From Monovariants to Zeckendorf Decompositions and Games, and Random Matrix Theory

Steven J Miller
Department of Math/Stats, Williams College

sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu
http://www.williams.edu/Mathematics/sjmiller

Williams SMALL REU 7/14/2021 and Texas Tech REU 7/29/2021
Summand Minimality
with Cordwell, Hlavacek, Huynh, Peterson, Vu
Previous Results

Fibonacci Numbers: \( F_{n+1} = F_n + F_{n-1}; \)
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots
Previous Results

Fibonacci Numbers: \( F_{n+1} = F_n + F_{n-1} \);
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... .

Zeckendorf’s Theorem
Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.
Fibonacci Numbers: \( F_{n+1} = F_n + F_{n-1} \);
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

**Zeckendorf’s Theorem**
Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

**Example:** 51 =?
Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ..., 

Zeckendorf’s Theorem
Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: 51 = 34 + 17 = $F_8 + 17$. 
Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Zeckendorf’s Theorem
Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 4 = F_8 + F_6 + 4$. 
Fibonacci Numbers: \( F_{n+1} = F_n + F_{n-1} \);
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . . .

Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: \( 51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + 1 \).
Fibonacci Numbers: \( F_{n+1} = F_n + F_{n-1} \);
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...  

**Zeckendorf’s Theorem**
Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

**Example:** \( 51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1 \).
**Previous Results**

**Fibonacci Numbers:**  $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

**Zeckendorf’s Theorem**
Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

**Example:** $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1$.
**Example:** $83 = 55 + 21 + 5 + 2 = F_9 + F_7 + F_4 + F_2$.
**Observe:** 51 miles $\approx$ 82.1 kilometers.
Introduction

Fibonaccis: $F_0 = 1, F_1 = 1, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example: $2021 = 1597 + 377 + 34 + 13 = F_{16} + F_{13} + F_8 + F_6$.

Conversely, we can construct the Fibonacci sequence using this property:

1
Introduction

Fibonaccis: \( F_0 = 1, F_1 = 1, F_{n+2} = F_{n+1} + F_n. \)

Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example: \( 2021 = 1597 + 377 + 34 + 13 = F_{16} + F_{13} + F_8 + F_6. \)

Conversely, we can construct the Fibonacci sequence using this property:

1, 2
Introduction

Fibonaccis: \( F_0 = 1, F_1 = 1, F_{n+2} = F_{n+1} + F_n \).

**Zeckendorf’s Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

**Example**: \( 2021 = 1597 + 377 + 34 + 13 = F_{16} + F_{13} + F_8 + F_6 \).

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3
Introduction

Fibonacci sequence:

\[ F_0 = 1, \ F_1 = 1, \ F_{n+2} = F_{n+1} + F_n. \]

**Zeckendorf’s Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

**Example:** \( 2021 = 1597 + 377 + 34 + 13 = F_{16} + F_{13} + F_8 + F_6. \)

Conversely, we can construct the Fibonacci sequence using this property:

\[ 1, 2, 3, 5 \]
Introduction

Fibonacci: \( F_0 = 1, \ F_1 = 1, \ F_{n+2} = F_{n+1} + F_n. \)

**Zeckendorf’s Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

**Example:** \( 2021 = 1597 + 377 + 34 + 13 = F_{16} + F_{13} + F_8 + F_6. \)

Conversely, we can construct the Fibonacci sequence using this property:

\( 1, 2, 3, 5, 8 \)
Fibonaccis: \( F_0 = 1, F_1 = 1, F_{n+2} = F_{n+1} + F_n. \)

### Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

**Example:** \( 2021 = 1597 + 377 + 34 + 13 = F_{16} + F_{13} + F_8 + F_6. \)

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...
Combinatorial Proof: The Cookie Problem

The Cookie Problem

The number of ways of dividing $C$ identical cookies among $P$ distinct people is $\binom{C+P-1}{P-1}$.
Combinatorial Proof: The Cookie Problem

**The Cookie Problem**

The number of ways of dividing $C$ identical cookies among $P$ distinct people is $\binom{C+P-1}{P-1}$.

**Proof**: Consider $C + P - 1$ cookies in a line. **Cookie Monster** eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do. Divides the cookies into $P$ sets.
Combinatorial Proof: The Cookie Problem

The Cookie Problem
The number of ways of dividing $C$ identical cookies among $P$ distinct people is $\binom{C+P-1}{P-1}$.

Proof: Consider $C + P - 1$ cookies in a line. Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do. Divides the cookies into $P$ sets.
Example: 8 cookies and 5 people ($C = 8$, $P = 5$):
The number of ways of dividing $C$ identical cookies among $P$ distinct people is $\binom{C + P - 1}{P - 1}$.

**Proof**: Consider $C + P - 1$ cookies in a line. **Cookie Monster** eats $P - 1$ cookies: $\binom{C + P - 1}{P - 1}$ ways to do. Divides the cookies into $P$ sets.

**Example**: 8 cookies and 5 people ($C = 8$, $P = 5$):
The Cookie Problem

The number of ways of dividing \( C \) identical cookies among \( P \) distinct people is \( \binom{C+P-1}{P-1} \).

