Mono-variants

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Goals

The goal of this talk is to introduce the wonderful concept of a mono-variant, and show how it can be used to solve a variety of problems.

It is a mix of math and art – it is a wonderful example of a subject where it is not too bad to follow step by step, but one is left with the feeling "how did they think to look at it that way!"

Invariants

A quantity is invariant if it does not change throughout the process.

Examples:

- Think of mass and energy in classical physics.
- If you travel on a straight line from 0 to 10 it doesn't matter how many stops you make, the total distance traveled is always 10.
- If you are given 1 meter and bend it in two places to make a triangle, the area of the triangles can differ but all will have a perimeter of 1.

Mono-variants

A mono-variant is a quantity that can change in only one way; it is either non-decreasing (so it can stay the same or increase) or it is non-increasing (so it can stay the same or decrease).

Examples:

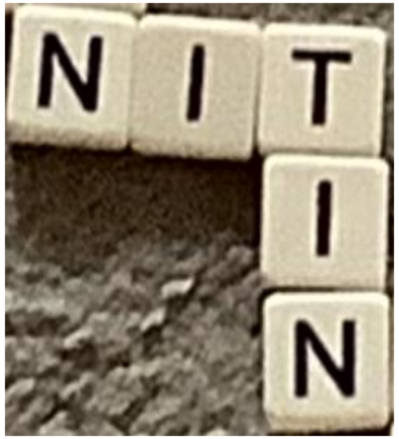
- The number of pieces on the board in a game of chess or checkers.
- The scores in a sports contest.
- The distance traveled by a cannonball (unless we have a very strong wind!).

Inspiration

This lecture was inspired by a challenge my son gave himself: use all 144 tiles in a bananagram set and have one connected board, with every word exactly three letters…. A subset is below. Is it possible?

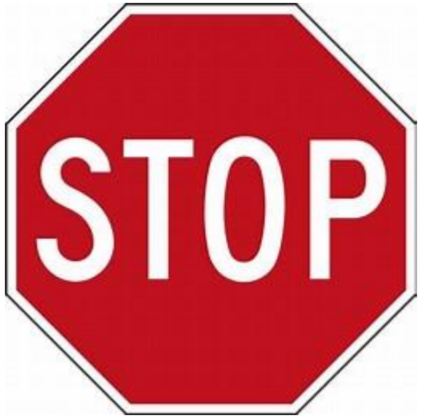
Bananagram Game

In the simplest way of trying to do the challenge, one starts with a three letter word, and then constantly adds two tiles at a time to form new three letter words.

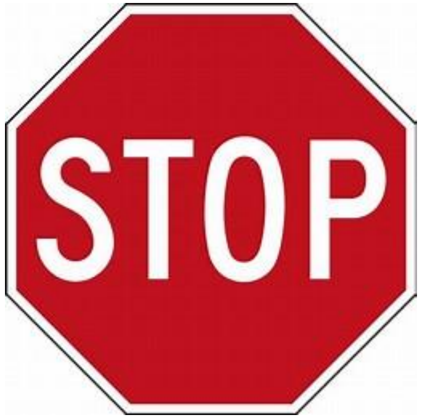


Will it be possible to win, adding two tiles at a time?

Remember we start with 144 tiles, and initially place a three letter word.



SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!



Bananagram Game Solution

We need to introduce a good mono-variant for this problem.

We start with 144 tiles, place 3, and there are 141 tiles left.

Note after the first word there is an ODD number of tiles on the board.

What happens after we add our next word, which costs us two tiles?

Bananagram Game Solution

We need to introduce a good mono-variant for this problem.

We start with 144 tiles, place 3, and there are 141 tiles left.

Note after the first word there is an ODD number of tiles on the board.

After we make our second word there is still an ODD number of tiles on the board.

What happens after we add the third word, which costs us two tiles?

Bananagram Game Solution

We need to introduce a good mono-variant for this problem.

We start with 144 tiles, place 3, and there are 141 tiles left.

Note after the first word there is an ODD number of tiles on the board.

After we make our second word there is still an ODD number of tiles on the board.

After the third word there is still an ODD number of tiles on the board!

Bananagram Game Solution

We need to introduce a good mono-variant for this problem.

We start with 144 tiles, place 3, and there are 141 tiles left.

Note after the first word there is an ODD number of tiles on the board.

After we make our second word there is still an ODD number of tiles on the board.

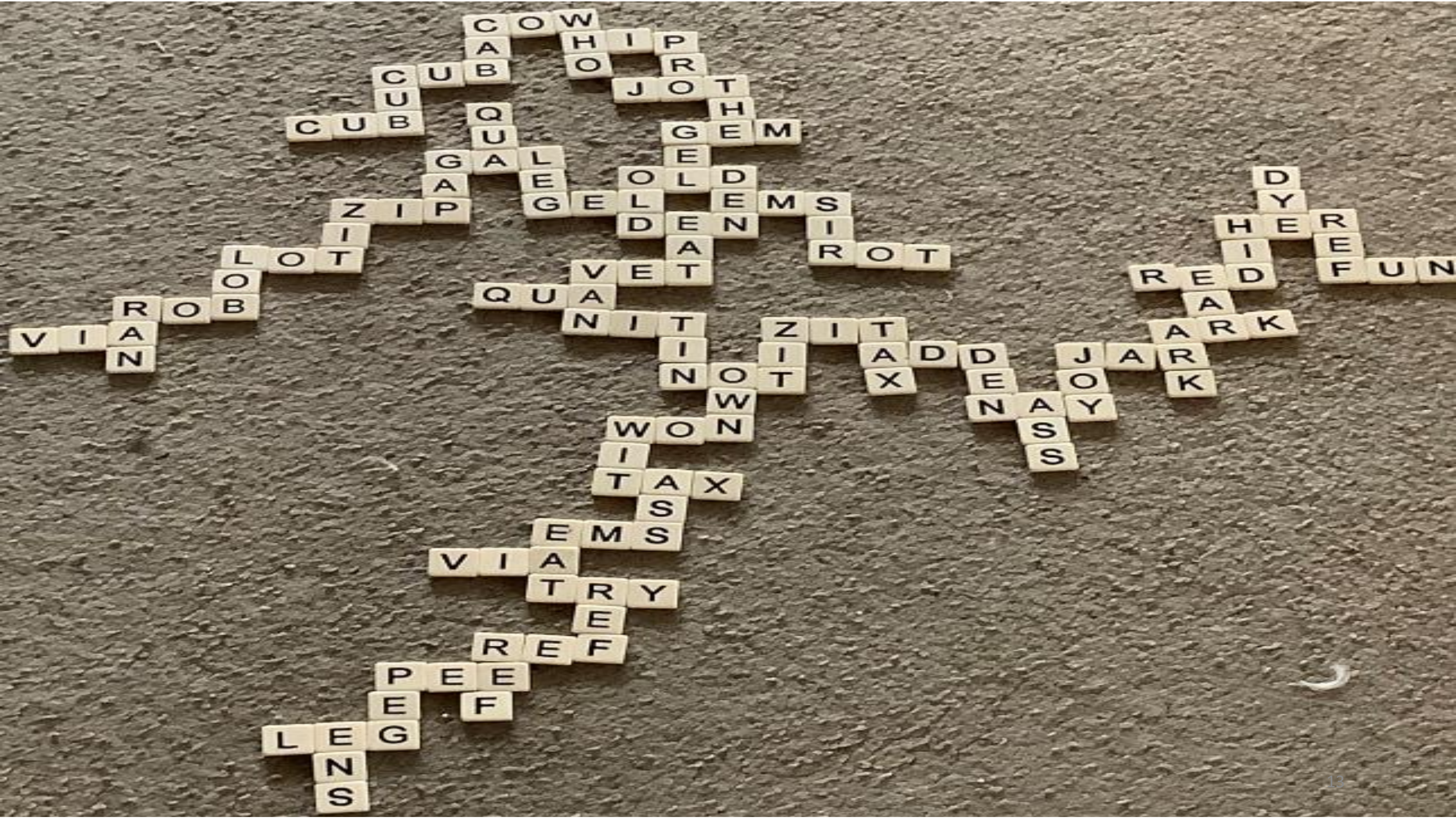
After the third word there is still an ODD number of tiles on the board!

There will always be an odd number of tiles – we have found a good invariant! Thus it is impossible to place all the tiles if we always add two.

Bananagram Game Modification

We can do it if we change how we place tiles.

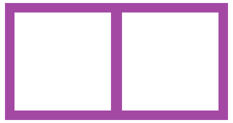
We must allow other patterns, we must allow for configurations where we place just one tile down to get a word, for example….

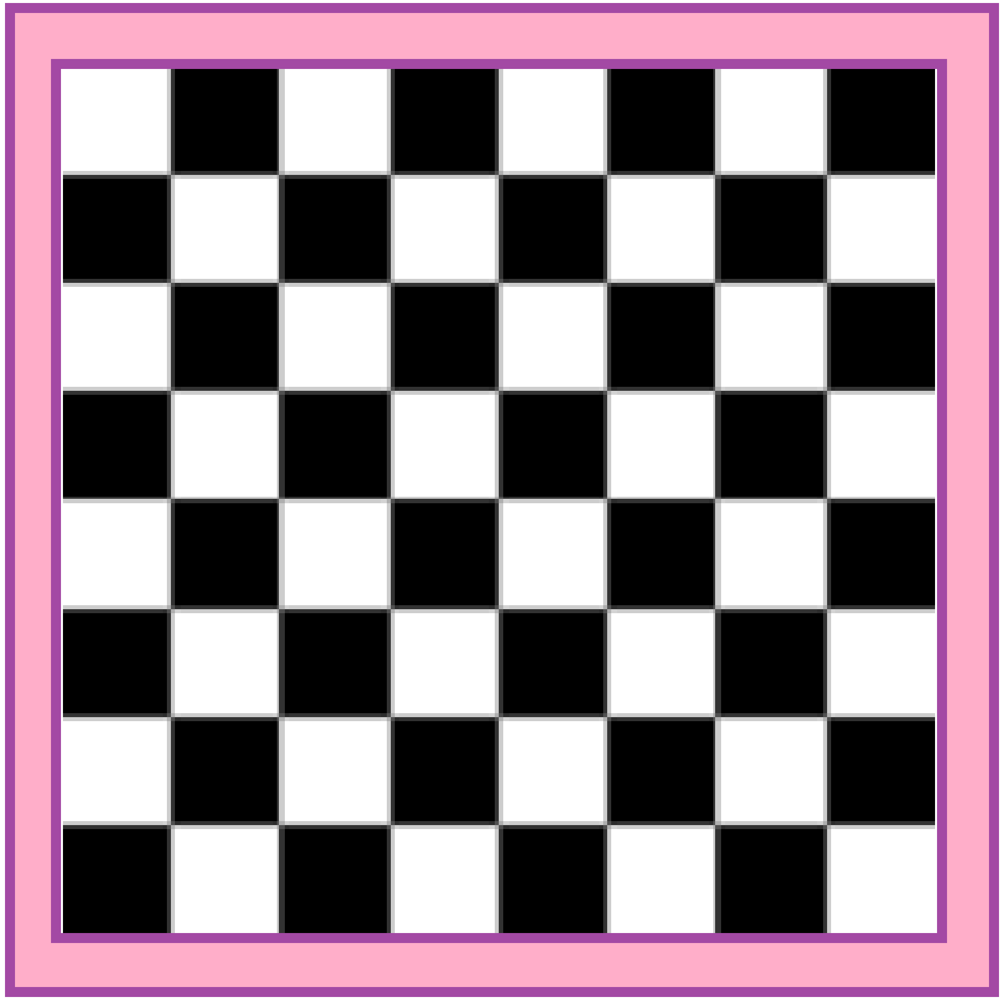


Checkerboard Problem

Consider an 8x8 Chessboard.

You want to place 2x1 tiles on it so that you cover the entire board. Is it possible?





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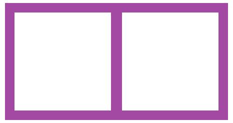
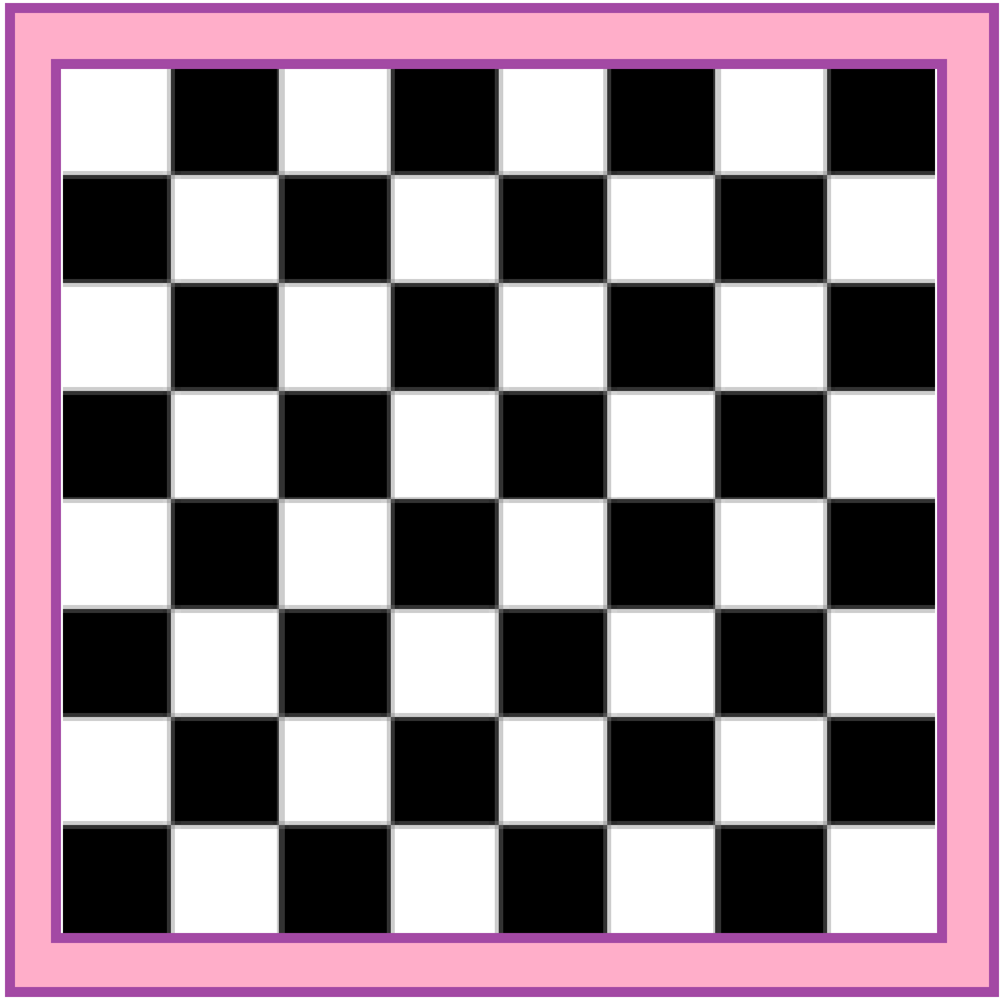


YES – just do row by row!

