Multidimensional Zeckendorf Decompositions

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Fibonacci and Zeckendorf

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Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1$.

Generalized Recurrences

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Definition (M. Kologlu, G.S. Kopp, S. J. Miller, Y. Wang, 2011)

For a positive coefficient vector $\vec{\mathbf{c}} = (c_1, \ldots, c_k) \in \mathbb{Z}^k$, we define a sequence $(X_n) \subseteq \mathbb{Z}$ such that:

1. $X_1 = 1$;
2. $X_{n+1} = c_1 X_n + \cdots + c_n X_1 + 1$, for $1 \leq n < k$;
3. $X_{n+1} = c_1 X_n + \cdots + c_k X_{n-k+1}$, for $n \geq k$.

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\begin{aligned}
X_1 &= 1, \\
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- **Sequence:** $1, 3, 8, 20, 51, 130, \dots$.

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Definition (AB-J, 2011)

- $\vec{\mathbf{X}}_0 := \vec{\mathbf{0}}$;
- $\vec{\mathbf{X}}_{-i} := \vec{\mathbf{e}}_i$, for $i < k$;
- $\vec{\mathbf{X}}_n := c_1\vec{\mathbf{X}}_{n-1} + \cdots + c_k\vec{\mathbf{X}}_{n-k}$, for $n \in \mathbb{Z}$.

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$$\vec{\mathbf{X}}_n = \vec{\mathbf{X}}_{n+k} - \sum_{i=1}^{k-1} c_i \vec{\mathbf{X}}_{n+k-i}$$

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- $\vec{v_1} := 2\vec{\mathbf{X}}_{-1} + 4\vec{\mathbf{X}}_{-2} + 2\vec{\mathbf{X}}_{-3} + 1\vec{\mathbf{X}}_{-4} \longrightarrow$ 2, 4, 2, 1

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- $\vec{v_3} := 0\vec{\mathbf{X}}_{-1} + 2\vec{\mathbf{X}}_{-2} + 1\vec{\mathbf{X}}_{-3} + 0\vec{\mathbf{X}}_{-4} + 1\vec{\mathbf{X}}_{-5} \longrightarrow$ 0, 2, 1, 0, 1

Anderson and Bicknell-Johnson

Theorem (AB-J)

Every $\vec{\mathbf{v}} \in \mathbb{Z}^{k-1}$ *has a unique* $\vec{\mathbf{c}} = (1, \ldots, 1)$*-satisfying representation.*

Weakly Decreasing Coefficients

Definition (Weakly decreasing)

Vector $\vec{\mathbf{c}} = (c_1, \ldots, c_k)$ is weakly decreasing if $c_n \geq c_{n+1}$.

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If $\vec{\mathbf{c}} = (c_1, c_2, \ \ldots, \ c_k)$ is weakly decreasing and $c_k = 1$, every $\vec{\mathbf{v}} \in \mathbb{Z}^{k-1}$ has a **unique $\vec{\mathbf{c}}$-satisfying representation**.

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- Prove uniqueness.
- Prove existence.

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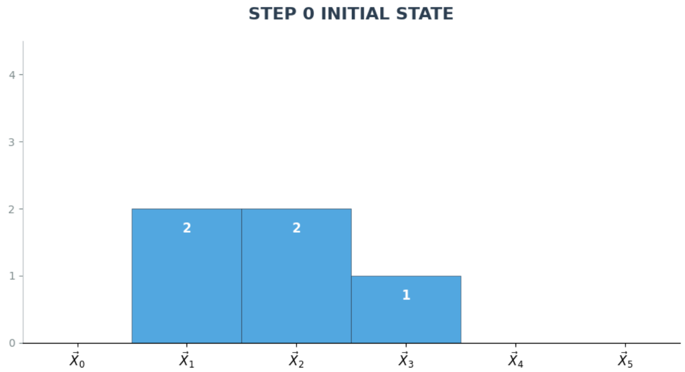
How to turn a *non-satisfying representation* into a satisfying one?