**Proof**: Consider \( C + P - 1 \) cookies in a line. **Cookie Monster** eats \( P - 1 \) cookies: \( \binom{C+P-1}{P-1} \) ways to do. Divides the cookies into \( P \) sets.

**Example**: 8 cookies and 5 people (\( C = 8, P = 5 \)):
Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is

\[
\binom{C+P-1}{P-1}.
\]
Combinatorial Proof: The Cookie Problem: Reinterpretation

**Reinterpreting the Cookie Problem**

The number of solutions to \( x_1 + \cdots + x_P = C \) with \( x_i \geq 0 \) is \( \binom{C+P-1}{P-1} \).

Let \( p_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \} \).
Combinatorial Proof: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to \( x_1 + \cdots + x_P = C \) with \( x_i \geq 0 \) is \( \binom{C+P-1}{P-1} \).

Let \( p_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \} \).

For \( N \in [F_n, F_{n+1}) \), the largest summand is \( F_n \).

\[
N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n, \\
1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \ i_j - i_{j-1} \geq 2.
\]
Reinterpreting the Cookie Problem

The number of solutions to \( x_1 + \cdots + x_P = C \) with \( x_i \geq 0 \) is \( \binom{C + P - 1}{P - 1} \).

Let \( p_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \} \).

For \( N \in [F_n, F_{n+1}) \), the \text{largest summand} is \( F_n \).

\[
N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,
\]
\[
1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \ i_j - i_{j-1} \geq 2.
\]
\[
d_1 := i_1 - 1, \ d_j := i_j - i_{j-1} - 2 \ (j > 1).
\]
\[
d_1 + d_2 + \cdots + d_k = n - 2k + 1, \ d_j \geq 0.
\]
Combinatorial Proof: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

Let $p_{n,k} = \# \{ N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands} \}$.

For $N \in [F_n, F_{n+1})$, the largest summand is $F_n$.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \ i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1, \ d_j := i_j - i_{j-1} - 2 \ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1, \ d_j \geq 0.$$

Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1+k-1}{k-1} = \binom{n-k}{k-1}$. 
Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

*The Zeckendorf decomposition is *summand minimal.*

Overall Question

What other recurrences are summand minimal?
Positive Linear Recurrence Sequences

**Definition**

A **positive linear recurrence sequence (PLRS)** is the sequence given by a recurrence \( \{a_n\} \) with

\[
a_n := c_1 a_{n-1} + \cdots + c_t a_{n-t}
\]

and each \( c_i \geq 0 \) and \( c_1, c_t > 0 \). We use **ideal initial conditions** \( a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1 \) and call \( (c_1, \ldots, c_t) \) the **signature of the sequence**.

**Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)**

*For a PLRS with signature \( (c_1, c_2, \ldots, c_t) \), the Generalized Zeckendorf Decompositions are summand minimal if and only if*

\[
c_1 \geq c_2 \geq \cdots \geq c_t.
\]
Proof Preliminaries: Invariant

A quantity is **invariant** if it does not change throughout the process.

Examples:

- Think of mass and energy in classical physics.
- If you travel on a straight line from 0 to 10 it doesn’t matter how many stops you make, the total distance traveled is always 10.
- If you are given 1 meter and bend it in two places to make a triangle, the area of the triangles can differ but all will have a perimeter of 1.
A **mono-variant** is a quantity that can change in only one way; it is either non-decreasing (so it can stay the same or increase) or it is non-increasing (so it can stay the same or decrease).

Examples:

- The number of pieces on the board in a game of chess or checkers.
- The scores in a sports contest.
- The distance traveled by a cannonball (unless we have a very strong wind!).
Proof Preliminaries: Applications of Mono-variants

- 1-dimensional Sperner’s Lemma game and Fixed Point Theorems.

- Zombie Apocalypse: Spread of infection.

- Conway Soldier / Checker Problem.

- 2 × 1 dominoes tiling an $n \times n$ square with upper right and bottom left corners removed.


Why I love Monovariants: From Zombies to Conway’s Soldiers via the Golden Mean: https://youtu.be/LWWc4q3e-RY.
Proof for Fibonacci Case

Idea of proof:

- $D = b_1 F_1 + \cdots + b_n F_n$ decomposition of $N$, set
  $\text{Ind}(D) = b_1 \cdot 1 + \cdots + b_n \cdot n$. 
Proof for Fibonacci Case

Idea of proof:

- $D = b_1 F_1 + \cdots + b_n F_n$ decomposition of $N$, set $\text{Ind}(D) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

- Move to $D'$ by
  - $2F_k = F_{k+1} + F_{k-2}$ (and $2F_2 = F_3 + F_1$).
  - $F_k + F_{k+1} = F_{k+2}$ (and $F_1 + F_1 = F_2$).
Proof for Fibonacci Case

Idea of proof:

- $D = b_1F_1 + \cdots + b_nF_n$ decomposition of $N$, set $\text{Ind}(D) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

- Move to $D'$ by
  - $2F_k = F_{k+1} + F_{k-2}$ (and $2F_2 = F_3 + F_1$).
  - $F_k + F_{k+1} = F_{k+2}$ (and $F_1 + F_1 = F_2$).