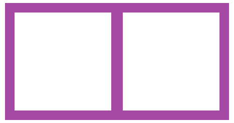
OK, need to modify to make it interesting…..

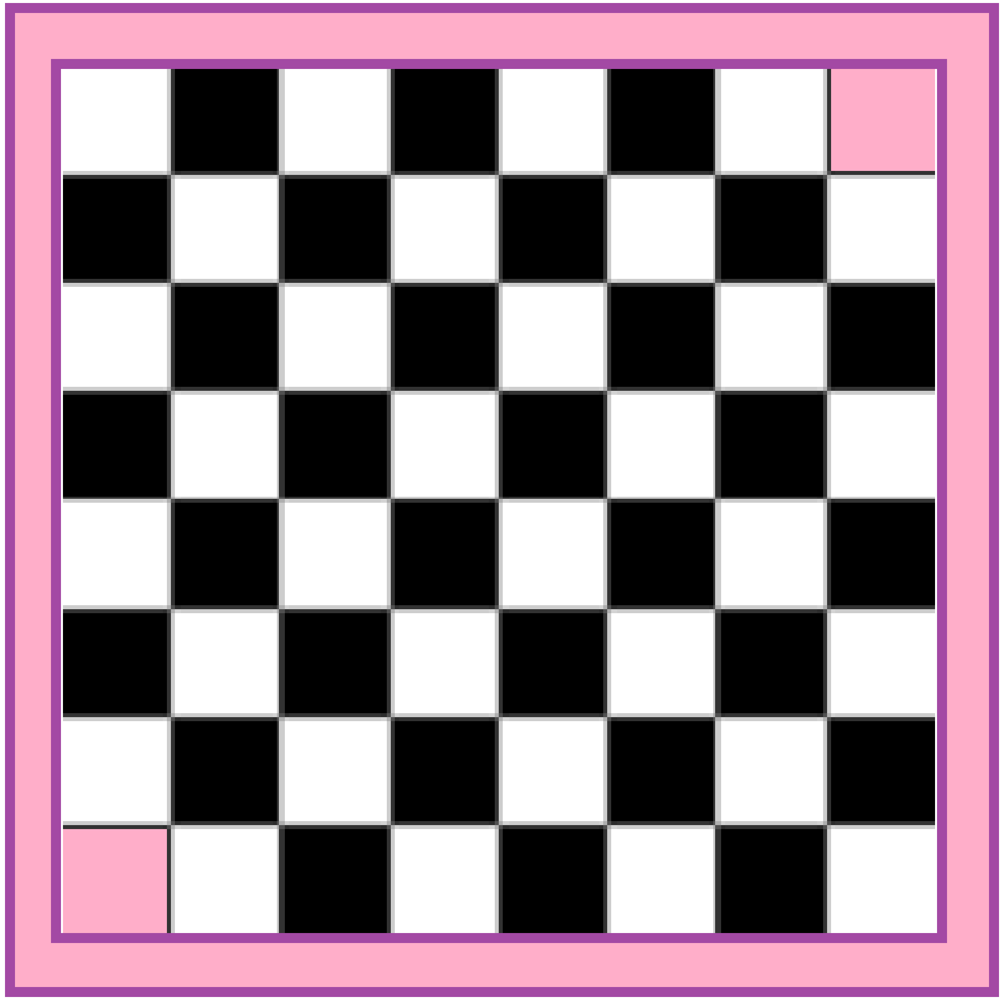


Checkerboard Problem

Consider an 8x8 Chessboard.

You want to place 2x1 tiles on it so that you cover the entire board; however, now the upper right and bottom left corners are missing. Is it possible?



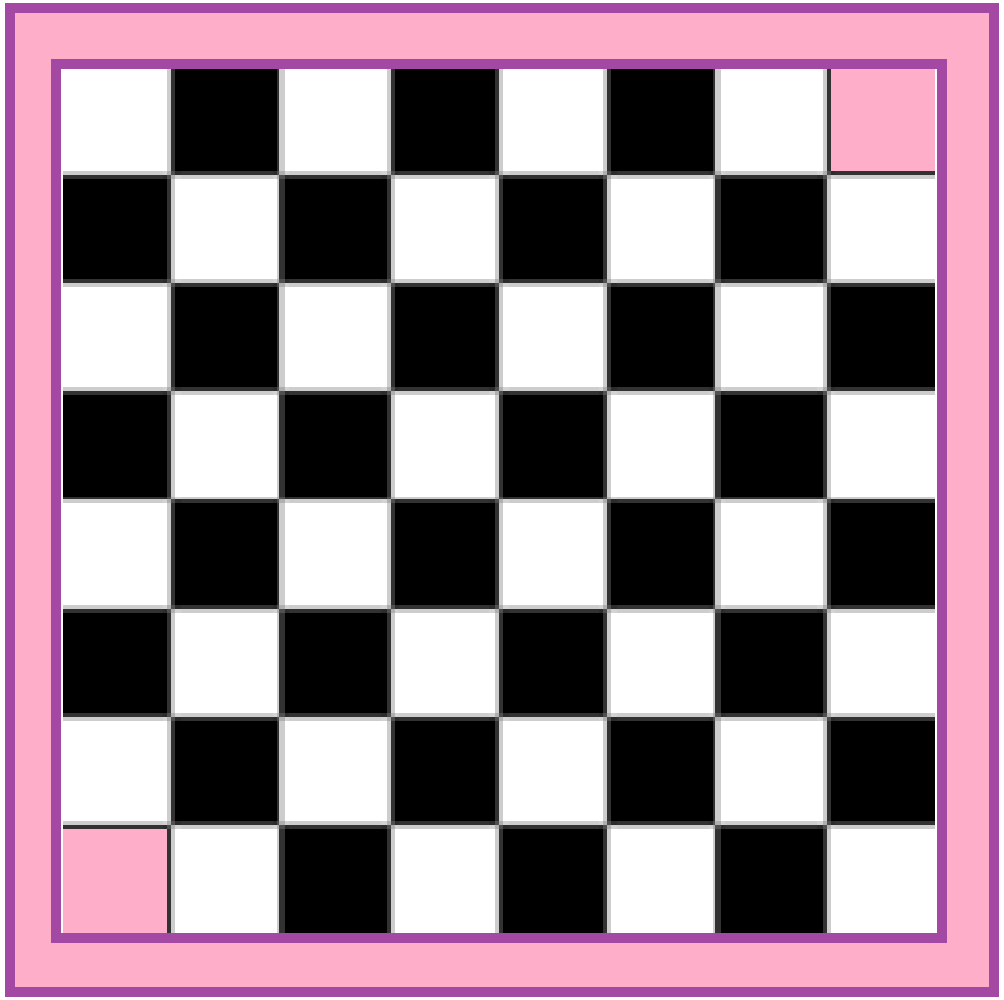


Checkerboard Problem

How would you go about solving this problem?

Often if a problem is hard, try to do a simpler one first.

What do you think is easier?



Checkerboard Problem

How would you go about solving this problem?

Often if a problem is hard, try to do a simpler one first.

What do you think is easier?

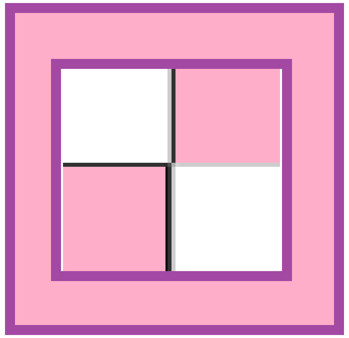
Try a 1x1? No, nothing left!

Checkerboard Problem

How would you go about solving this problem?

Often if a problem is hard, try to do a simpler one first.

What do you think is easier?



Try a 2x2 version….

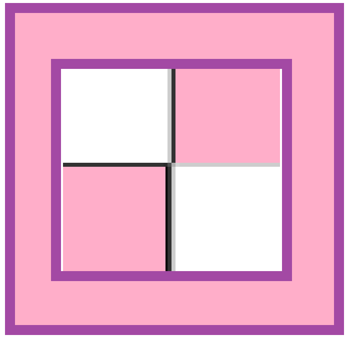
Checkerboard Problem

How would you go about solving this problem?

Often if a problem is hard, try to do a simpler one first.

What do you think is easier?

Try a 2x2 version…. Impossible!



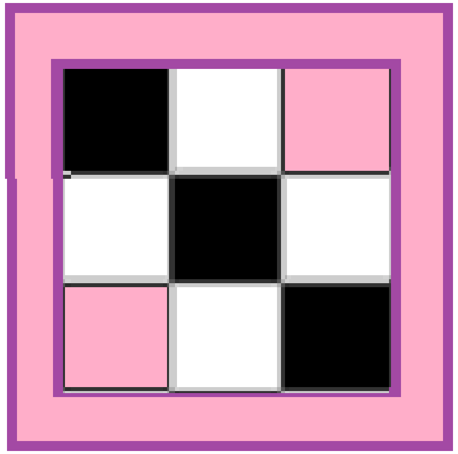
Checkerboard Problem

How would you go about solving this problem?

Often if a problem is hard, try to do a simpler one first.

What do you think is easier?

Try a 3x3 version….



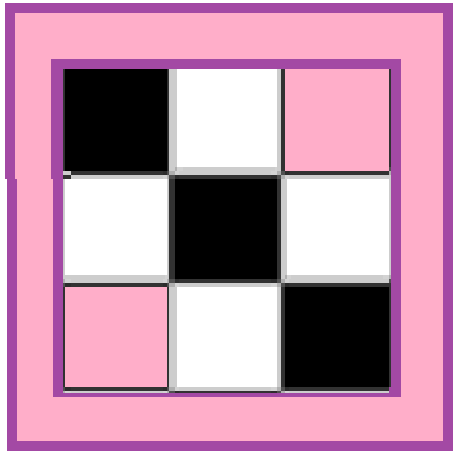
Checkerboard Problem

How would you go about solving this problem?

Often if a problem is hard, try to do a simpler one first.

What do you think is easier?

Try a 3x3 version…. Impossible!



Cannot do as each tile covers up two squares, and thus we always cover up an EVEN number of squares, but on any ODD by ODD board there are an odd number of squares left to cover!

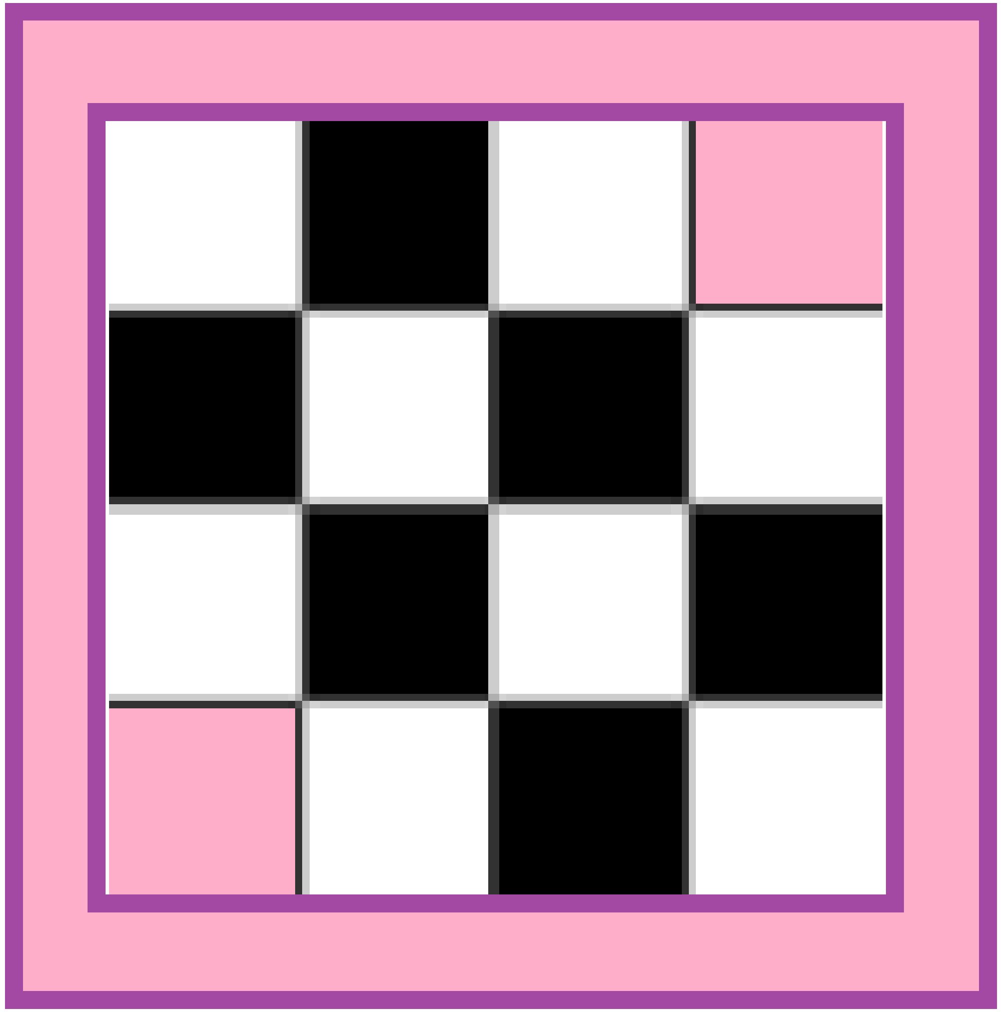
Checkerboard Problem

How would you go about solving this problem?

