. \end{aligned}$$

- Generating function: $\sum_{n,k>0} p_{n,k} x^k y^n = \frac{y}{1-y-xy^2}$.
- Partial fraction expansion:

$$\frac{y}{1-y-xy^2} = -\frac{y}{y_1(x)-y_2(x)}\left(\frac{1}{y-y_1(x)} - \frac{1}{y-y_2(x)}\right)$$

where $y_1(x)$ and $y_2(x)$ are the roots of $1-y-xy^2 = 0$.

Coefficient of y^n: $g(x) = \sum_{k>0} p_{n,k} x^k$.

New Approach: Case of Fibonacci Numbers (Continued)

K_n: the corresponding random variable associated with k.

$g(x) = \sum_{k>0} p_{n,k} x^k$.

- Differentiating identities:

  $g(1) = \sum_{k>0} p_{n,k} = F_{n+1} - F_n,$

  $g'(x) = \sum_{k>0} k p_{n,k} x^{k-1}$, $g'(1) = g(1) E[K_n]$,

  $(xg'(x))' = \sum_{k>0} k^2 p_{n,k} x^{k-1}$,

  $(xg'(x))' \mid_{x=1} = g(1) E[K_n^2]$,

  $\left(x\,(xg'(x))'\right)' \mid_{x=1} = g(1) E[K_n^3]$, ...

  Similar results hold for the centralized K_n:

  $K_n' = K_n - E[K_n]$.

- Method of moments (for normalized K_n'):

  $E[(K_n')^{2m}]/(SD(K_n'))^{2m} \to (2m-1)!!$,

  $E[(K_n')^{2m-1}]/(SD(K_n'))^{2m-1} \to 0$. $\Rightarrow K_n \to$ Gaussian.

New Approach: General Case

Let $p_{n,k} = \#\{N \in [H_n, H_{n+1})$: the generalized Zeckendorf decomposition of N has exactly k summands$\}$.

- Recurrence relation:

  Fibonacci: $p_{n+1,k+1} = p_{n,k+1} + p_{n,k}$.

  General: $p_{n+1,k} = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} p_{n-m,k-j}$.

  where $s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m$.

- Generating function:

  Fibonacci: $\frac{y}{1-y-xy^2}$.

  General:

$$\frac{\sum_{n\le L} p_{n,k}x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n<L-m} p_{n,k}x^k y^n}{1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}}.$$

New Approach: General Case (Continued)

- Partial fraction expansion:

Fibonacci: $-\frac{y}{y_1(x)-y_2(x)}\left(\frac{1}{y-y_1(x)}-\frac{1}{y-y_2(x)}\right).$

General:

$$-\frac{1}{\sum_{j=s_{L-1}}^{s_L-1} x^j}\sum_{i=1}^{L}\frac{B(x,y)}{(y-y_i(x))\prod_{j\neq i}(y_j(x)-y_i(x))}.$$

$$B(x,y)=\sum_{n\leq L} p_{n,k}x^k y^n-\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}\sum_{n<L-m} p_{n,k}x^k y^n,$$

$$y_i(x)\text{: root of } 1-\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}=0.$$

Coefficient of y^n: $g(x)=\sum_{n,k>0} p_{n,k}x^k$.

- Differentiating identities
- Method of moments: implies $K_n \to$ Gaussian.