# Modeling the Vanishing of $L$-functions at the Central Point

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http://web.williams.edu/Mathematics/sjmiller/public_html/

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- Motivation
- Background
- Main question
- The model
- One-level density
- Pair-correlation

## Motivation

**Conjecture (Montgomery-Dyson, 1970s)**

High on the critical line, spacings between

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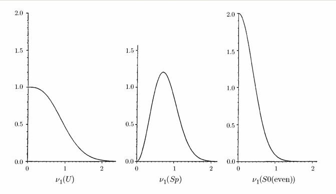
$$\text{zeros of the Riemann zeta function} \longleftrightarrow \text{eigenvalues of the Gaussian Unitary Ensemble.}$$

## Conjecture (Katz-Sarnak, 1990s)

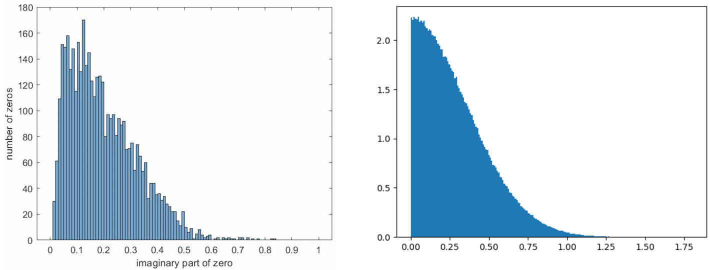
Katz and Sarnak conjectured that the following distributions match in the correct asymptotic limit:

- lowest-lying zeros at the critical point of families of $L$-functions,
- eigenvalues of random matrices from classical compact groups.




Source: N. M. Katz and P. Sarnak, *Zeros of zeta functions and symmetry*, Bulletin of the American Mathematical Society (1) **36** (1999), pages 1-26.

- In 2005, M. noticed a repulsion of the lowest-lying zeros near the central point of a family of even twists of a fixed elliptic curve $L$-function with finite conductor.




Comparison of distribution of lowest zeros for twists of an elliptic curve $L$-function and the corresponding eigenvalues from SO(even)

# An Excised Orthogonal Model

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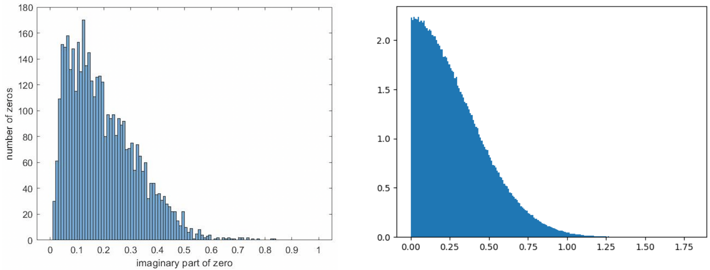


Comparison of distribution of lowest zeros for twists of an elliptic curve $L$-function and the corresponding eigenvalues from SO(even)

- In 2011, E. Duenez, D.K. Huynh, J.P. Keating, M., and N.C. Snaith proposed an excised orthogonal model to capture the behavior of this repulsion in the elliptic curve case.

Can this model be generalized or adapted to $L$-functions arising from cuspidal newforms?

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**Motivating Question**

How accurately do eigenvalues of random matrices from classical compact groups model the lowest-lying zeros of families of $L$-functions associated to a cuspidal newform?

Let $f \in S_k^{\text{new}}(M, \chi_f)$ be a cuspidal newform of level an odd prime $M$, weight $k$, and nebentypus $\chi_f$. Then, $f$ has Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}$$

at the cusp $\infty$.

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Put $\lambda_f(n) = a_f(n)/n^{(k-1)/2}$. Then, for $\text{Re}(s) > 1$, the $L$-function attached to $f$ is given by the Dirichlet series

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$$L(f, s) \; = \; \prod_p \left(1 - \lambda_f(p) p^{-s} + \chi_f(p) p^{-2s}\right)^{-1}.$$

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The functional equation of the completed $L$-function is given by

$$\Lambda(f,s) \;=\; \epsilon_f \Lambda(\overline{f}, 1-s),$$

where $\epsilon_f$ is the root number.