- Monovariant: Note $\text{Ind}(D') \leq \text{Ind}(D)$.
  - $2F_k = F_{k+1} + F_{k-2}$: $2k$ vs $2k - 1$.
  - $F_k + F_{k+1} = F_{k+2}$: $2k + 1$ vs $k + 2$. 
Proof for Fibonacci Case

Idea of proof:

- $\mathcal{D} = b_1 F_1 + \cdots + b_n F_n$ decomposition of $N$, set $\text{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

- Move to $\mathcal{D}'$ by
  - $2F_k = F_{k+1} + F_{k-2}$ (and $2F_2 = F_3 + F_1$).
  - $F_k + F_{k+1} = F_{k+2}$ (and $F_1 + F_1 = F_2$).

- Monovariant: Note $\text{Ind}(\mathcal{D}') \leq \text{Ind}(\mathcal{D})$.
  - $2F_k = F_{k+1} + F_{k-2}$: $2k$ vs $2k - 1$.
  - $F_k + F_{k+1} = F_{k+2}$: $2k + 1$ vs $k + 2$.

- If not at Zeckendorf decomposition can continue, if at Zeckendorf cannot. Better: $\text{Ind}'(\mathcal{D}) = b_1 \sqrt{1} + \cdots + b_n \sqrt{n}$. 
The Zeckendorf Game
with Paul Baird-Smith, Alyssa Epstein and Kristen Flint
Rules

- Two player game, alternate turns, last to move wins.
Rules

- Two player game, alternate turns, last to move wins.
- Bins $F_1$, $F_2$, $F_3$, \ldots, start with $N$ pieces in $F_1$ and others empty.
Rules

- Two player game, alternate turns, last to move wins.

- Bins $F_1, F_2, F_3, \ldots$, start with $N$ pieces in $F_1$ and others empty.

- A turn is one of the following moves:
  - If have two pieces on $F_k$ can remove and put one piece at $F_{k+1}$ and one at $F_{k-2}$
    (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - If pieces at $F_k$ and $F_{k+1}$ remove and add one at $F_{k+2}$.
Rules

- Two player game, alternate turns, last to move wins.

- Bins $F_1, F_2, F_3, \ldots$, start with $N$ pieces in $F_1$ and others empty.

- A turn is one of the following moves:
  - If have two pieces on $F_k$ can remove and put one piece at $F_{k+1}$ and one at $F_{k-2}$ (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - If pieces at $F_k$ and $F_{k+1}$ remove and add one at $F_{k+2}$.

Questions:

- Does the game end? How long?
- For each $N$ who has the winning strategy?
- What is the winning strategy?
Sample Game

Start with 10 pieces at $F_1$, rest empty.

<table>
<thead>
<tr>
<th>10</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[F_1 = 1]$</td>
<td>$[F_2 = 2]$</td>
<td>$[F_3 = 3]$</td>
<td>$[F_4 = 5]$</td>
<td>$[F_5 = 8]$</td>
</tr>
</tbody>
</table>

Next move: Player 1: $F_1 + F_1 = F_2$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
8 & 1 & 0 & 0 & 0 & \\
\end{array}
\]

Next move: Player 2: $F_1 + F_1 = F_2$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

<table>
<thead>
<tr>
<th>6</th>
<th>2</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[F_1 = 1]$</td>
<td>$[F_2 = 2]$</td>
<td>$[F_3 = 3]$</td>
<td>$[F_4 = 5]$</td>
<td>$[F_5 = 8]$</td>
</tr>
</tbody>
</table>

Next move: Player 1: $2F_2 = F_3 + F_1$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
7 & 0 & 1 & 0 & 0 \\
\end{array}
\]

Next move: Player 2: $F_1 + F_1 = F_2$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

$$
\begin{array}{cccccc}
5 & 1 & 1 & 0 & 0 \\
\end{array}
$$

Sample Game

Start with 10 pieces at $F_1$, rest empty.

$$
\begin{array}{cccccc}
5 & 0 & 0 & 1 & 0 \\
\end{array}
$$

Next move: Player 2: $F_1 + F_1 = F_2$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>$F_2$</td>
<td>$F_3$</td>
<td>$F_4$</td>
<td>$F_5$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

Next move: Player 1: $F_1 + F_1 = F_2$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>0</th>
<th>1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[F_1 = 1]$</td>
<td>$[F_2 = 2]$</td>
<td>$[F_3 = 3]$</td>
<td>$[F_4 = 5]$</td>
<td>$[F_5 = 8]$</td>
</tr>
</tbody>
</table>

Next move: Player 2: $F_1 + F_2 = F_3$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
0 & 1 & 1 & 1 & 0 \\
\end{array}
\]

Next move: Player 1: $F_3 + F_4 = F_5$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>1</th>
<th></th>
<th>0</th>
<th></th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

$[F_1 = 1]$  $[F_2 = 2]$  $[F_3 = 3]$  $[F_4 = 5]$  $[F_5 = 8]$  

No moves left, Player One wins.
Sample Game

Player One won in 9 moves.

