Mind the Gap: Distribution of Gaps in Generalized Zeckendorf Decompositions

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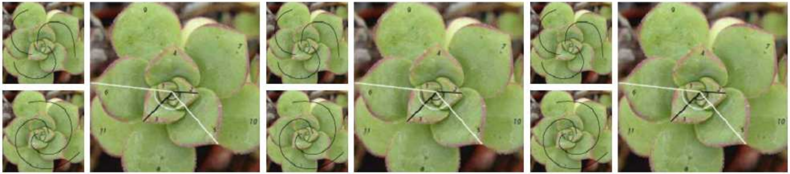
Introduction

Goals of the Talk

- Combinatorial perspective.
- Asking for help: completing elementary proof.
- New results on longest gap.
- Techniques: Generating fns, partial fractions, Rouche.



Joint with Olivia Beckwith, Iddo Ben-Ari, Amanda Bower, Louis Gaudet, Rachel Insoft, Shiyu Li, Philip Tosteson.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $2013 = 1597 + 377 + 34 + 5 = F_{16} + F_{13} + F_8 + F_4$.

Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Old Results

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.

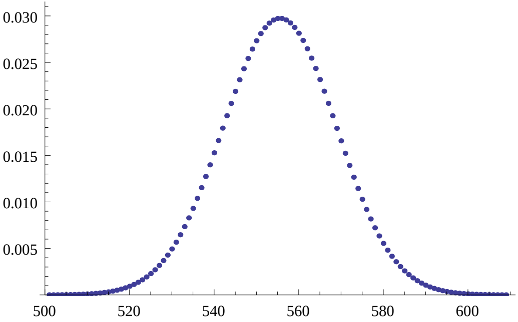


Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

New Results: Bulk Gaps: $m \in [F_n, F_{n+1})$ and $\phi = \frac{1+\sqrt{5}}{2}$

$$m \;=\; \sum_{j=1}^{k(m)=n} F_{i_j}, \quad \nu_{m;n}(x) \;=\; \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Theorem (Zeckendorf Gap Distribution)

Gap measures $\nu_{m;n}$ converge to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.

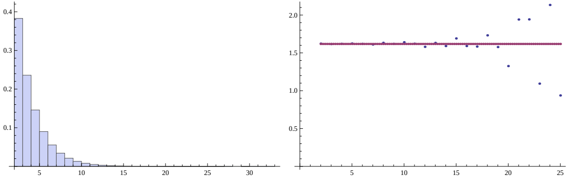


Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{2010} \approx 10^{208}$.

New Results: Longest Gap

Fair coin: largest gap tightly concentrated around $\log n / \log 2$.

Theorem (Longest Gap)

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\text{Prob}\left(L_n(m) \le f(n)\right) \;\approx\; e^{-e^{\log n - f(n) \cdot \log \phi}}$$

- $\mu_n = \frac{\log\left(\frac{\phi^2}{\phi^2+1)}n\right)}{\log \phi} + \frac{\gamma}{\log \phi} - \frac{1}{2} +$ Small Error.

- If $f(n)$ grows ***slower*** (resp. ***faster***) than $\log n / \log \phi$, then $\text{Prob}(L_n(m) \le f(n))$ goes to **0** (resp. ***1***).

Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

Let $p_{n,k} = \#\{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n,\ i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1,\ d_j := i_j - i_{j-1} - 2\ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1,\ d_j \geq 0.$$

Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1\ +\ k-1}{k-1} = \binom{n-k}{k-1}$.

Gaps in the Bulk

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \dots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

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Can ask similar questions about binary or other expansions: $2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2$.

Main Result

Theorem (Distribution of Bulk Gaps (SMALL 2012))

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ be a positive linear recurrence of length L where $c_i \geq 1$ for all $1 \leq i \leq L$. Then

$$P(j) = \begin{cases} 1 - (\frac{a_1}{C_{Lek}})(2\lambda_1^{-1} + a_1^{-1} - 3) & : j = 0 \\ \lambda_1^{-1}(\frac{1}{C_{Lek}})(\lambda_1(1 - 2a_1) + a_1) & : j = 1 \\ (\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-j} & : j \geq 2. \end{cases}$$

Special Cases

Theorem (Base B Gap Distribution (SMALL 2011))

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$, $P(k) = c_B B^{-k}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution (SMALL 2011))

For Zeckendorf decompositions, $P(k) = 1/\phi^k$ for $k \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.

Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

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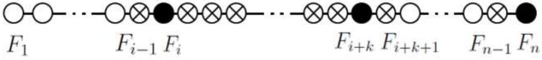
$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Calculating $X_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?



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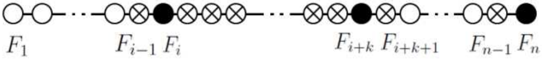

For the indices less than i: F_{i-1} choices. Why? Have F_i as largest summand and follows by Zeckendorf: $\#[F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}$.