Often if a problem is hard, try to do a simpler one first.

What do you think is easier?

Try a 4x4 version….



Checkerboard Problem

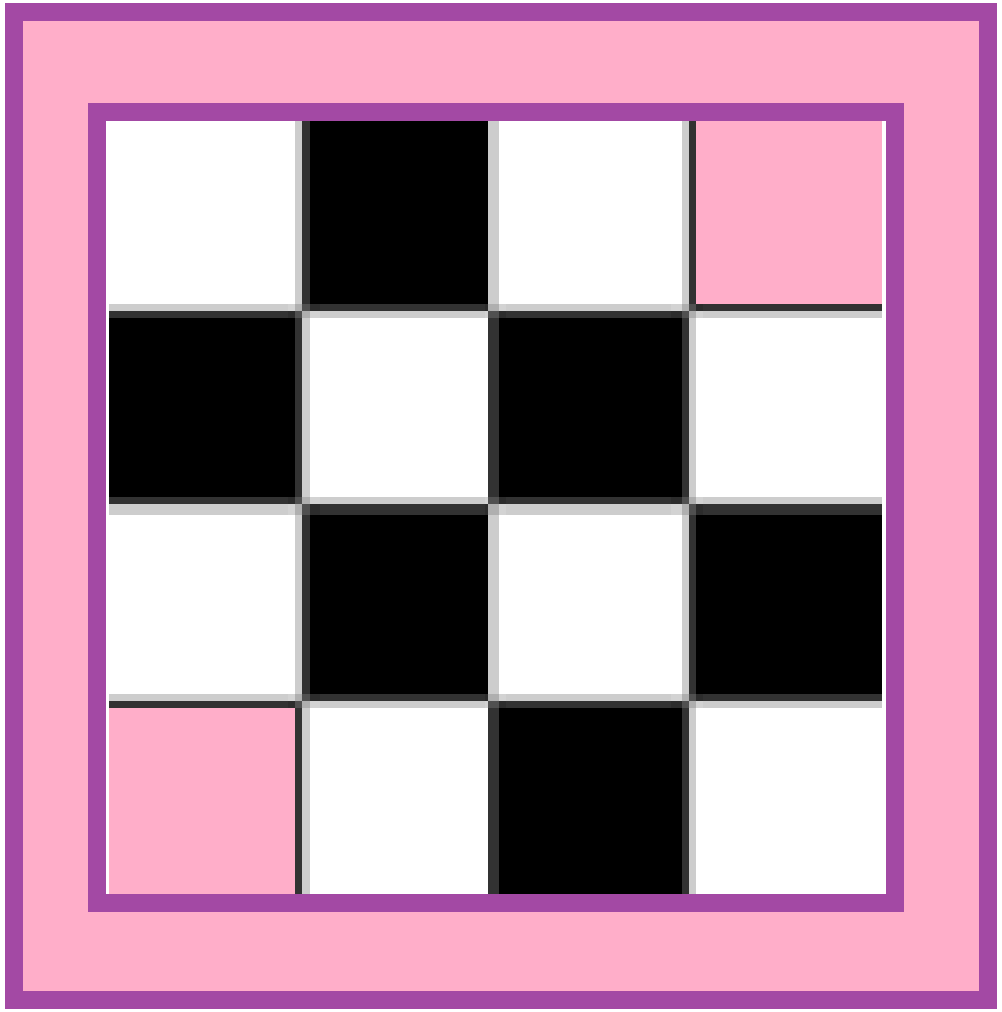
How would you go about solving this problem?

Often if a problem is hard, try to do a simpler one first.

What do you think is easier?

Try a 4x4 version…. Impossible!

Though could take awhile to show no approach will work.



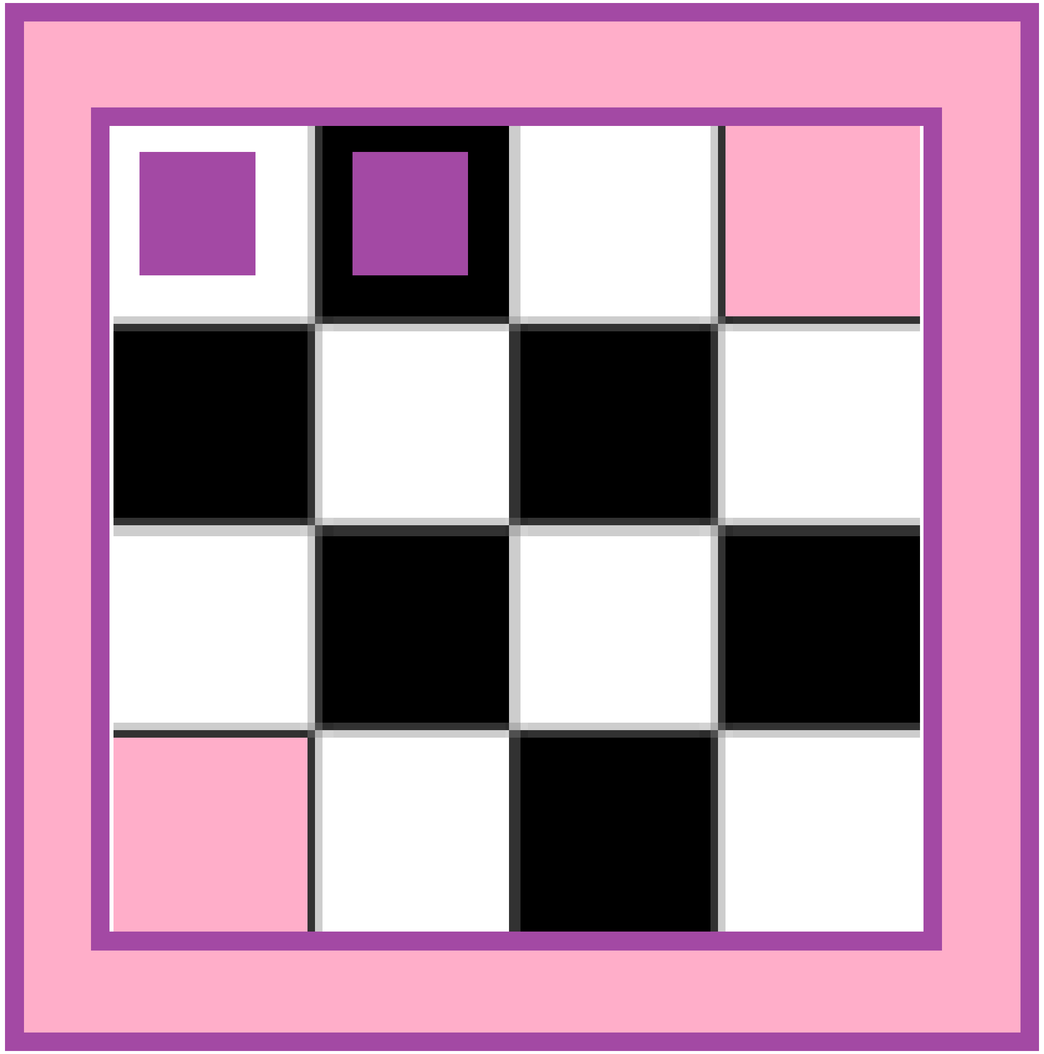
Checkerboard Problem

How would you go about solving this problem?

Try a 4x4 version…. Impossible!

Though could take awhile to show no approach will work.

By symmetry, can assume first tile is in top row going East-West. Where is next?



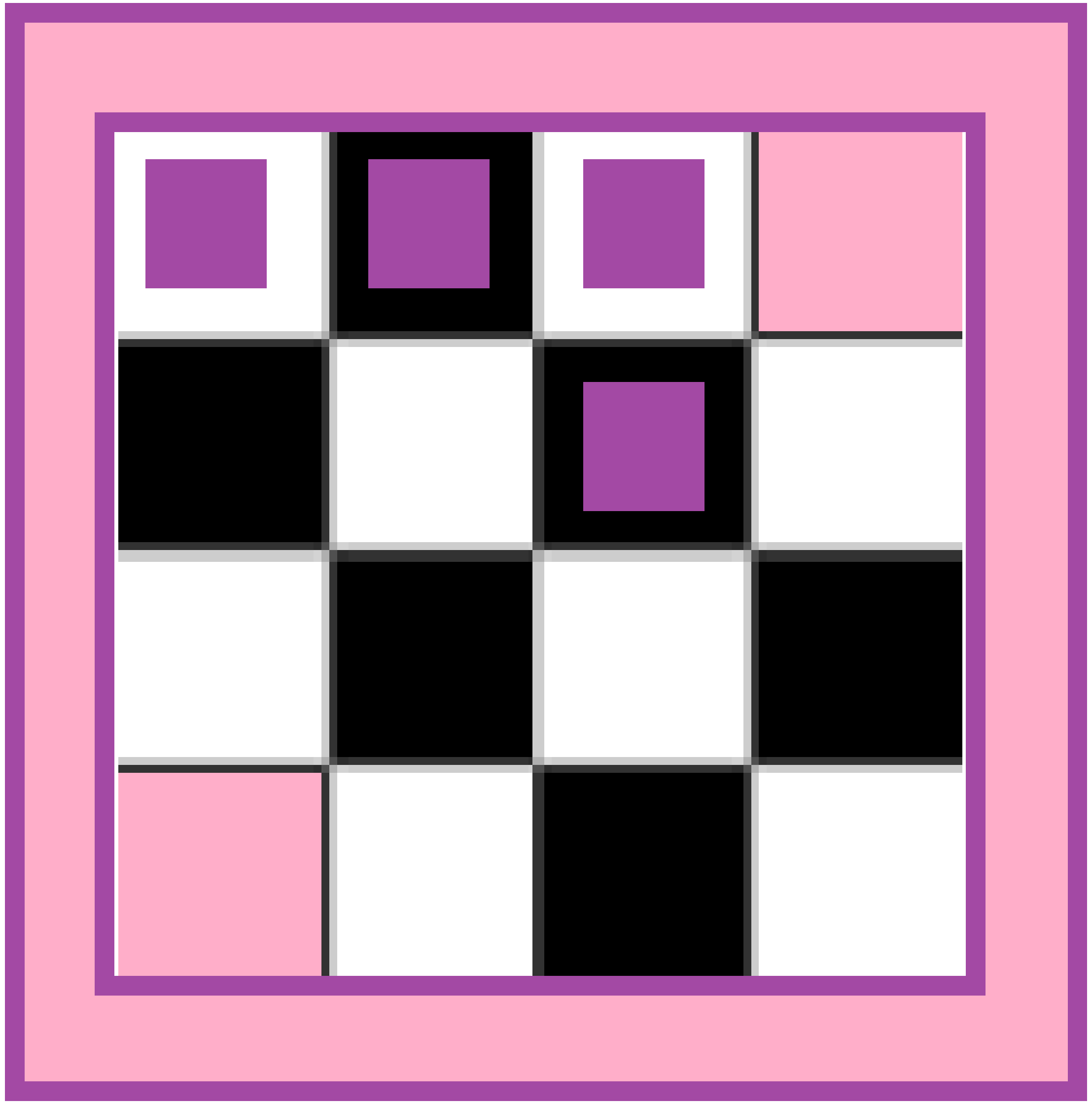
Checkerboard Problem

By symmetry, can assume first tile is in top row going East-West. Where is next?

Has to go down from top....

If we do not have one here, we cannot get the third square in the top row.

Where is next?