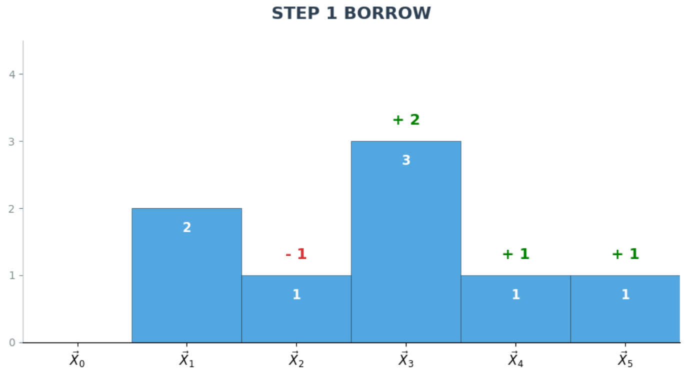
- **Borrowing:** If a coefficient is too high, decrease it by expanding the recursion.
- **Carrying:** If we have a copy of the recursion we can absorb or "condense" it into the previous term.

STEP 0 INITIAL STATE

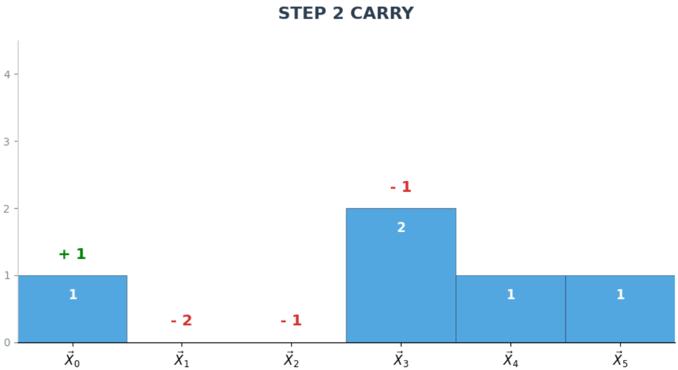
Let $\vec{\mathbf{c}} = (2, 1, 1)$.



STEP 1 BORROW



STEP 2 CARRY


STEP 2 CARRY
4
3
2
1
0
+ 1
1
- 2
- 1
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2
1
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$\vec{X}_0$
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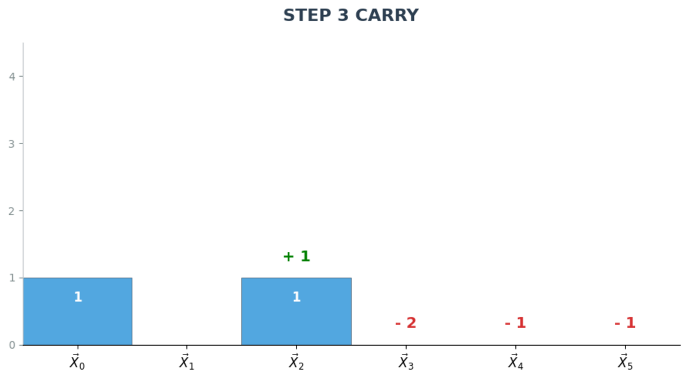
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If the algorithm terminates, we get a $\vec{\mathbf{c}}$-satisfying representation.

Algorithm Termination

Does the algorithm always terminate?

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Let $\vec{\mathbf{c}} = (1, 3, 1)$ and consider the representation $0, 2$.

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Let $\vec{\mathbf{c}} = (1, 3, 1)$ and consider the representation $0, 2$.

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C. $0, 1, 2, 0, 0, 5, 2;$

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...eventually we get $0, 1, 2, 0, 1, 3, 0, 1, 3, 0, 0, 5, 2$.

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...eventually we get $0, 1, 2, 0, 1, 3, 0, 1, 3, 0, 0, 5, 2$.

Observation: The prefix $0, 1, 2, 0$ and suffix $0, 5, 2$ remain fixed, while the middle gains an extra $1, 3, 0, 1, 3, 0$ block each cycle. The algorithm will therefore loop forever, inserting more and more terms in the middle without ever terminating.

Back to Main Theorem

Theorem (Main Theorem)

If $\vec{\mathbf{c}} = (c_1, c_2, \ \ldots, \ c_k)$ *is weakly decreasing and* $c_k = 1$*, then for every vector* $\vec{\mathbf{u}} \in \mathbb{Z}^{k-1}$ *there is always a representation for which the algorithm terminates.*

Special Properties

Central Limit Type Theorem (Lekkerkerker, 1952)

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for $m \in [F_n, F_{n+1})$ is **Gaussian**.