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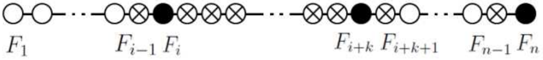

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For the indices greater than $i+k$: $F_{n-k-i-2}$ choices. Why? Shift. Choose summands from $\{F_1, \dots, F_{n-k-i+1}\}$ with $F_1, F_{n-k-i+1}$ chosen. Decompositions with largest summand $F_{n-k-i+1}$ minus decompositions with largest summand F_{n-k-i}.

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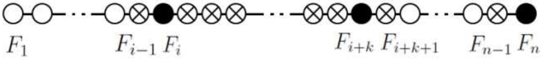

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So total choices number of choices is $F_{n-k-2-i}F_{i-1}$.

Determining $P(k)$

Recall

$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Use Binet's formula. Sums of geometric series: $P(k) = 1/\phi^k$.

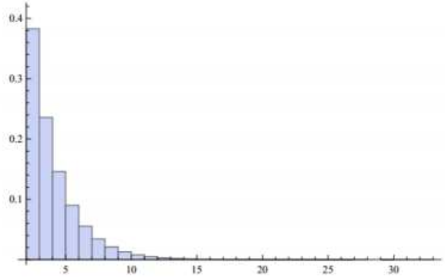

Figure: Distribution of summands in $[F_{1000}, F_{1001})$.

Individual Gaps

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Theorem (Distribution of Individual Gaps (SMALL 2012))

Gap measures $\nu_{m;n}$ converge to average gap measure.

Proof Sketch of Individual Gap Measures

- $\mu_{m,n}(t) = \int x^t \mathrm{d}\nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} (i_j - i_{j-1})^t.$

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Key ideas:
(1) Replace $k(m)$ with average (Gaussianity);
(2) use $X_{i,i+g_1,j,j+g_2}$.

Future Research

Future Research

- Finish elementary proof of convergence of individual gap measures (maybe probabilities instead of moments). Email sjm1@williams.edu if interested.

- Extend to recurrences with coefficients that can be zero: SMALL '13.

- Generalize to signed decompositions, ℓ largest gaps, ... SMALL '13.

Longest Gap

Fibonacci Case Generating Function

$G_{n,k,f}$ be the number of $m \in [F_n, F_{n+1})$ with k nonzero summands and all gaps less than $f(n)$.

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Let $m = F_n + F_{n-g_1} + F_{n-g_1-g_2} + \cdots + F_{n-g_1-\cdots-g_{n-1}}$, then

- Each gap is ≥ 2.
- Each gap is $< f(n)$.
- The sum of the gaps of x is $\leq n$.

Gaps uniquely identify m by Zeckendorf's Theorem.

The Combinatorics

$G_{n,k,f}$ is the n^{th} coefficient of

$$\frac{1}{1-x}\left[x^2+\cdots+x^{f(n)-2}\right]^{k-1}=\frac{x^{2(k-1)}}{1-x}\left(\frac{1-x^{f(n)-3}}{1-x}\right)^{k-1}.$$

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For fixed k hard to analyze, but only care about sum over k.

The Generating Function

Sum over k gives number of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$, call it $G_{n,f}$.

It's the n^{th} coefficient (up to potentially small algebra errors!) of

$$F(x) = \frac{1}{1-x}\sum_{k=1}^{\infty}\left(\frac{x^2 - x^{f-2}}{1-x}\right)^{k-1} = \frac{x}{1-x-x^2+x^{f(n)}}.$$

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Use **partial fractions** and **Rouché's Theorem** to find CDF.

Partial Fractions

Write the roots of $x^f - x^2 - x - 1$ as $\{\alpha_i\}_{i=1}^f$, generating function is

$$F(x) = \frac{x}{1 - x - x^2 + x^{f(n)}} = \sum_{i=1}^{f(n)} \frac{-\alpha_i}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i} \sum_{j=1}^{\infty} \left(\frac{x}{\alpha_i}\right)^j.$$

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Take the n^{th} coefficient to find the number of m with gaps less than $f(n)$.

Partial Fractions

Divide the number of $m \in [F_n, F_{n+1})$ with longest gap $< f(n)$ by the number of m, which is

$$F_{n+1} - F_n \;=\; F_{n-1} \;=\; 5^{-1/2}\left(\phi^{n-1} - (1/\phi)^{n-1}\right).$$

Theorem

The proportion of $m \in [F_n, F_{n+1})$ with $L(x) < f(n)$ is exactly

$$\sum_{i=1}^{f(n)} \frac{-\sqrt{5}(\alpha_i)}{f(n)\alpha_i^{f(n)} - 2\alpha_i^2 - \alpha_i}\left(\frac{1}{\alpha_i}\right)^{n+1}\frac{1}{(\phi^n - (-1/\phi)^n)}$$

Now study the roots of $x^f - x^2 - x + 1$.