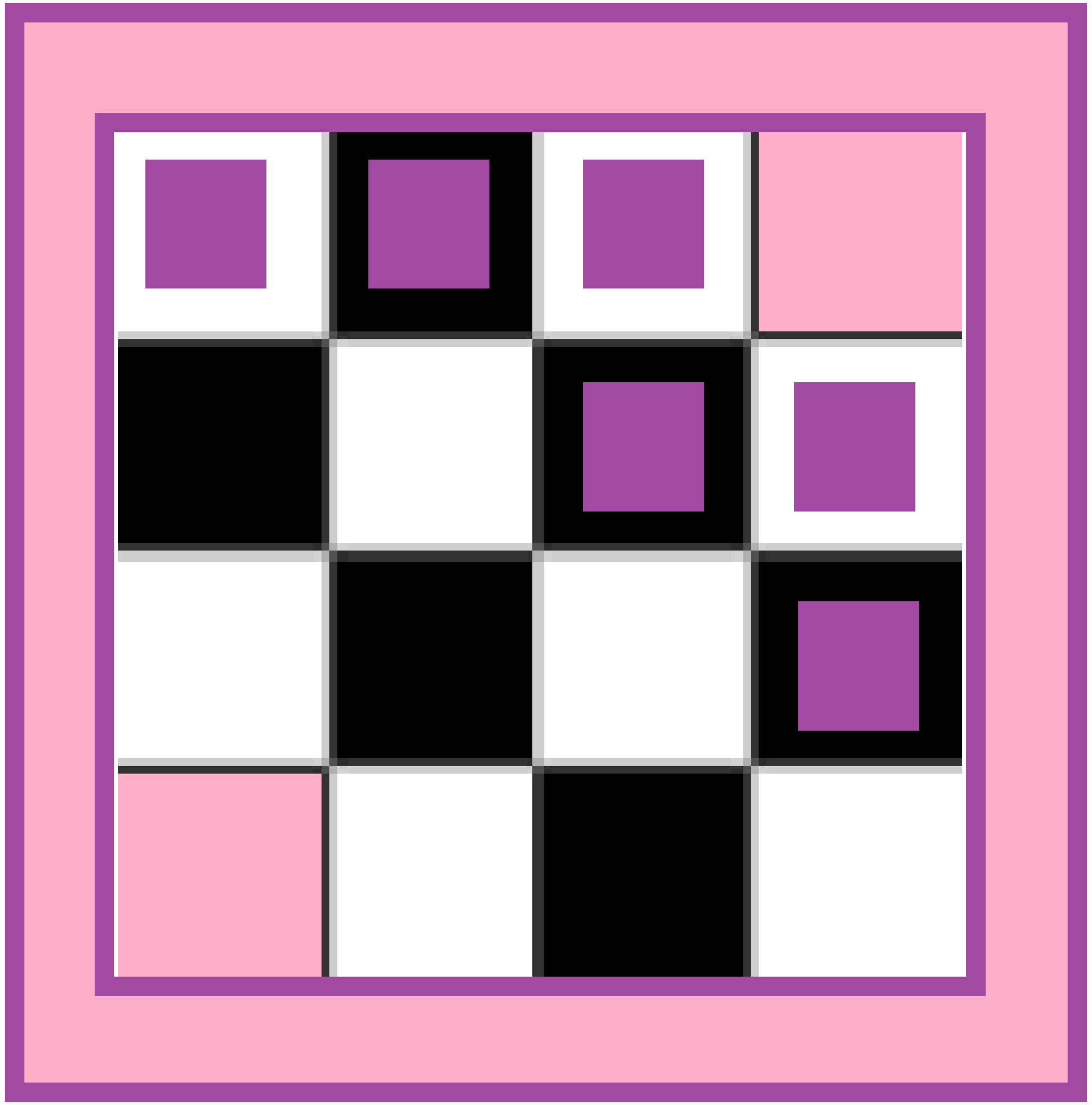


Checkerboard Problem

By symmetry, can assume first tile is in top row going East-West. Where is next?

Has to go down from top…. Where is next?

Has to be in the last column…. Where is next?



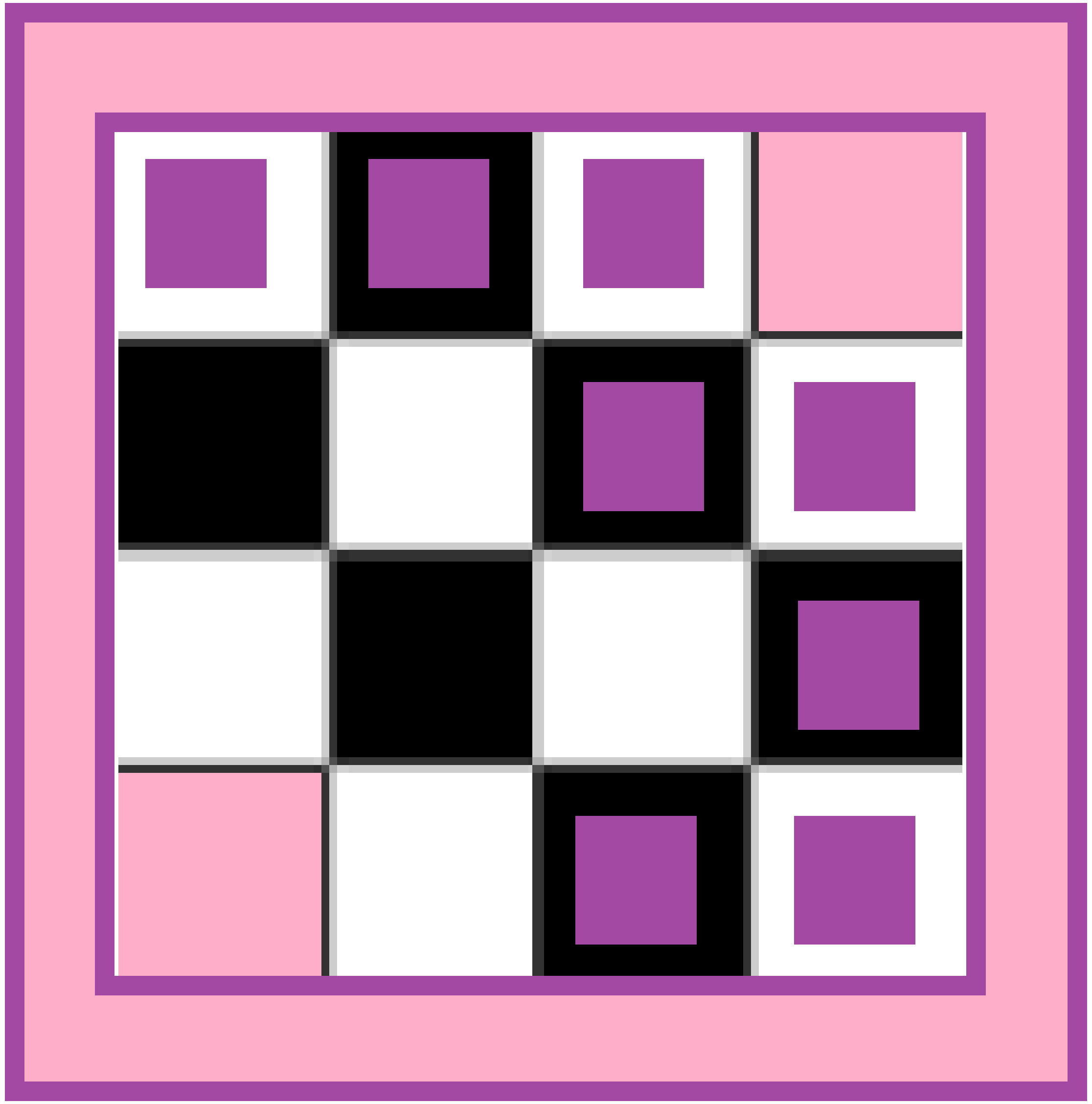
Checkerboard Problem

By symmetry, can assume first tile is in top row going East-West. Where is next?

Has to go down from top…. Where is next?

Has to be in the last column…. Where is next?

Has to be in the last row…. Where is next?



Checkerboard Problem

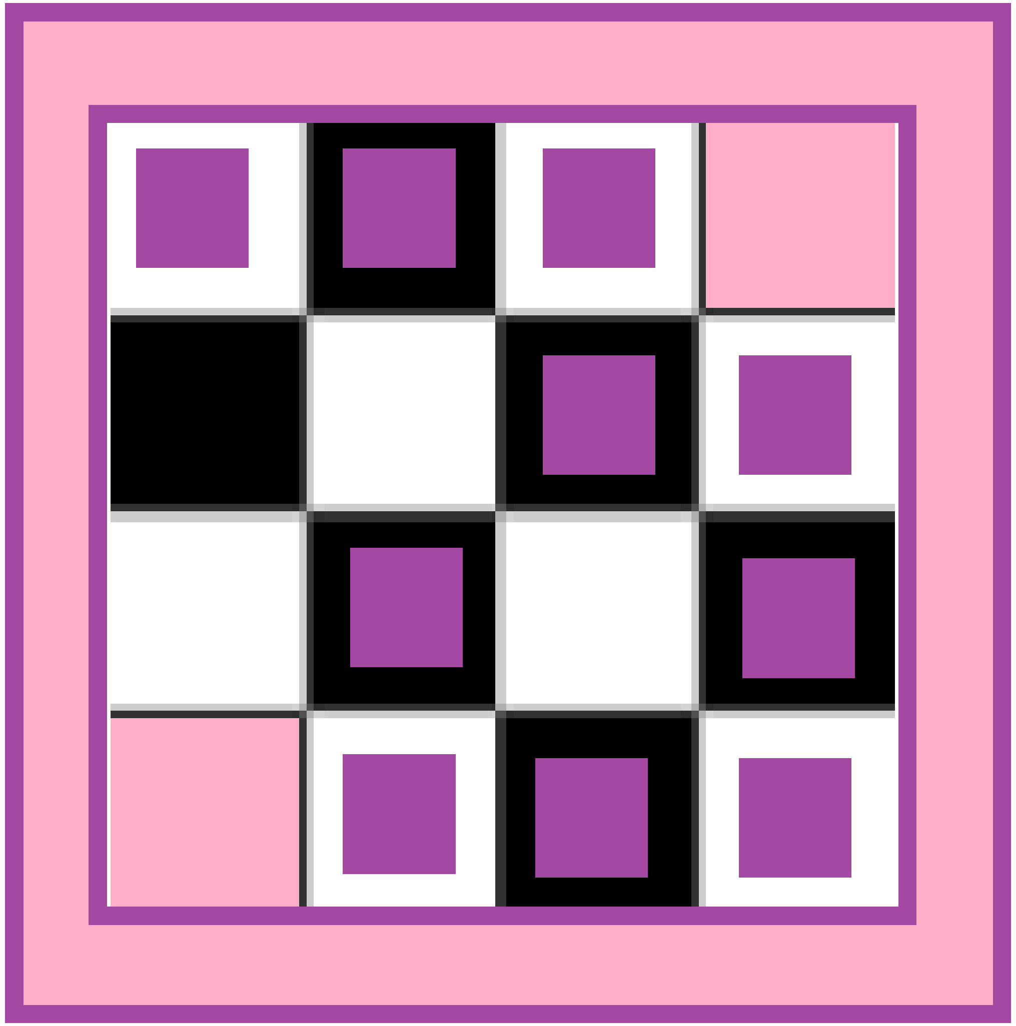
By symmetry, can assume first tile is in top row going East-West. Where is next?

Has to go down from top…. Where is next?

Has to be in the last column…. Where is next?

Has to be in the last row…. Where is next?

Has to come up from last row, now in trouble. Cannot get the surrounded white square.

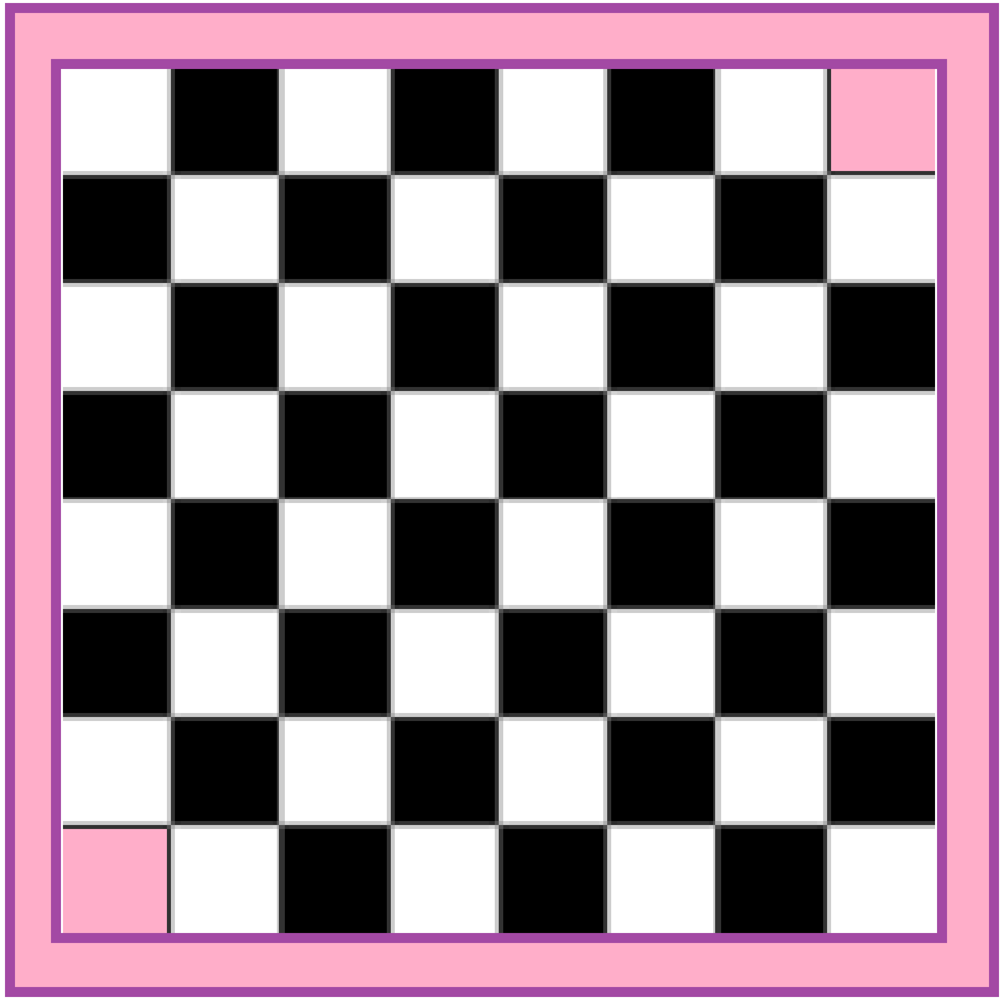


Checkerboard Problem

We need to look at all the other even boards.