Fix a cuspidal newform $f$, and consider its $L$-function $L(f,s)$. Given a quadratic character $\psi_d$, we create a twist by

$$L(f\otimes\psi_d,s) \;=\; \sum_{n=1}^{\infty}\frac{\psi_d(n)\lambda_f(n)}{n^s} \;=\; \prod_p\left(1-\psi_d(p)\lambda_f(p)p^{-s}+\psi_d(p)\chi_f(p)p^{-2s}\right)^{-1}.$$

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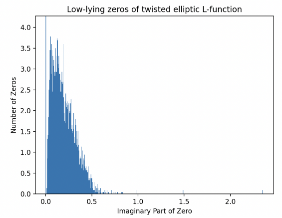
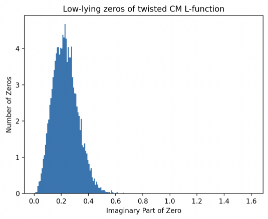
Fix $\Delta\in\{\pm1\}$. We create a family of $L$-functions by taking twists of $L(f,s)$ with positive fundamental discriminants $d\in\mathcal{D}^+$ ranging over

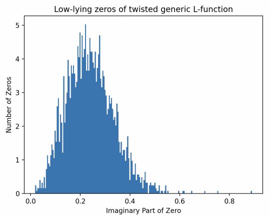
$$\mathcal{D}_f^+(X) \coloneqq \begin{cases} \{d\in\mathcal{D}^+ \mid 0<d\le X,\;\; \psi_d(-M)\epsilon_f \;=\; +1\} & \chi_f \text{ principal, even twists,}\\ \{d\in\mathcal{D}^+ \mid 0<d\le X,\;\; \psi_d(-M)\epsilon_f \;=\; -1\} & \chi_f \text{ principal, odd twists,}\\ \{d\in\mathcal{D}^+ \mid 0<d\le X,\;\; \psi_d(-M) \;=\; \Delta\} & \chi_f \text{ non-principal, } f=\overline{f},\\ \{d\in\mathcal{D}^+ \mid 0<d\le X\} & \chi_f \text{ non-principal, } f\neq\overline{f}. \end{cases}$$

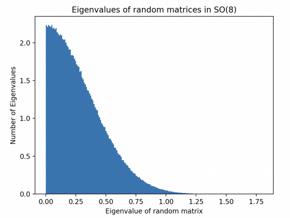
## Motivating question, revisited

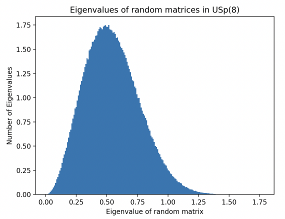
Is there any unexpected behavior that appears when we try to model the lowest-lying zeros of our family?

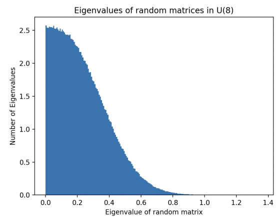


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We computed the one-level density of our family and compared it to that of the groups U, Sp, and SO to determine the model:

$$\begin{aligned}
\text{Principal nebentype, even twists} &\longleftrightarrow \text{SO(even)}\\
\text{Principal nebentype, odd twists} &\longleftrightarrow \text{SO(odd)}\\
\text{Non-principal nebentype and self-dual} &\longleftrightarrow \text{Sp}\\
\text{Generic} &\longleftrightarrow \text{U}
\end{aligned}$$

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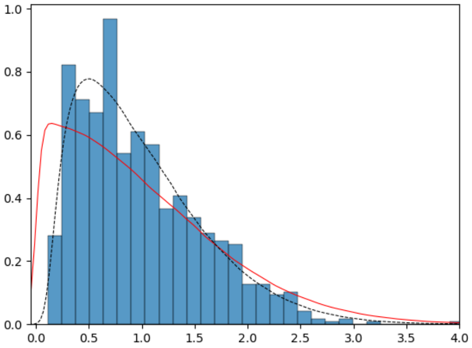
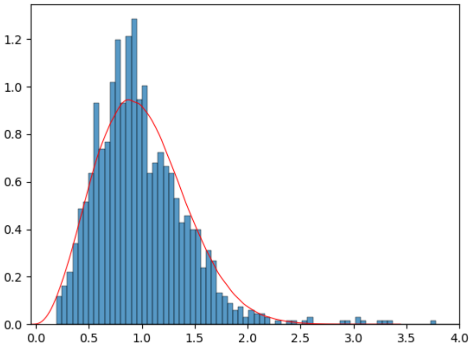
When do we need a cutoff value?