Rouché and Roots

When $f(n)$ is large, $z^{f(n)}$ is very small for $|z| < 1$. Thus, by Rouché's theorem:

Lemma

For $f \in \mathbb{N}$ and $f \geq 4$, the polynomial $p_f(z) = z^f - z^2 - z + 1$ has exactly one root z_f with $|z_f| < .9$. Further, $z_f \in \mathbb{R}$ and $z_f = \frac{1}{\phi} + \left|\frac{z_f^f}{z_f + \phi}\right|$, so as $f \to \infty$, z_f converges to $\frac{1}{\phi}$.

We only care about the **smallest root**.

Getting the CDF

As f grows, only one root goes to $1/\phi$. The other roots don't matter. So,

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Theorem

If $\lim_{n\to\infty} f(n) = \infty$, *the proportion of* m *with* $L(m) < f(n)$ *is, as* $n \to \infty$

$$\lim_{n\to\infty} (\phi z_f)^{-n} = \lim_{n\to\infty} \left(1 + \left|\frac{\phi z_f^{f(n)}}{\phi + z_f}\right|\right)^{-n}.$$

If $f(n)$ *is bounded, then* $P_f = 0$.

Take logarithms, Taylor expand, result follows from algebra.

Algebra increases greatly for general recurrence.

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Generalizations

Positive Linear Recurrence Sequences

This method can be greatly generalized to **Positive Linear Recurrence Sequences**: linear recurrences with non-negative coefficients:

$$H_{n+1} = c_1 H_{n-(j_1=0)} + c_2 H_{n-j_2} + \cdots + c_L H_{n-j_L}.$$

Theorem (Zeckendorf's Theorem for *PLRS* recurrences)

*Any $b \in \mathbb{N}$ has a unique **legal** decomposition into sums of H_n, $b = a_1 H_{i_1} + \cdots + a_{i_k} H_{i_k}$.*

Here **legal** reduces to non-adjacency of summands in the Fibonacci case.

Messier Combinatorics

The **number** of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of x^{n-s} in the generating function:

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$$\frac{1}{1-x}\left(c_1 - 1 + c_2 x^{t_2} + \cdots + c_L x^{t_L}\right) \times$$

$$\times \sum_{k \geq 0} \left[\left((c_1 - 1)x^{t_1} + \cdots + (c_L - 1)x^{t_L}\right) \left(\frac{x^{s+1} - x^f}{1-x}\right) + \right.$$

$$\left. + x^{t_1}\left(\frac{x^{s+t_2-t_1+1} - x^f}{1-x}\right) + \cdots + x^{t_{L-1}}\left(\frac{x^{s+t_L-t_{L-1}} + 1 - x^f}{1-x}\right)\right]^k.$$

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The **number** of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of x^{n-s} in the generating function:

$$\frac{1}{1-x}\left(c_1 - 1 + c_2 x^{t_2} + \cdots + c_L x^{t_L}\right) \times$$

$$\times \sum_{k\geq 0}\Bigg[\left((c_1-1)x^{t_1} + \cdots + (c_L-1)x^{t_L}\right)\left(\frac{x^{s+1}-x^f}{1-x}\right) +$$

$$+x^{t_1}\left(\frac{x^{s+t_2-t_1+1}-x^f}{1-x}\right) + \cdots + x^{t_{L-1}}\left(\frac{x^{s+t_L-t_{L-1}}+1-x^f}{1-x}\right)\Bigg]^k.$$

A geometric series!

Let $f > j_L$. The number of $x \in [H_n, H_{n+1})$, with longest gap $< f$ is given by **the coefficient of** s^n in the generating function

$$F(s) = \frac{1 - s^{j_L}}{\mathcal{M}(s) + s^f \mathcal{R}(s)},$$

where

$$\mathcal{M}(s) = 1 - c_1 s - c_2 s^{j_2+1} - \cdots - c_L s^{j_L+1},$$

and

$$\mathcal{R}(s) = c_{j_1+1} s^{j_1} + c_{j_2+1} s^{j_2} + \cdots + (c_{j_L+1} - 1) s^{j_L}.$$

and c_i and j_i are defined **as above**.

The **coefficients** in the **partial fraction** expansion might blow up from multiple roots.

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Theorem (Mean and Variance for "Most Recurrences")

For x in the interval $[H_n, H_{n+1})$, the mean longest gap μ_n and the variance of the longest gap σ_n^2 are given by

$$\mu_n = \frac{\log\left(\frac{\mathcal{R}(\frac{1}{\lambda_1})}{\mathcal{G}(\frac{1}{\lambda_1})}n\right)}{\log \lambda_1} + \frac{\gamma}{\log \lambda_1} - \frac{1}{2} + \textit{Small Error} + \epsilon_1(n),$$

and

$$\sigma_n^2 = \frac{\pi^2}{6 \log \lambda_1} - \frac{1}{12} + \textit{Small Error} + \epsilon_2(n),$$

where $\epsilon_i(n)$ tends to zero in the limit, and Small Error comes from the Euler-Maclaurin Formula.