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[Theorem 3.5]

For weakly decreasing $\vec{\mathbf{c}}$ with $c_k = 1$: As $n \to \infty$ the distribution of number of summands in the general Zeckendorf representation for $\vec{\mathbf{v}} \in R_n$ (generalized regions) is **Gaussian**.

Illustrations

Illustrations for $\vec{\mathbf{c}} = (2, 1, 1)$.

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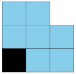

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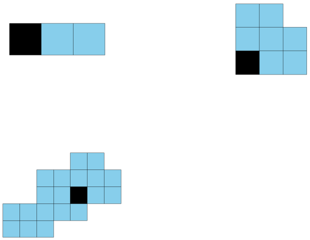

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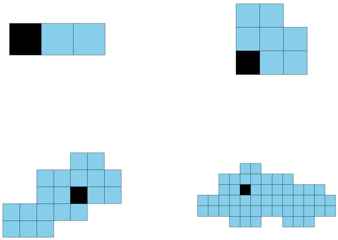

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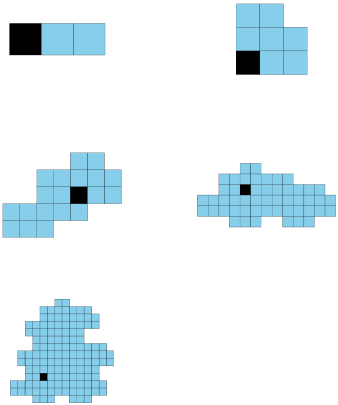

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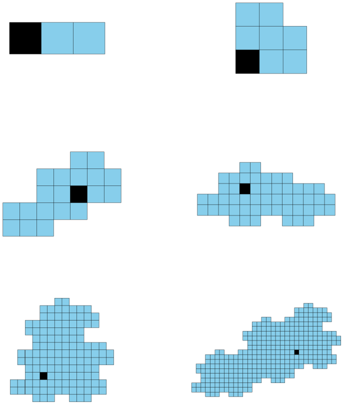

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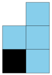

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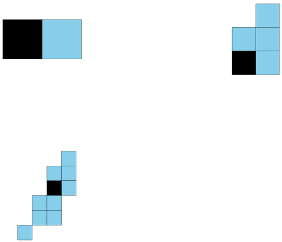

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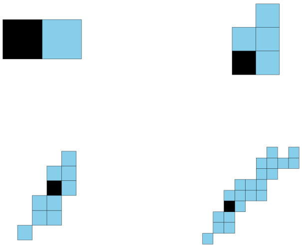

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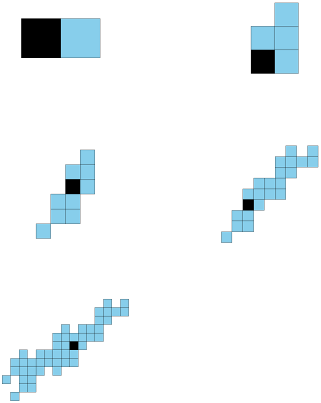

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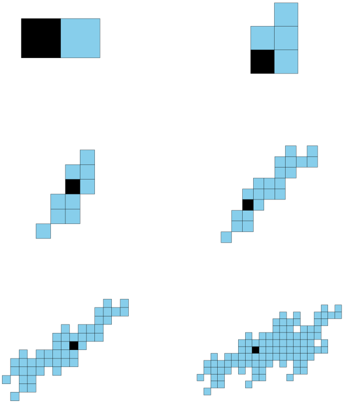

Further Research

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2. How "quickly" do these representations fill the entire space depending on the coefficient vector $\vec{\mathbf{c}}$?

Thank you!

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Steven J. Miller: `sjm1@williams.edu`

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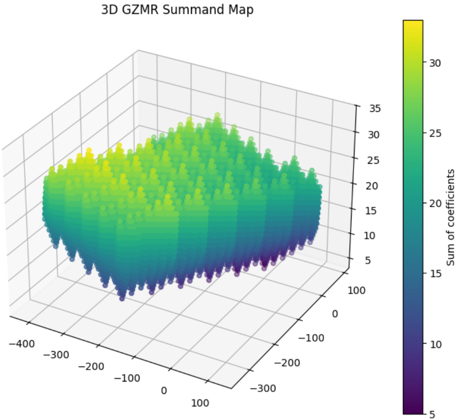
3D illustrations

Let $\vec{\mathbf{c}} = (5, 1, 1)$.

3D illustrations



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