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ F_1 = 1 \] \[ F_2 = 2 \] \[ F_3 = 3 \] \[ F_4 = 5 \] \[ F_5 = 8 \]
Sample Game

Player Two won in 10 moves.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

$[F_1 = 1] \quad [F_2 = 2] \quad [F_3 = 3] \quad [F_4 = 5] \quad [F_5 = 8]$
Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: \( \left( \sqrt{k} + \sqrt{k} \right) - \sqrt{k} + 2 < 0. \)
- Splitting: \( 2\sqrt{k} - \left( \sqrt{k+1} + \sqrt{k+1} \right) < 0. \)
- Adding 1's: \( 2\sqrt{1} - \sqrt{2} < 0. \)
- Splitting 2's: \( 2\sqrt{2} - \left( \sqrt{3} + \sqrt{1} \right) < 0. \)
Games Lengths: I

Upper bound: At most $n \log_\phi (n\sqrt{5} + 1/2)$ moves (improved last year to order $n$).

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in $n$’s Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).
Games Lengths: II

Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....
Winning Strategy

Theorem

*Payer Two Has a Winning Strategy*

Idea is to show if not, Player Two could steal Player One’s strategy.

Non-constructive!

Will highlight idea with a simpler game.
Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at \((i, j)\) and coloring every dot \((m, n)\) with \(i \leq m\) and \(j \leq n\).

Once all dots colored game ends; whomever goes last loses.

Proof Player 1 has a winning strategy. If have, play; if not, steal.
Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at \((i, j)\) and coloring every dot \((m, n)\) with \(i \leq m\) and \(j \leq n\).

Once all dots colored game ends; whomever goes last loses.

Proof Player 1 has a winning strategy. If have, play; if not, steal.
Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at \((i, j)\) and coloring every dot \((m, n)\) with \(i \leq m\) and \(j \leq n\).

Once all dots colored game ends; whomever goes last loses.

Proof Player 1 has a winning strategy. If have, play; if not, steal.
Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at \((i, j)\) and coloring every dot \((m, n)\) with \(i \leq m\) and \(j \leq n\).

Once all dots colored game ends; whomever goes last loses.

Proof Player 1 has a winning strategy. If have, play; if not, steal.
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two's Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
The Abstract Zeckendorf Game (AZG)

**Definition**

The AZG game is played on an infinite tape. In each index is a (possibly empty) stack of tokens which can be moved in two ways:

- **The Combine Move** \((x_1 \geq 1, x_2 \geq 1)\): 
  \[
  (\ldots, x_1, x_2, x_3, \ldots) \xrightarrow{C} (\ldots, x_1 - 1, x_2 - 1, x_3 + 1, \ldots)
  \]

- **The Split Move** \((x_3 \geq 2)\): 
  \[
  (\ldots, x_1, x_2, x_3, x_4, \ldots) \xrightarrow{S} (\ldots, x_1 + 1, x_2, x_3 - 2, x_4 + 1, \ldots)
  \]

Two players take turns making either of these two moves, the last player to move wins.
Base $\varphi$ Decompositions

- Base $\varphi$ Decompositions $\leftrightarrow$ Zeckendorf Decompositions $\sim$
  Abstract Zeckendorf Game $\leftrightarrow$ Zeckendorf Game
- For example, play the game on the tuple $(6)$ and keep track of the starting index

$$(6) \xrightarrow{S} (1, 0, 4, 1) \xrightarrow{C} (1, 0, 3, 0, 1) \xrightarrow{S} (2, 0, 1, 1, 1)$$
$$\xrightarrow{S} (1, 0, 0, 1, 1, 1, 1) \xrightarrow{C} (1, 0, 0, 1, 1, 0, 0, 1)$$
$$\xrightarrow{C} (1, 0, 0, 0, 0, 1, 0, 1)$$

- Notice that $6 = \varphi^3 + \varphi + \varphi^{-4}$. Why?
The game terminates in $O(n^2 + bn)$ moves, where $n$ is the total number of summands in your initial configuration $I$ and $b$ is the width of $I$.

The average number of summands in the base $\varphi$ decomposition in the interval $[L_i, L_{i+1}]$ is linear in $i$. Conjectured to be Gaussian in the limit.

The AZG is hard: if the game board is a general directed acyclic graph (DAG) instead of a tape, it is PSPACE-hard. Over a wide family of DAGs, it is instead PSPACE-complete.
Classical Random Matrix Theory

With Olivia Beckwith, Leo Goldmakher, Chris Hammond, Steven Jackson, Cap Khoury, Murat Koloğlu, Gene Kopp, Victor Luo, Adam Massey, Eve Ninsuwan, Vincent Pham, Karen Shen, Jon Sinsheimer, Fred Strauch, Nicholas Triantafillou, Wentao Xiong
Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.
Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.

Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

\[ H \psi_n = E_n \psi_n \]

\( H \) : matrix, entries depend on system
\( E_n \) : energy levels
\( \psi_n \) : energy eigenfunctions
Origins of Random Matrix Theory

- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\bar{A}^T = A$).
Random Matrix Ensembles

\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji} \]

Fix \( p \), define

\[ \text{Prob}(A) = \prod_{1 \leq i \leq j \leq N} p(a_{ij}). \]

This means

\[ \text{Prob}(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]) = \prod_{1 \leq i \leq j \leq N} \int_{x_{ij} = \alpha_{ij}}^{\beta_{ij}} p(x_{ij}) \, dx_{ij}. \]

Want to understand eigenvalues of \( A \).
Eigenvalue Distribution

\[ \delta(x - x_0) \text{ is a unit point mass at } x_0: \int f(x)\delta(x - x_0)dx = f(x_0). \]
Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at $x_0$: $\int f(x)\delta(x - x_0)dx = f(x_0)$.

To each $A$, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)$$
Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at $x_0$: $\int f(x)\delta(x - x_0)dx = f(x_0)$.

To each $A$, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)$$

$$\int_{a}^{b} \mu_{A,N}(x)dx = \frac{\# \left\{ \lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a, b] \right\}}{N}$$
δ(x − x₀) is a unit point mass at x₀: \( \int f(x)\delta(x - x₀)dx = f(x₀) \).

To each \( A \), attach a probability measure:

\[
\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)
\]

\[
\int_{a}^{b} \mu_{A,N}(x)dx = \frac{\# \left\{ \lambda_i: \frac{\lambda_i(A)}{2\sqrt{N}} \in [a, b] \right\}}{N}
\]

\[
\text{k}^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N} \lambda_i(A)^k}{2^k N^{k/2 + 1}} = \frac{\text{Trace}(A^k)}{2^k N^{k/2 + 1}}.
\]
Wigner’s Semi-Circle Law

\( N \times N \) real symmetric matrices, entries i.i.d.r.v. from a fixed \( p(x) \) with mean 0, variance 1, and other moments finite. Then for almost all \( A \), as \( N \to \infty \)

\[
\mu_{A,N}(x) \to \begin{cases} 
\frac{2}{\pi} \sqrt{1 - x^2} & \text{if } |x| \leq 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Numerical examples

500 Matrices: Gaussian $400 \times 400$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
Numerical examples

The eigenvalues of the Cauchy distribution are NOT semicircular.

Cauchy Distribution: \( p(x) = \frac{1}{\pi (1+x^2)} \)


SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of $A$, but choose the matrix elements randomly and independently.

**Eigenvalue Trace Lemma**

Let $A$ be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) = \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1} a_{i_2} \cdots a_{i_N}.$$
SKETCH OF PROOF: Correct Scale

\[
\text{Trace}(A^2) = \sum_{i=1}^{N} \lambda_i(A)^2.
\]

By the Central Limit Theorem:

\[
\text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} a_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \sim N^2
\]

\[
\sum_{i=1}^{N} \lambda_i(A)^2 \sim N^2
\]

Gives \(N \text{Ave}(\lambda_i(A)^2) \sim N^2\) or \(\text{Ave}(\lambda_i(A)) \sim \sqrt{N}\).
SKETCH OF PROOF: Averaging Formula

Recall $k$-th moment of $\mu_{A,N}(x)$ is $\text{Trace}(A^k)/2^k N^{k/2+1}$.

Average $k$-th moment is

$$\int \cdots \int \frac{\text{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of $k$-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).
SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

\[
\frac{1}{2^2 N^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \cdot p(a_{11}) \, da_{11} \cdots p(a_{NN}) \, da_{NN}
\]

Integration factors as

\[
\int_{a_{ij} = -\infty}^{\infty} a_{ij}^2 p(a_{ij}) \, da_{ij} \cdot \prod_{(k,l) \neq (i,j), k < l} \int_{a_{kl} = -\infty}^{\infty} p(a_{kl}) \, da_{kl} = 1.
\]

Higher moments involve more advanced combinatorics (Catalan numbers).
SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

\[
\frac{1}{2^k N^{k+1}} \prod_{i \leq j} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} a_{i_1} \cdots a_{i_k} \cdot \prod_{i \leq j} p(a_{ij}) \, da_{ij}.
\]

Main contribution when the \(a_{i_\ell i_{\ell+1}}\)'s matched in pairs, not all matchings contribute equally (if did would get a Gaussian and not a semi-circle; this is seen in Real Symmetric Palindromic Toeplitz matrices).


http://arxiv.org/abs/math/0512146
GOE Conjecture:

As $N \to \infty$, the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of $p$.

Until recently only known if $p$ is a Gaussian.