Is there a better approach?

Think about what happens every time you put down a tile. What is covered up?

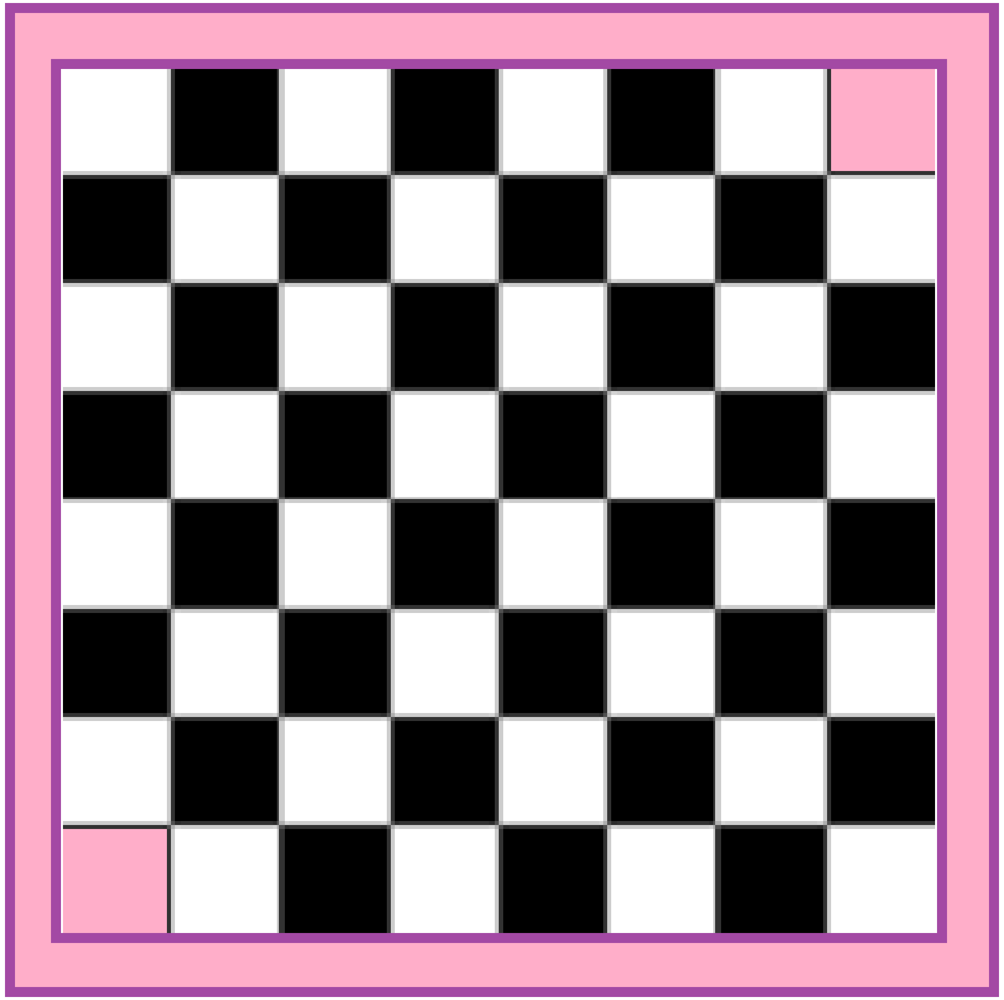


Checkerboard Problem

Think about what happens every time you put down a tile. What is covered up?

Every time we put down a tile we cover up ONE white square and ONE black square.

How does this help us?

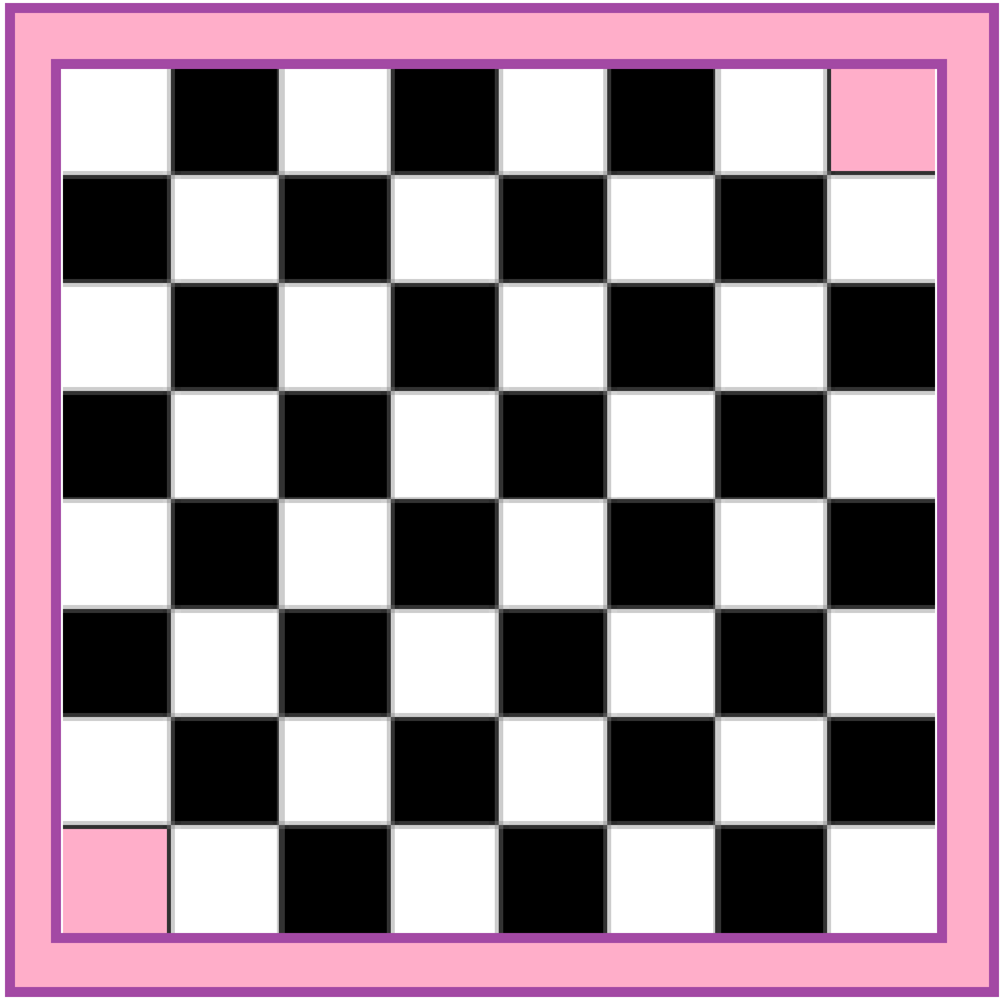


Checkerboard Problem

Think about what happens every time you put down a tile. What is covered up?

Every time we put down a tile we cover up ONE white square and ONE black square.

As we removed both white corners, we start with two more black squares than white – it is impossible!

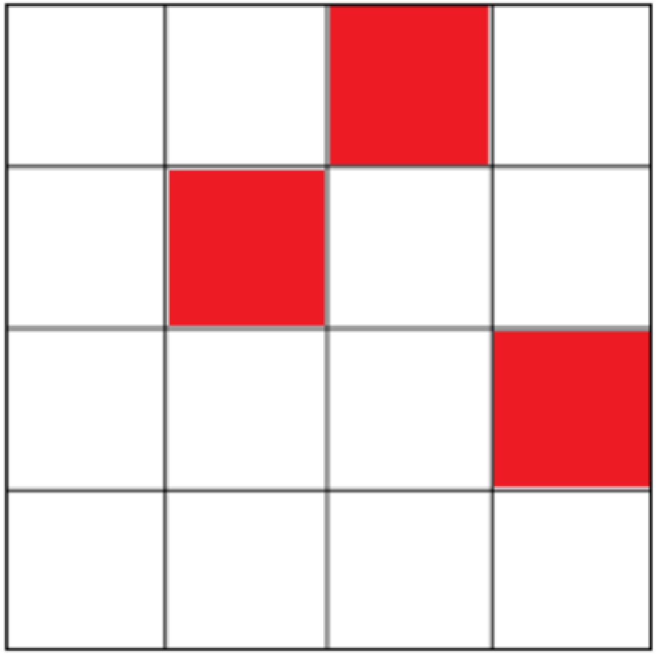


Zombies

(though perhaps we should change it to people being infected with covid-19, given the present circumstances)

Zombine Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



Initial Configuration

Zombine Infection: Rules

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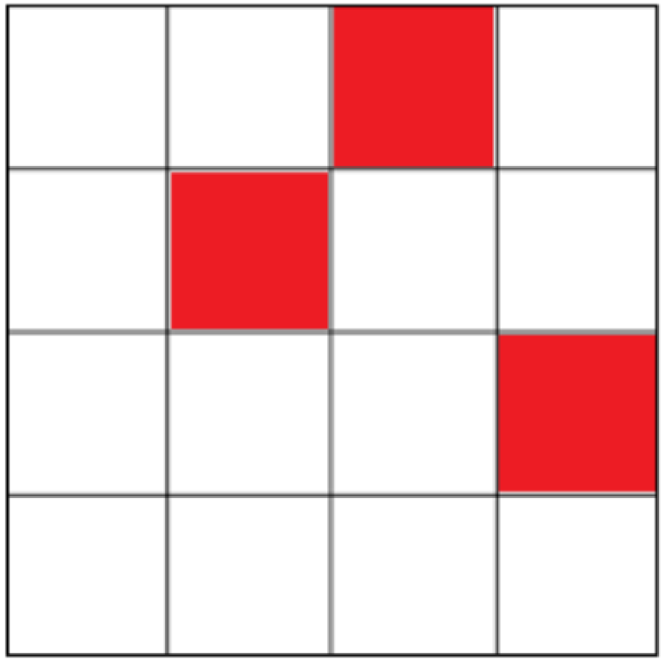
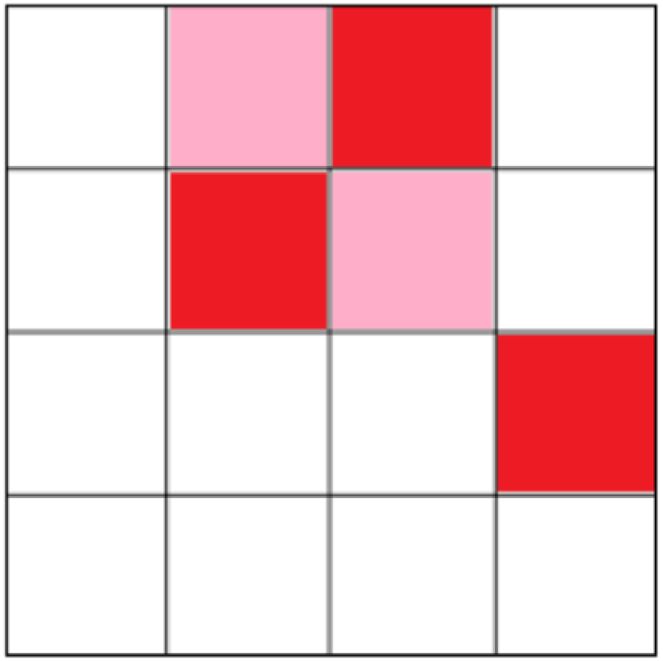


Initial Configuration



One moment later

Zombine Infection: Rules

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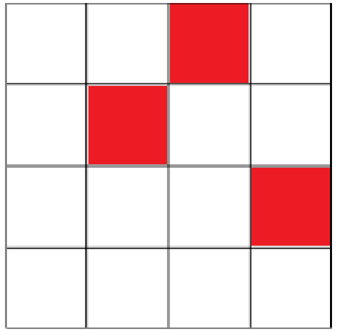
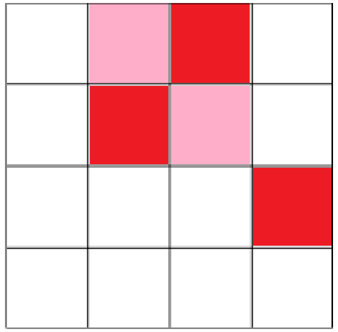
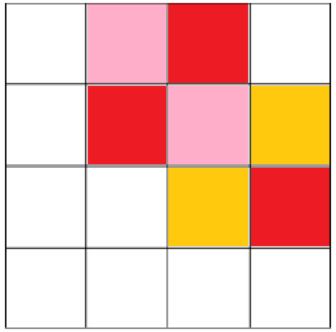


Initial Configuration



One moment later



Two moments later

Zombine Infection: Rules

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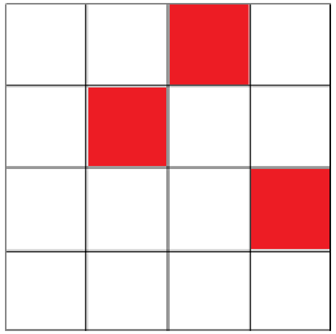
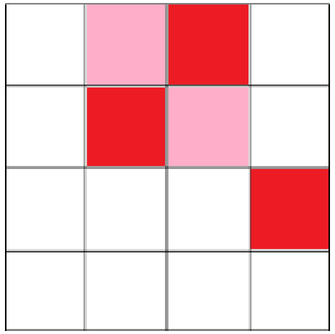
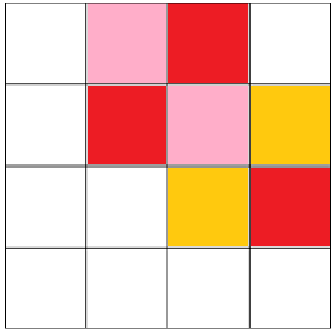