| Lowest zeros (even twists) of 11.2.a.a | Second lowest zeros (odd twists) of 11.2.a.a |
|---|---|
| Eigenvalues from random matrices of SO(18) | Eigenvalues from random matrices of SO(19) |



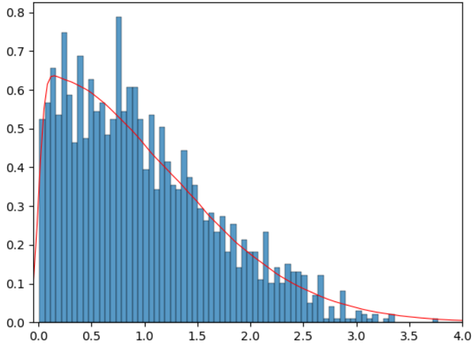



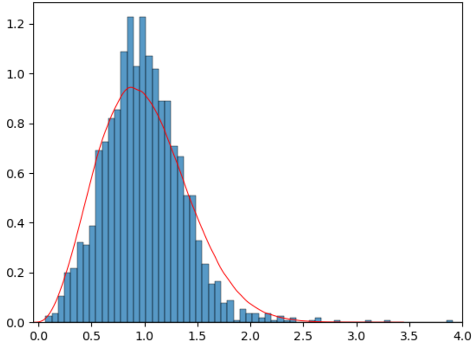
Lowest zeros (even twists) of 5.4.a.a

Eigenvalues from random matrices of SO(18)

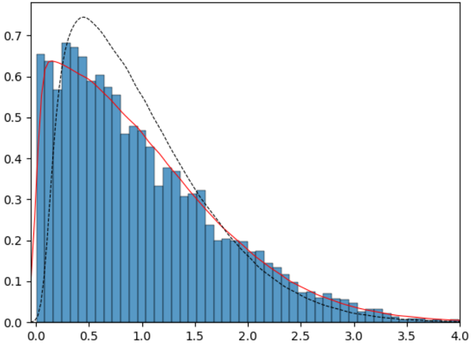
Second lowest zeros (odd twists) of 5.4.a.a

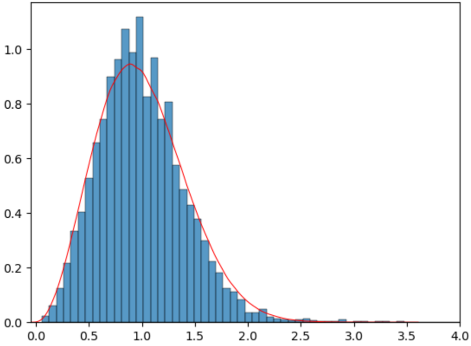
Eigenvalues from random matrices of SO(19)



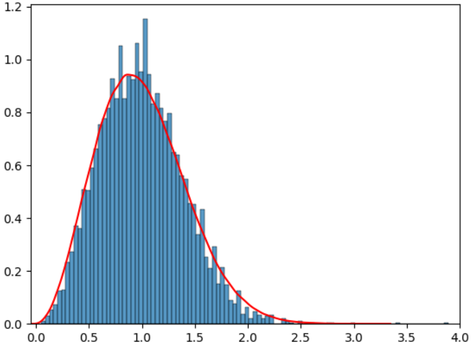
| Lowest zeros (even twists) of 7.4.a.a | Lowest zeros (odd twists) of 7.4.a.a |
| --- | --- |
| Eigenvalues of random matrices of SO(20) | Eigenvalues of random matrices of SO(21) |



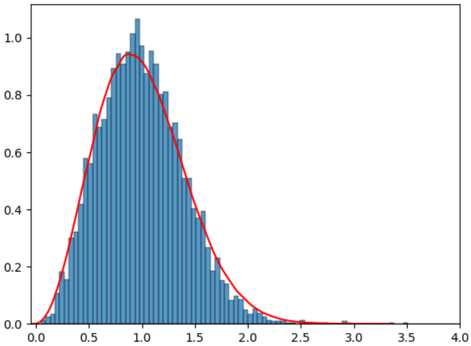
Lowest zeros ($\Delta = +1$) of 3.7.b.a

Eigenvalues of random matrices of Sp(20)