$\text{GOE}(x) \approx \frac{\pi}{2} xe^{-\pi x^2/4}$. 
Numerical Experiment: Uniform Distribution

Let \( p(x) = \frac{1}{2} \) for \( |x| \leq 1 \).
Cauchy Distribution

Let \( p(x) = \frac{1}{\pi(1+x^2)} \).
Random Graphs

Degree of a vertex = number of edges leaving the vertex.
Adjacency matrix: $a_{ij} =$ number edges b/w Vertex $i$ and Vertex $j$.

$$A = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 2 \\
1 & 0 & 2 & 0
\end{pmatrix}$$

These are Real Symmetric Matrices.
McKay’s Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for $d$-regular graphs

$$f(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)} \sqrt{4(d-1)-x^2} & |x| \leq 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$
McKay’s Law (Kesten Measure) with $d = 6$

Fat Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \to \infty$ recover semi-circle).
3-Regular Graph with 2000 Vertices: Comparison with the GOE

Spacings between eigenvalues of 3-regular graphs and the GOE:
Block Circulant Ensemble

With Murat Koloğlu, Gene Kopp, Fred Strauch and Wentao Xiong.
The Ensemble of $m$-Block Circulant Matrices

Symmetric matrices periodic with period $m$ on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

\[
\begin{pmatrix}
  c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
  c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\
  c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
  c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
  c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
  d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
  c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
  d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0 
\end{pmatrix}.
\]

Choose distinct entries i.i.d.r.v.
Oriented Matchings and Dualization

Compute moments of eigenvalue distribution (as $m$ stays fixed and $N \to \infty$) using the combinatorics of pairings. Rewrite:

$$M_n(N) = \frac{1}{N^{n+1}} \sum_{1 \leq i_1, \ldots, i_n \leq N} \mathbb{E}(a_{i_1} a_{i_2} a_{i_3} \cdots a_{i_n})$$

$$= \frac{1}{N^{n+1}} \sum_{\sim} \eta(\sim) m_{d_1}(\sim) \cdots m_{d_l}(\sim).$$

where the sum is over oriented matchings on the edges \{(1, 2), (2, 3), \ldots, (n, 1)\} of a regular $n$-gon.
Figure: An oriented matching in the expansion for $M_n(N) = M_6(8)$. 

\[
\begin{pmatrix}
    c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
    c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
    c_2 & d_1 & c_0 & c_1 & c_2 & d_3 \\
    c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
    c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
    d_3 & d_4 & c_3 & d_2 & c_1 & d_0 \\
    c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
    d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\
\end{pmatrix}
\]
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.
Think of pairings as topological identifications; the contributing ones give rise to orientable surfaces.

Contribution from such a pairing is $m^{-2g}$, where $g$ is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.
Computing the Even Moments

**Theorem: Even Moment Formula**

\[ M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k \left( \frac{1}{N} \right), \]

with \( \varepsilon_g(k) \) the number of pairings of the edges of a \((2k)\)-gon giving rise to a genus \( g \) surface.

J. Harer and D. Zagier (1986) gave generating functions for the \( \varepsilon_g(k) \).
Harer and Zagier

\[\left\lfloor \frac{k}{2} \right\rfloor \sum_{g=0}^{\left\lfloor k/2 \right\rfloor} \varepsilon g(k) r^{k+1-2g} = (2k - 1)!! c(k, r)\]

where

\[1 + 2 \sum_{k=0}^{\infty} c(k, r) x^{k+1} = \left(\frac{1 + x}{1 - x}\right)^r.\]

Thus, we write

\[M_{2k} = m^{-(k+1)}(2k - 1)!! c(k, m).\]
A multiplicative convolution and Cauchy’s residue formula yield

**Theorem: Koloğlu, Kopp and Miller**

Limiting spectral density $f_m(x)$ of the real symmetric $m$-block circulant ensemble is

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(\frac{1}{2}\right)^s (mx^2)^r.$$ 

As $m \to \infty$, $f_m(x)$ approaches the semicircle distribution.
Figure: Plot for $f_1$ and histogram of eigenvalues of 100 circulant matrices of size $400 \times 400$. 
Results (continued)

Figure: Plot for \( f_2 \) and histogram of eigenvalues of 100 2-block circulant matrices of size 400 \( \times \) 400.
Results (continued)

**Figure:** Plot for $f_3$ and histogram of eigenvalues of 100 3-block circulant matrices of size $402 \times 402$. 
Figure: Plot for $f_4$ and histogram of eigenvalues of 100 4-block circulant matrices of size $400 \times 400$. 
Results (continued)

**Figure:** Plot for $f_8$ and histogram of eigenvalues of 100 8-block circulant matrices of size $400 \times 400$. 
Results (continued)

**Figure:** Plot for $f_{20}$ and histogram of eigenvalues of 100 20-block circulant matrices of size $400 \times 400$. 
Results (continued)

**Figure:** Plot of convergence to the semi-circle.

Checkerboard Matrices

- Second paper with Ryan Chen, Yujin Kim, Jared Lichtman, Shannon Sweitzer, and Eric Winsor (Michigan).
- Third paper with Fangu Chen (Michigan), Yuxin Lin and Jiahui Yu.
Checkerboard Matrices: $N \times N (k, w)$-checkerboard ensemble

Matrices $M = (m_{ij}) = M^T$ with $a_{ij}$ iidrv, mean 0, variance 1, finite higher moments, $w$ fixed and

$$m_{ij} = \begin{cases} a_{ij} & \text{if } i \not\equiv j \text{ mod } k \\ w & \text{if } i \equiv j \text{ mod } k. \end{cases}$$