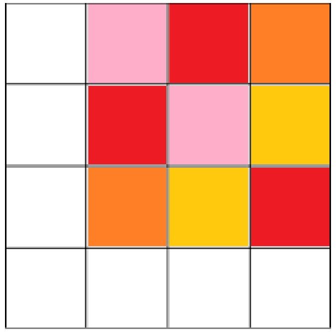
Initial Configuration



One moment later



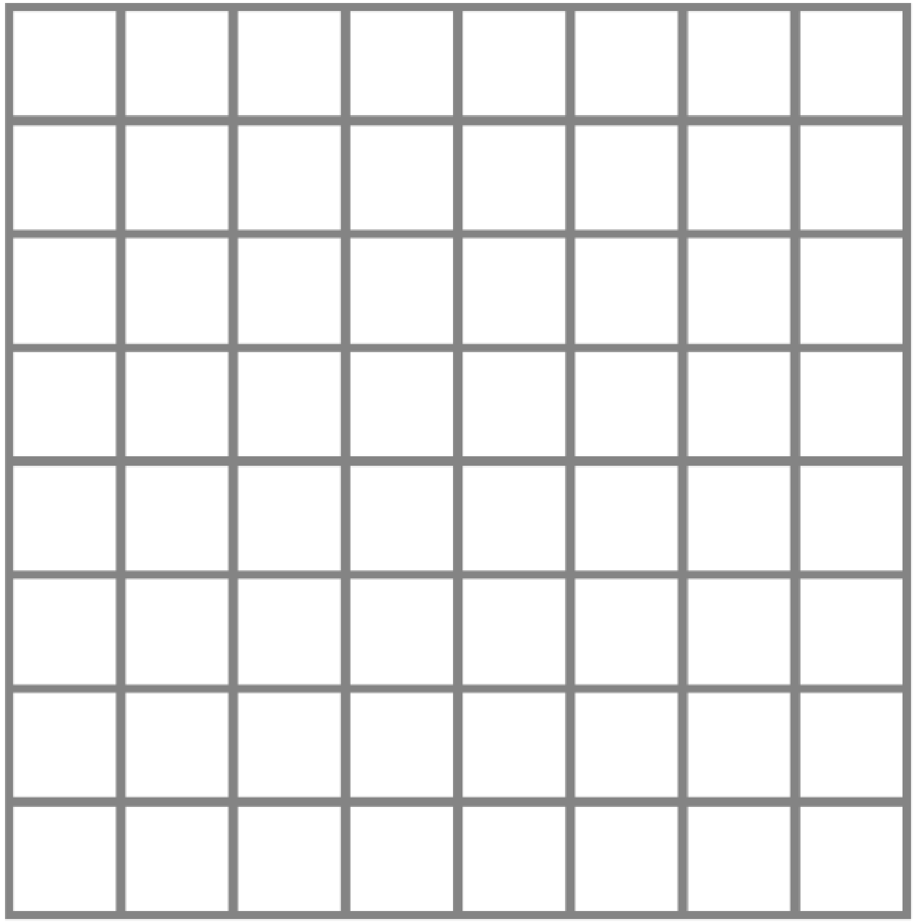
Two moments later



Three moments later

Zombine Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



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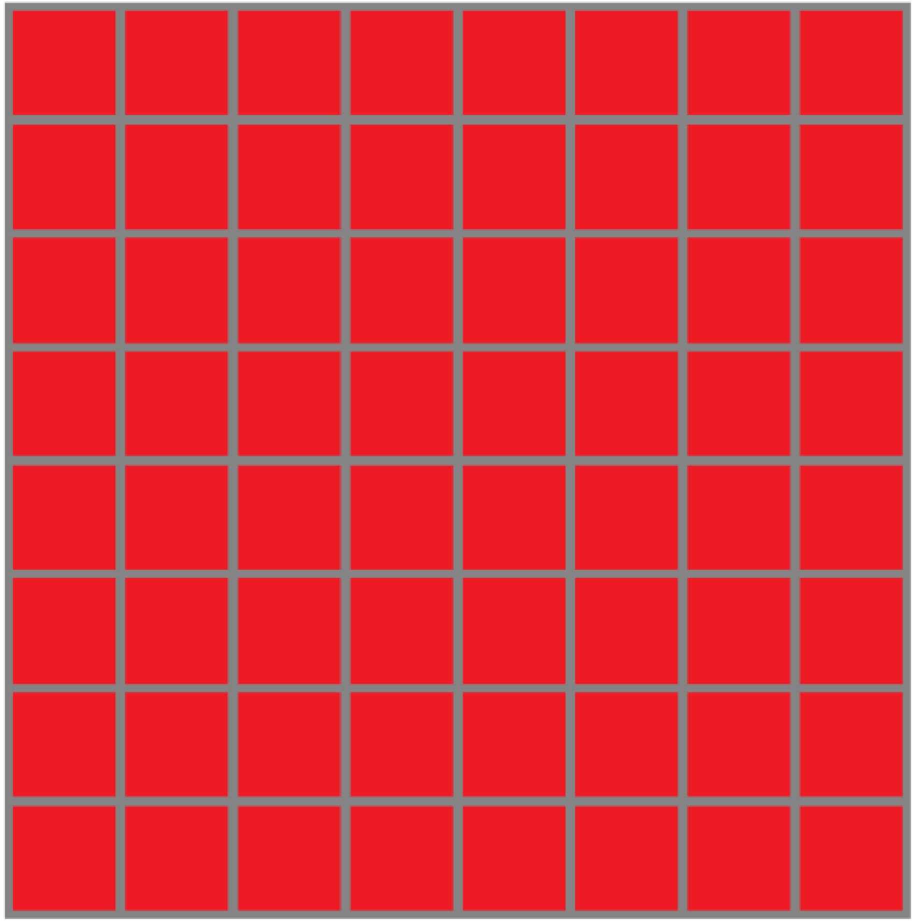
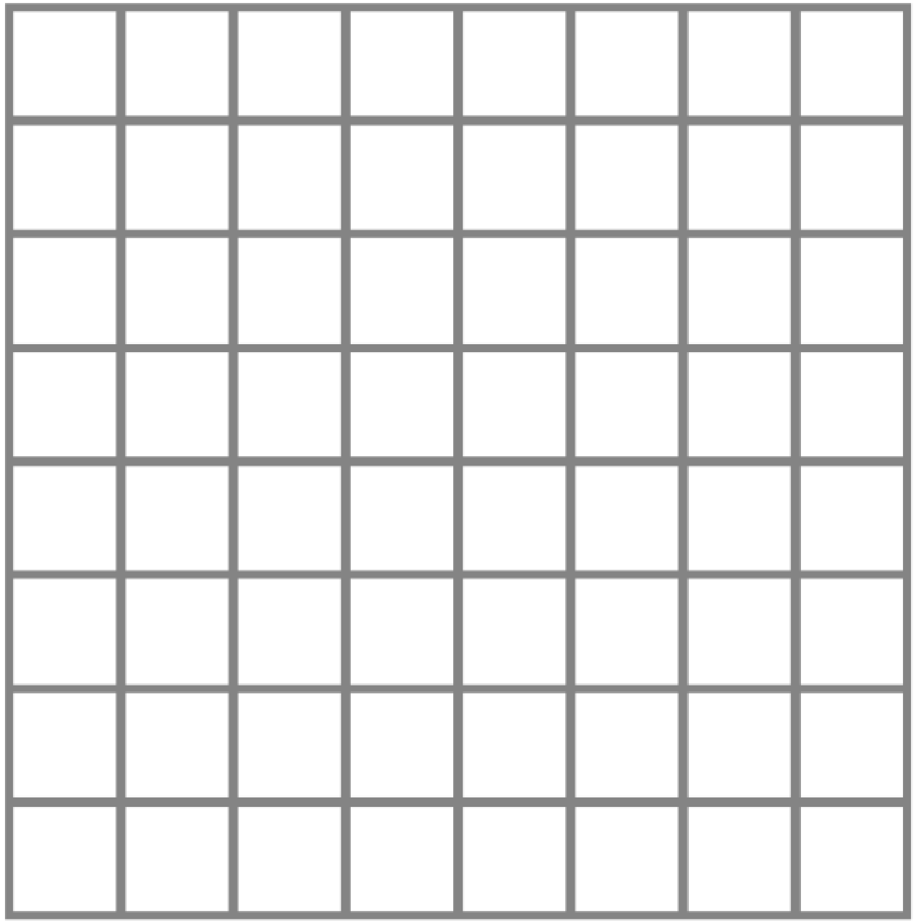


Zombine Infection: Conquering The World

Next simplest initial state ensuring all eventually infected...?



Zombine Infection: Conquering The World

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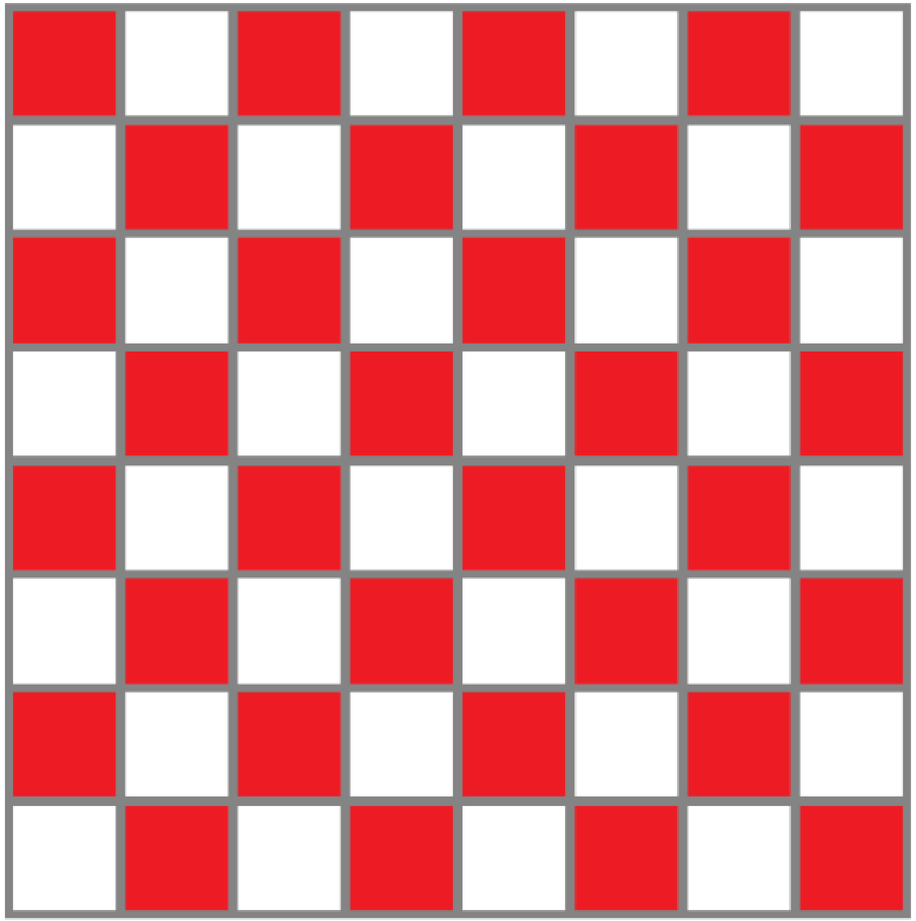
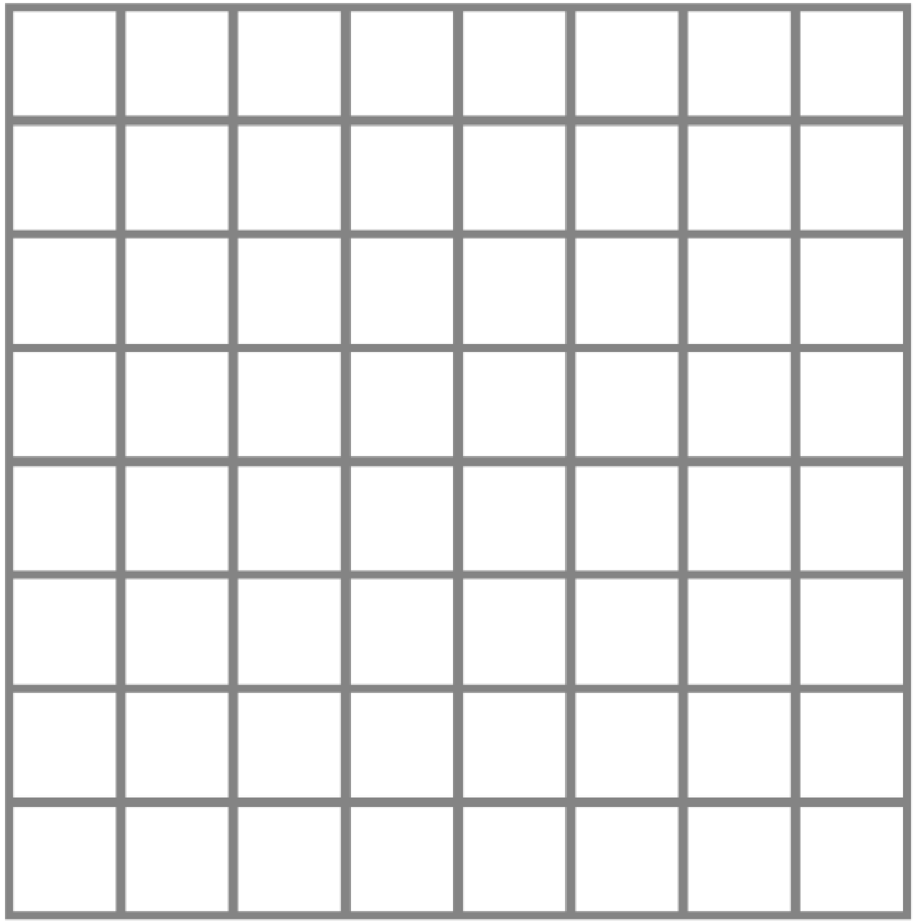


Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



Zombie Infection: Conquering The World

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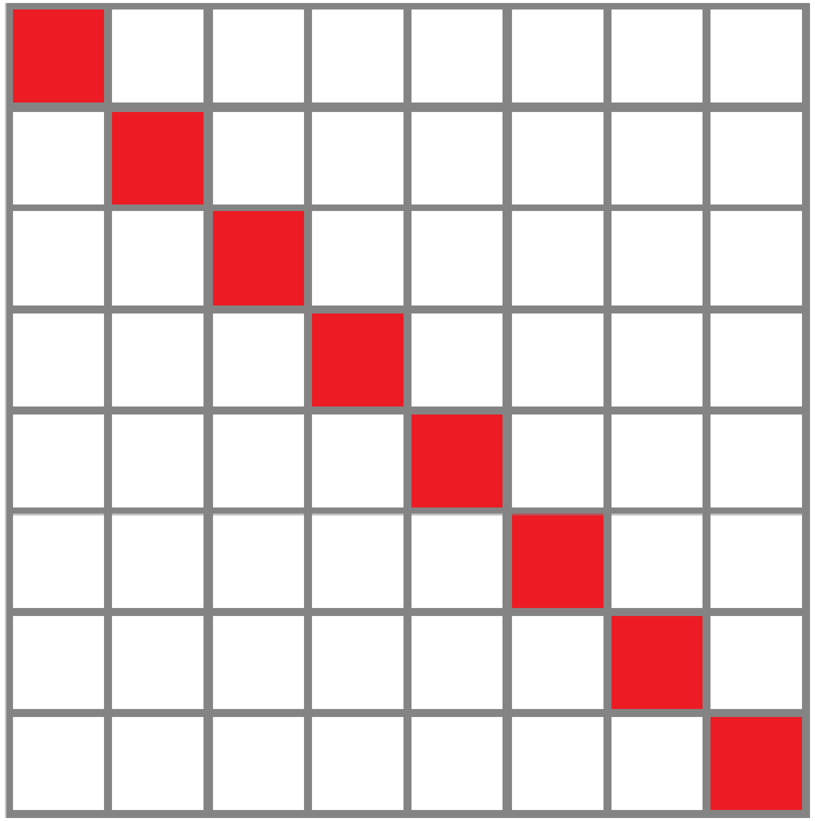

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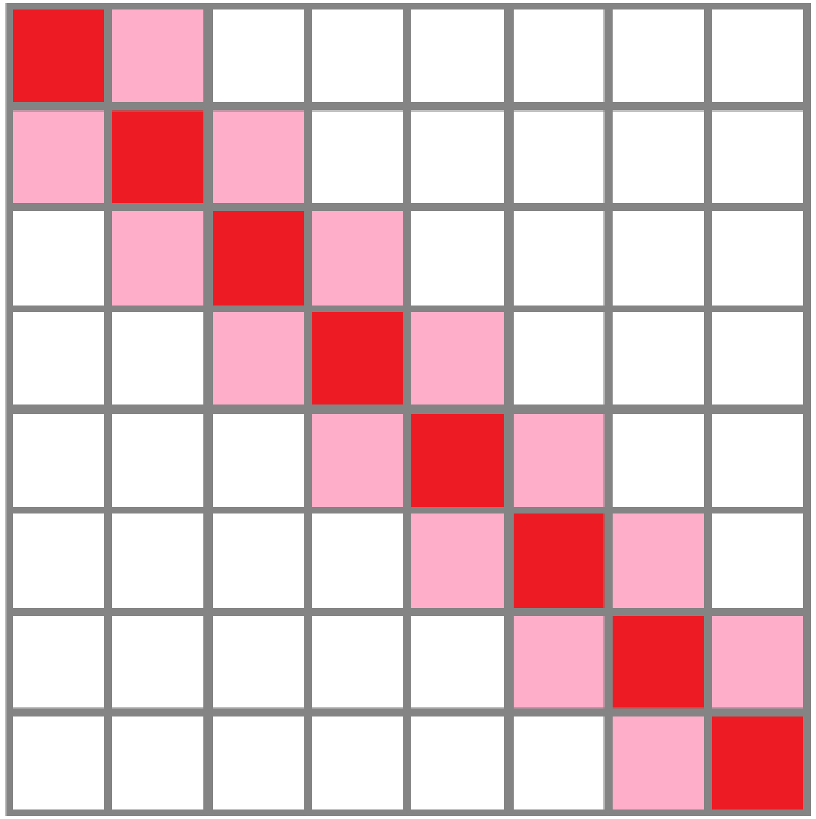

Zombie Infection: Conquering The World

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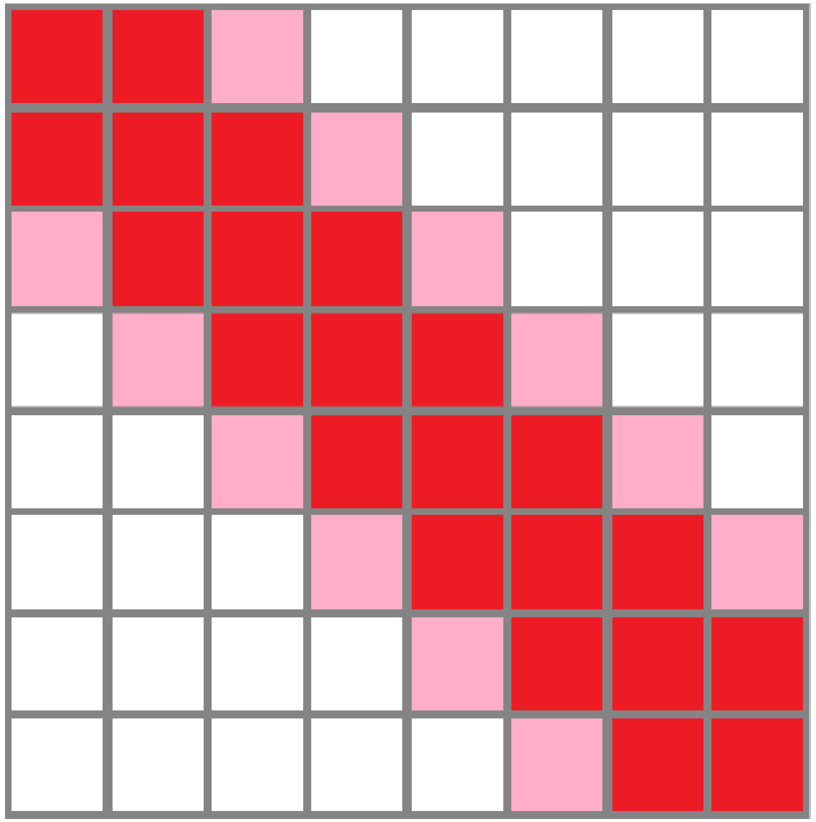

Zombie Infection: Conquering The World

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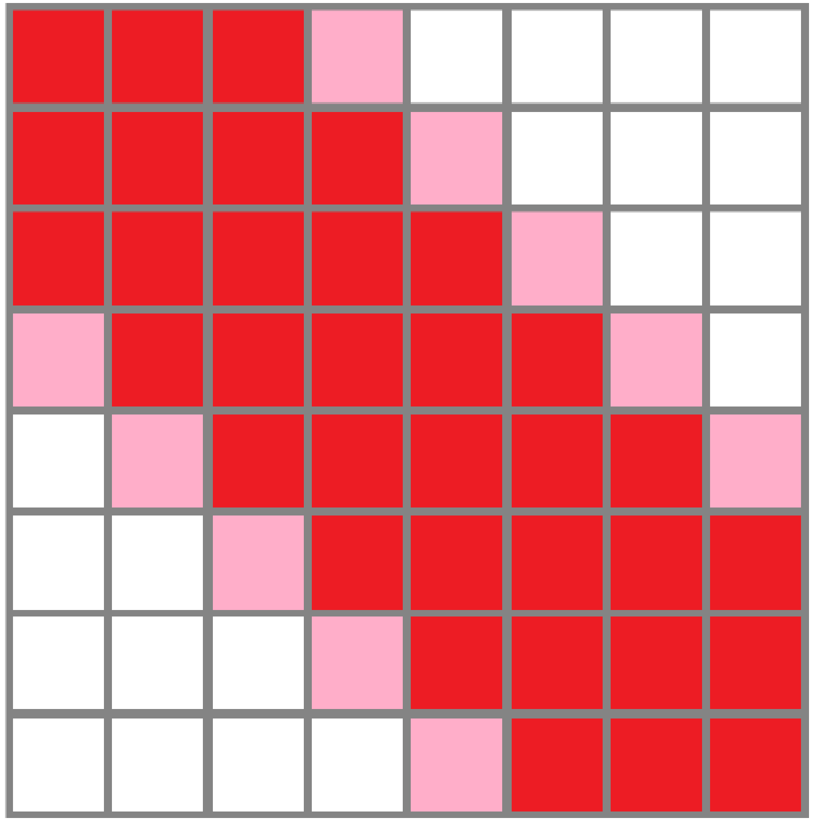
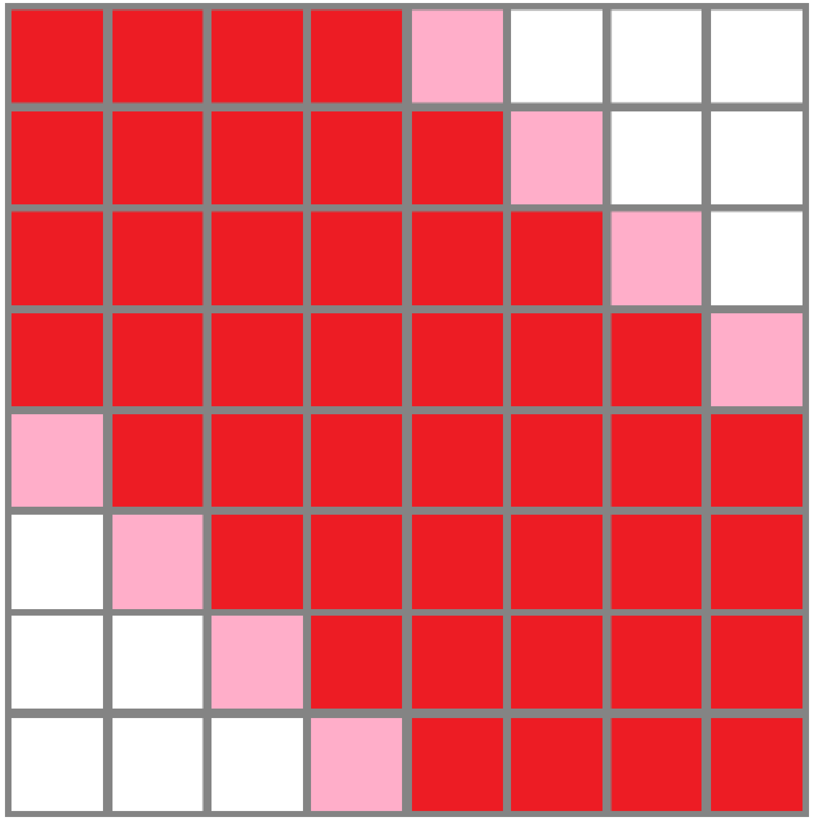


Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



Zombie Infection: Conquering The World

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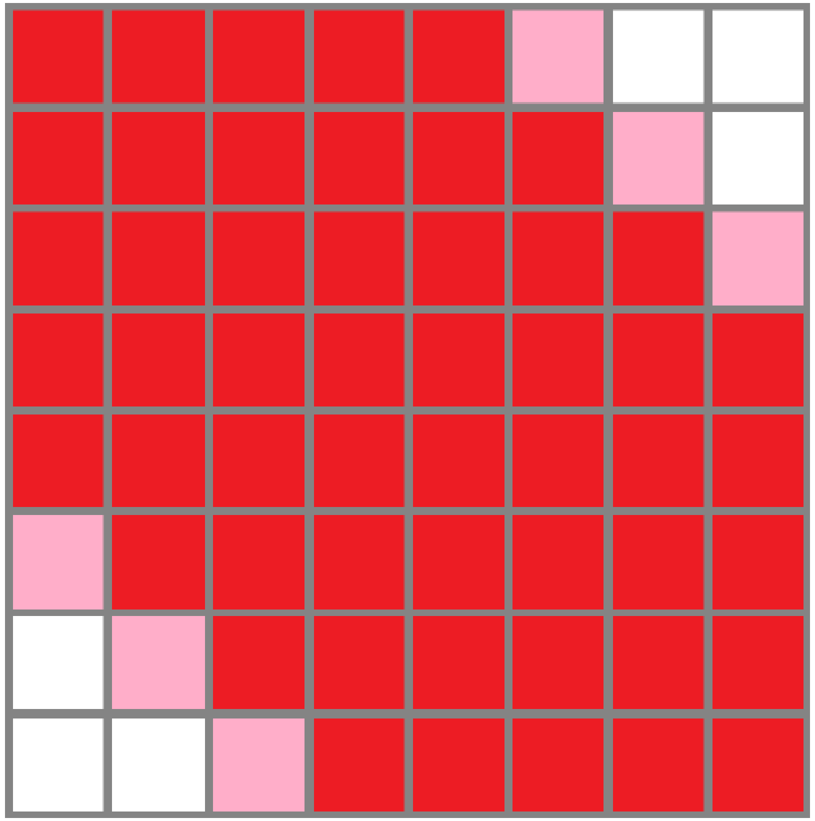
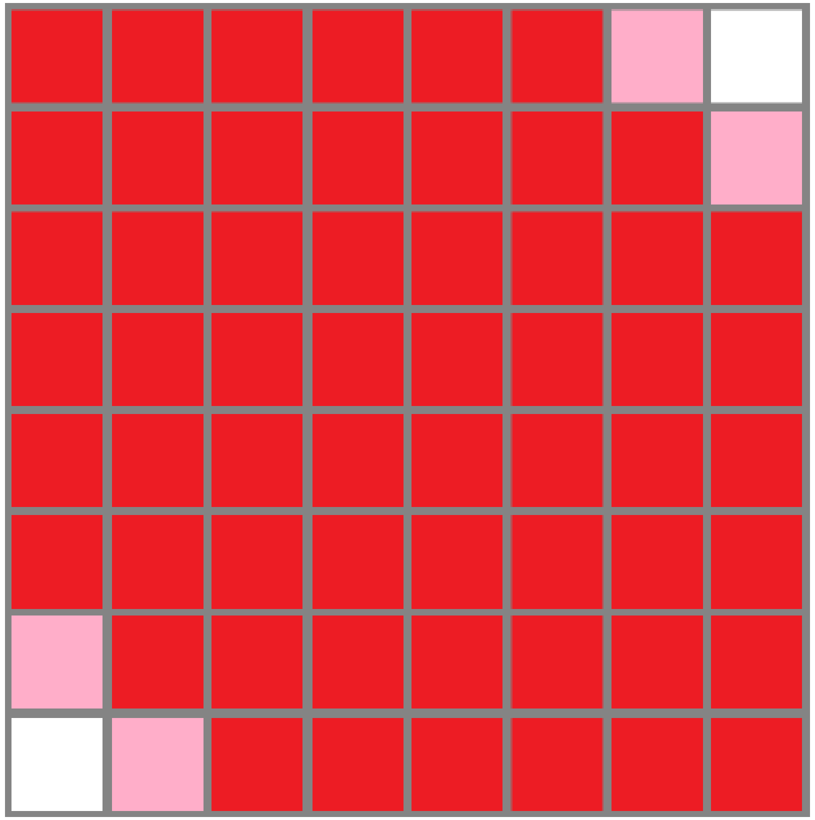


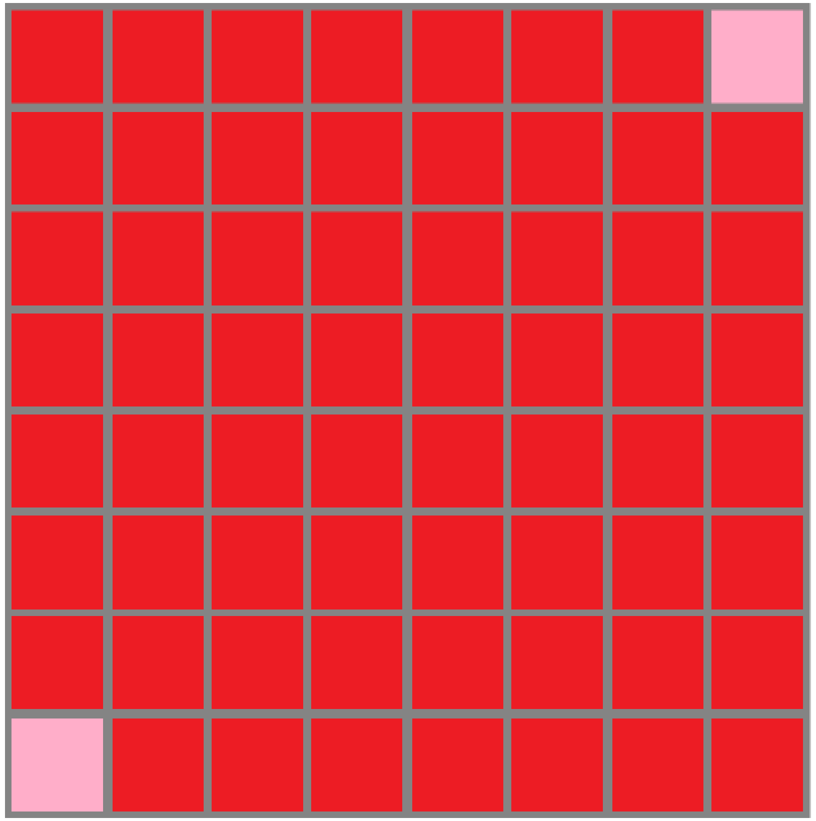
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



Zombie Infection: Can $n-1$ infect all?

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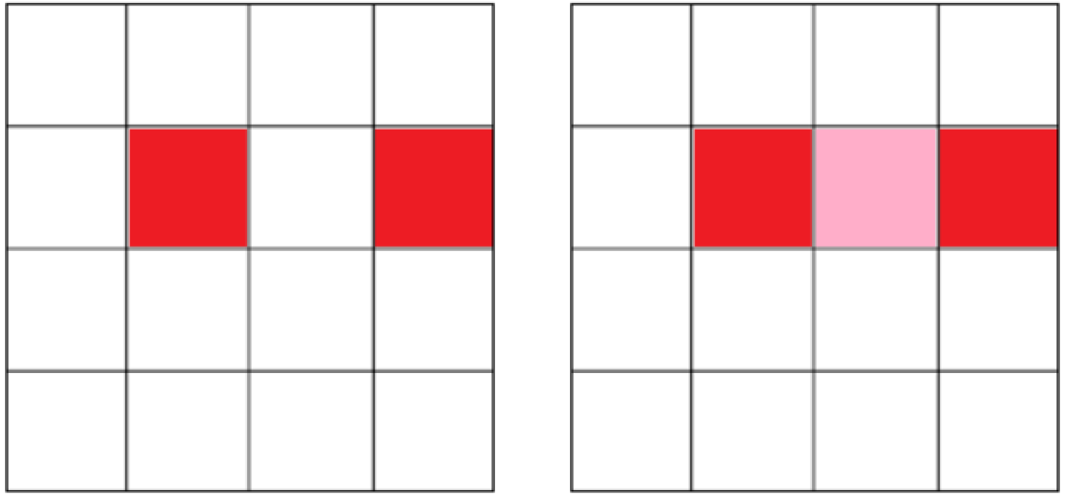
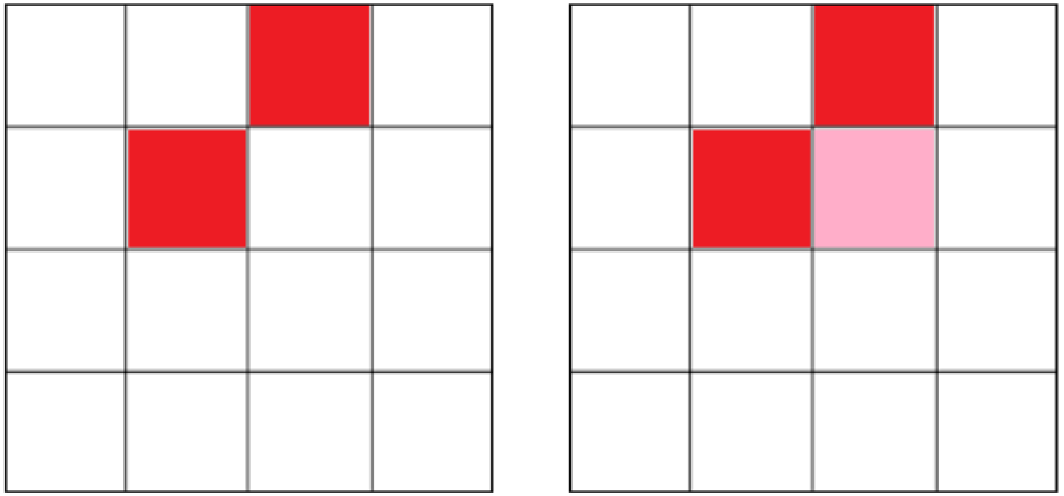


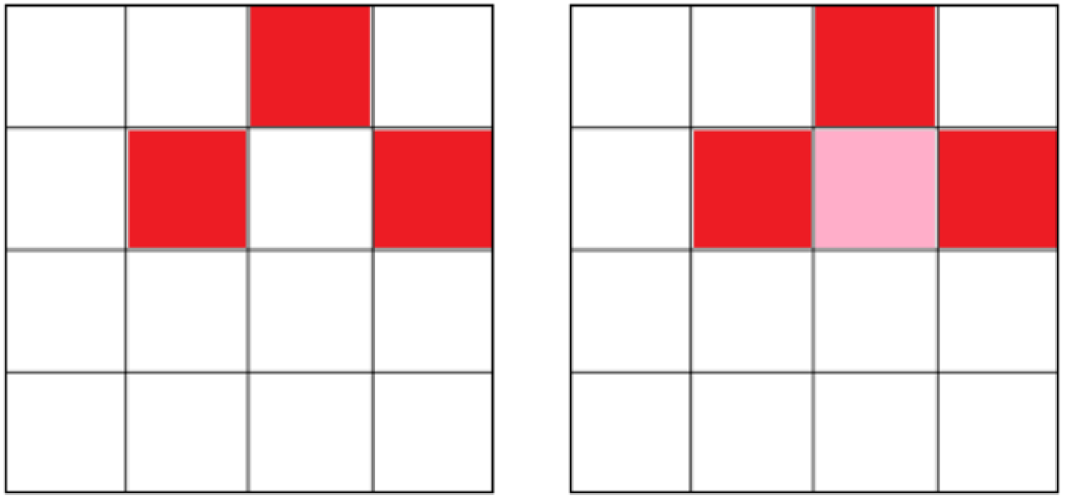
Perimeter of infection unchanged.

Zombie Infection: Can $n - 1$ infect all?



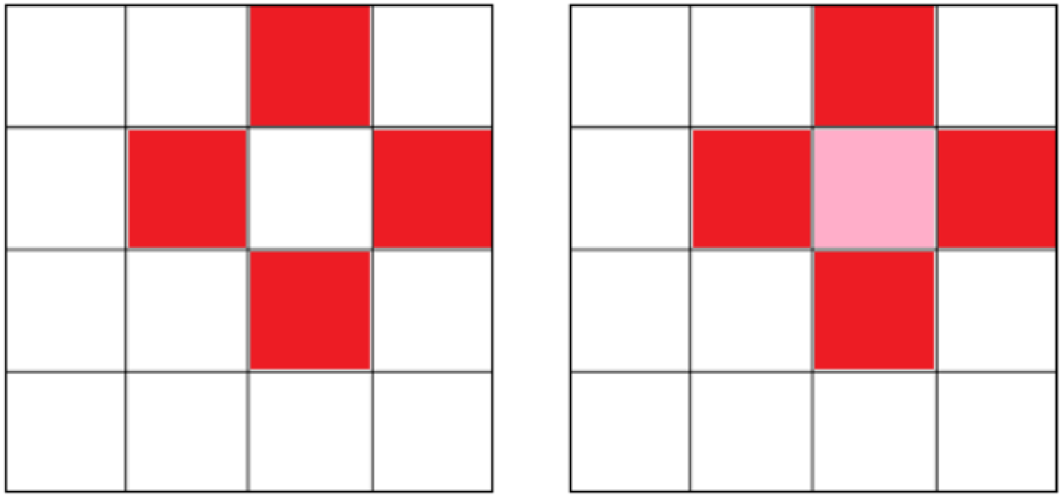
Perimeter of infection unchanged.

Zombie Infection: Can $n - 1$ infect all?



Perimeter of infection decreases by 2.

Zombie Infection: Can $n - 1$ infect all?



Perimeter of infection decreases by 4.

Zombie Infection: $n-1$ cannot infect all

- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.
- Mono-variant: As time passes, perimeter of infection never increases.
- Perimeter of $n \times n$ square is $4n$, so at least 1 square safe!

This is just the tip of the iceberg for problems using invariants or mono-variants. Another great one is the Conway checkers (or soldiers) problem. See https://en.wikipedia.org/wiki/Conway%27s_Soldiers for details, as well as this excellent video: https://www.youtube.com/watch?v=Or0uWM9bT5w .

However, before you go to these sites, look at the statement below, explore, conjecture, and see what you can do! Note: While there is a mono-variant solution, it is a lot more involved than the ones seen so far.

Conway's Soldiers

From Wikipedia, the free encyclopedia

Conway's Soldiers or the **checker-jumping problem** is a one-person mathematical game or puzzle devised and analyzed by mathematician John Horton Conway in 1961. A variant of peg solitaire, it takes place on an infinite checkerboard. The board is divided by a horizontal line that extends indefinitely. Above the line are empty cells and below the line are an arbitrary number of game pieces, or "soldiers". As in peg solitaire, a move consists of one soldier jumping over an adjacent soldier into an empty cell, vertically or horizontally (but not diagonally), and removing the soldier which was jumped over. The goal of the puzzle is to place a soldier as far above the horizontal line as possible.