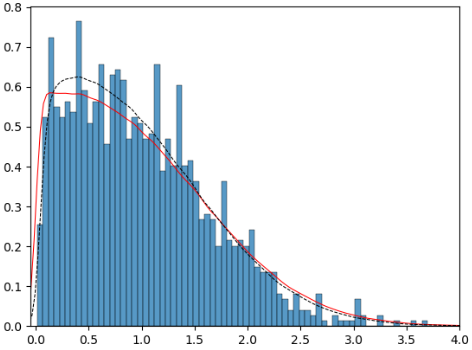

Lowest zeros ($\Delta = -1$) of 7.3.b.a

Eigenvalues of random matrices of Sp(20)



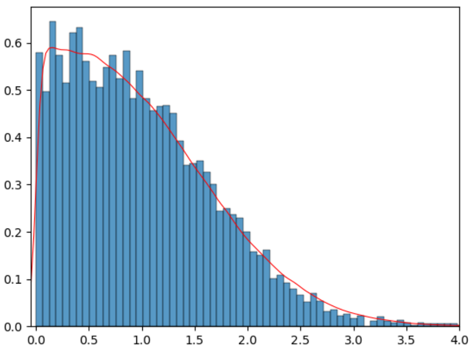
Lowest zeros (twists) of 13.2.e.a

Eigenvalues of random matrices of Sp(20)



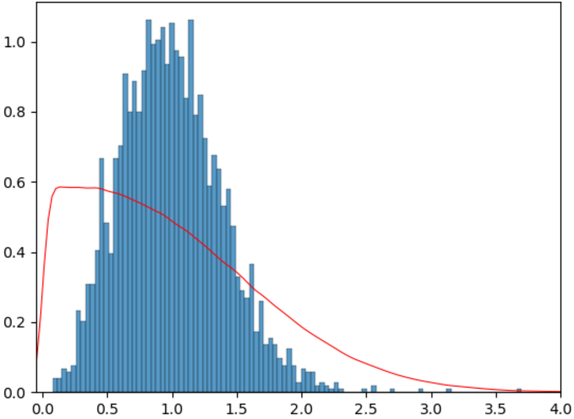

Lowest zeros (twists) of 17.2.d.a

Eigenvalues of random matrices of Sp(20)



Lowest zeros (twists) of 11.7.b.b

Eigenvalues of random matrices of U(9)



For large $T$, we denote the pair-correlation of a family of twists of a given form $f$ by

$$P(f \otimes \psi_d; \varphi) \;=\; \sum_{0<\gamma,\gamma'<T} \varphi(\gamma - \gamma'),$$

where the $\gamma$'s are the imaginary part of the zeros and $\varphi$ a (holomorphic) test function.

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$$P(f \otimes \psi_d; \varphi) = \sum_{0<\gamma,\gamma'<T} \varphi(\gamma - \gamma'),$$

where the $\gamma$'s are the imaginary part of the zeros and $\varphi$ a (holomorphic) test function. Using the ratios conjecture and analyticity, we expand the above using series expansions

$$\begin{aligned} P(f \otimes \psi_d; \varphi) := \frac{T}{2\pi} R\Bigg[h(0) + \int_{\mathbb{R}} h(y)\Bigg(1 - \left(\frac{\sin \pi y}{\pi y}\right)^2 \\ + \frac{e_1 - e_2 \sin^2 \pi y}{R^2} - \frac{e_3 \pi y \sin 2\pi y}{R^3} + O(R^{-4})\Bigg)\, dy\Bigg] + O(T^{\varepsilon + 1/2}), \end{aligned}$$

where

$$R = \log\left(\frac{\sqrt{M}|d|T}{2\pi e}\right).$$

Compare the U(N) pair-correlation

$$1 - \left(\frac{\sin \pi y}{\pi y}\right)^2 - \frac{\sin^2 \pi y}{3N^2},$$

to the pair-correlation for our form $f$, we compare the term

$$1 - \left(\frac{\sin \pi y}{\pi y}\right)^2 + \frac{e_1 - e_2 \sin^2 \pi y}{R^2} - e_3 \frac{\pi y \sin 2\pi y}{R^3}.$$

# Effective Matrix Size: Pair-correlation

Compare the U(N) pair-correlation

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to the pair-correlation for our form $f$, we compare the term

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## Conjecture (Montgomery, 1973)

High on the critical line, the spacing between pairs of the Riemann zeta function is asymptotically

$$1-\left(\frac{\sin \pi u}{\pi u}\right)^2.$$

# Acknowledgments

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