Example: $(3, w)$-checkerboard matrix:

$$
\begin{pmatrix}
  w & a_{0,1} & a_{0,2} & w & a_{0,4} & \cdots & a_{0,N-1} \\
  a_{1,0} & w & a_{1,2} & a_{1,3} & w & \cdots & a_{1,N-1} \\
  a_{2,0} & a_{2,1} & w & a_{2,3} & a_{2,4} & \cdots & w \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{0,N-1} & a_{1,N-1} & w & a_{3,N-1} & a_{4,N-1} & \cdots & w 
\end{pmatrix}
$$
**Split Eigenvalue Distribution**

**Figure:** Histogram of normalized eigenvalues: 2-checkerboard

$100 \times 100$ matrices, 100 trials.
**Figure:** Histogram of normalized eigenvalues: 2-checkerboard 150 × 150 matrices, 100 trials.
Split Eigenvalue Distribution

**Figure**: Histogram of normalized eigenvalues: 2-checkerboard $200 \times 200$ matrices, 100 trials.
Split Eigenvalue Distribution

Figure: Histogram of normalized eigenvalues: 2-checkerboard 250 × 250 matrices, 100 trials.
**Split Eigenvalue Distribution**

**Figure:** Histogram of normalized eigenvalues: 2-checkerboard 300 × 300 matrices, 100 trials.
Figure: Histogram of normalized eigenvalues: 2-checkerboard 350 × 350 matrices, 100 trials.
The Weighting Function

Use weighting function $f_n(x) = x^{2n}(x - 2)^{2n}$.

**Figure:** $f_n(x)$ plotted for $n \in \{1, 2, 3, 4\}$. 
The Weighting Function

Use weighting function $f_n(x) = x^{2n}(x - 2)^{2n}$.

Figure: $f_n(x)$ plotted for $n = 4^m$, $m \in \{0, 1, \ldots, 5\}$. 
Spectral distribution of hollow GOE

**Figure:** Hist. of eigenvals of 32000 (Left) $2 \times 2$ hollow GOE matrices, (Right) $3 \times 3$ hollow GOE matrices.

**Figure:** Hist. of eigenvals of 32000 (Left) $4 \times 4$ hollow GOE matrices, (Right) $16 \times 16$ hollow GOE matrices.
New Result: Preliminaries: Symmetric Hankel Matrices

Definition

A circulant Hankel matrix is a symmetric matrix constant along antidiagonals, which wrap about the matrix cyclically:

\[
\begin{bmatrix}
 x_0 & x_1 & x_2 & x_3 & x_4 \\
 x_1 & x_2 & x_3 & x_4 & x_0 \\
 x_2 & x_3 & x_4 & x_0 & x_1 \\
 x_3 & x_4 & x_0 & x_1 & x_2 \\
 x_4 & x_0 & x_1 & x_2 & x_3 \\
\end{bmatrix}
\]

Theorem (SMALL 2021: Dunn, Fleischmann, Jackson, Khunger, Nadjimzadah, Reifenberg, Shashkov, Willis.)

The distribution of the spectral measure of the ensemble of circulant Hankel matrices converges almost surely to the Laplace distribution \( f(x) = e^{\frac{|x|}{2}} \).
New Result: Swirl of a matrix $A$

Definition

$$\text{swirl}(A, X) := \begin{pmatrix}XA & A \\ XAX & AX \end{pmatrix}$$

Note: When $X^2 = I$, $\text{Trace}(\text{swirl}(A, X)^n) = \text{Trace}((XA)^n)$.

When $X^2 = I$ and $XA$ is circulant Hankel, the previous theorem tells us the distribution of the spectral measure is Laplace.

\[
\begin{pmatrix}
x_2 & x_1 & x_0 & x_3 & x_3 & x_0 & x_1 & x_2 \\
x_1 & x_0 & x_3 & x_2 & x_2 & x_3 & x_0 & x_1 \\
x_0 & x_3 & x_2 & x_1 & x_1 & x_2 & x_3 & x_0 \\
x_3 & x_2 & x_1 & x_0 & x_0 & x_1 & x_2 & x_3 \\
x_3 & x_2 & x_1 & x_0 & x_0 & x_1 & x_2 & x_3 \\
x_0 & x_3 & x_2 & x_1 & x_1 & x_2 & x_3 & x_0 \\
x_1 & x_0 & x_3 & x_2 & x_2 & x_3 & x_0 & x_1 \\
x_2 & x_1 & x_0 & x_3 & x_3 & x_0 & x_1 & x_2 \\
\end{pmatrix}
\]
Introduction to $L$-Functions
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]

Unique Factorization: \( n = p_1^{r_1} \cdots p_m^{r_m}. \)
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]

Unique Factorization: \( n = p_1^{r_1} \cdots p_m^{r_m}. \)

\[ \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots \right] \left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots \right] \cdots \]

\[ = \sum_n \frac{1}{n^s}. \]
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \# \{ p : p \text{ is prime, } p \leq x \} \]

Properties of \( \zeta(s) \) and Primes:
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{\rho} \left(1 - \frac{1}{\rho^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \#\{\rho : \rho \text{ is prime, } \rho \leq x\} \]

Properties of \( \zeta(s) \) and Primes:

- \( \lim_{s \to 1^+} \zeta(s) = \infty, \pi(x) \to \infty. \)
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \#\{p : p \text{ is prime, } p \leq x\} \]

Properties of \( \zeta(s) \) and Primes:

- \( \lim_{s \to 1^+} \zeta(s) = \infty, \pi(x) \to \infty. \)
- \( \zeta(2) = \frac{\pi^2}{6}, \pi(x) \to \infty. \)
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]

**Functional Equation:**

\[ \xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1 - s). \]

**Riemann Hypothesis (RH):**

All non-trivial zeros have \( \text{Re}(s) = \frac{1}{2} \); can write zeros as \( \frac{1}{2} + i\gamma \).

**Observation:** Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices \( \overline{A^T} = A \).
General $L$-functions

\[ L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s, f)^{-1}, \quad \text{Re}(s) > 1. \]

Functional Equation:

\[ \Lambda(s, f) = \Lambda_{\infty}(s, f)L(s, f) = \Lambda(1 - s, f). \]

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have Re$(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\bar{A}^T = A$. 
Nuclear spacings: Thorium

227 spacings b/w adjacent energy levels of Thorium.
Zeros of $\zeta(s)$ vs GUE

70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20}$th zero (from Odlyzko).
Elliptic Curves: Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.

Addition of distinct points $P$ and $Q$

Adding a point $P$ to itself

$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$
Elliptic curve $L$-function

$E : y^2 = x^3 + ax + b$, associate $L$-function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{\text{prime } p} L_E(p^{-s}),$$

where

$$a_E(p) = p - \# \{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \mod p\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s, E)$ at $s = 1/2$. 
Properties of zeros of $L$-functions

- infinitude of primes, primes in arithmetic progression.
- Chebyshev’s bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ ‘most’ of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from $L$-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.
Distribution of zeros

- $\zeta(s) \neq 0$ for $\Re(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.

- **GRH**: error terms.

- **GSH**: Chebyshev’s bias.

- Analytic rank, adjacent spacings: $h(D)$. 
Explicit Formula (Contour Integration)

\[- \frac{\zeta'(s)}{\zeta(s)} = - \frac{d}{ds} \log \zeta(s) = - \frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \]
Explicit Formula (Contour Integration)

\[- \frac{\zeta'(s)}{\zeta(s)} = - \frac{d}{ds} \log \zeta(s) = - \frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \]

\[= \frac{d}{ds} \sum_p \log (1 - p^{-s}) \]

\[= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).\]
Explicit Formula (Contour Integration)

\[ -\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \]

\[ = \frac{d}{ds} \sum_p \log (1 - p^{-s}) \]

\[ = \sum_p \log p \cdot \frac{p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s). \]

Contour Integration:

\[ \int -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \, ds \quad \text{vs} \quad \sum_p \log p \int \left( \frac{x}{p} \right)^s \frac{ds}{s}. \]
Explicit Formula (Contour Integration)

\[-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1}\]

\[= \frac{d}{ds} \sum_p \log (1 - p^{-s})\]

\[= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).\]

Contour Integration:

\[\int -\frac{\zeta'(s)}{\zeta(s)} \phi(s) ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) p^{-s} ds.\]
Explicit Formula (Contour Integration)

\[-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1}\]

\[= \frac{d}{ds} \sum_p \log (1 - p^{-s})\]

\[= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).\]

Contour Integration (see Fourier Transform arising):

\[\int -\frac{\zeta'(s)}{\zeta(s)} \phi(s) ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma \log p} e^{-it \log p} ds.\]

Knowledge of zeros gives info on coefficients.
Correspondences

Similarities between $L$-Functions and Nuclei:

Zeros $\leftrightarrow$ Energy Levels

Schwartz test function $\rightarrow$ Neutron

Support of test function $\leftrightarrow$ Neutron Energy.
References
References: Fibonacci


References: Random Matrix Theory

1. Distribution of eigenvalues for the ensemble of real symmetric Toeplitz matrices (with Christopher Hammond), Journal of Theoretical Probability 18 (2005), no. 3, 537–566. 
   http://arxiv.org/abs/math/0312215

   http://arxiv.org/abs/math/0512146

3. The distribution of the second largest eigenvalue in families of random regular graphs (with Tim Novikoff and Anthony Sabelli), Experimental Mathematics 17 (2008), no. 2, 231–244. 
   http://arxiv.org/abs/math/0611649


8. The expected eigenvalue distribution of large, weighted d-regular graphs (with Leo Goldmahker, Cap Khoury and Kesinee Ninsuwan), preprint.
Publications: $L$-Functions


Publications: Elliptic Curves


Publications: $L$-Function Ratio Conjecture


Acknowledgements

Supported by NSF Grants DMS1561945.