Within-perfect & near-perfect numbers

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A natural number \( n \) is **perfect** if \( \sigma(n) = 2n \), where 
\[
\sigma(n) := \sum_{d \mid n} d.
\]

Are there infinitely many perfect numbers? (6, 28, 496, 8128, \ldots?)

Does odd perfect number exists?
Analytic Progress

Let $V(x)$ be the number of perfect numbers up to $x$. As $x \to \infty$,

- (Volkmann 1955) $V(x) = O(x^{5/6})$
- (Hornfeck 1955) $V(x) = O(x^{1/2})$
- (Kanold 1956) $V(x) = o(x^{1/2})$
- (Erdős 1956) $V(x) = O(x^{1/2-\delta})$
- (Kanold 1957) $V(x) = O(x^{1/4} \frac{\log x}{\log \log x})$
- (Hornfeck & Wirsing 1957) $V(x) = O(x^{\epsilon})$
- (Wirsing 1959) $V(x) \leq x^{W/\log \log x}$

**Conjecture 1.1**

As $x \to \infty$,

$$V(x) \sim \frac{e^\gamma}{\log 2} \log \log x.$$
Definitions

Let $k : [1, \infty) \to [0, \infty)$ be an increasing function.

**Definition 1.2 ($k$-near-perfect)**

We say $n$ is $k$-near-perfect if

$$\sigma(n) = 2n + \sum_{d \in D_n} d,$$

where $D_n$ is a set of proper divisors of $n$ and $\#D_n \leq k(n)$.

**Definition 1.3 ($\ell; k$)-within-perfect**

Let $\ell > 1$. Then we say $n$ is ($\ell; k$)-within-perfect if

$$|\sigma(n) - \ell n| < k(n).$$
Notations

- $N(k)$: set of $k$-near-perfect numbers

- $N(k; x) = N(k) \cap [1, x]$

- $W(\ell; k)$: set of ($\ell; k$)-within-perfect numbers

- $W(\ell; k; x) := W(\ell; k) \cap [1, x]$
Our Results — Near-Perfect

Theorem 1.4 (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)

Let $N(k; x)$ be the set of $k$-near-perfect numbers in $[1, x]$. For $k \geq 4$ and $k \neq 2^{s+2} - 6, 2^{s+2} - 5$ ($s \geq 2$),

$$
\#N(k; x) \asymp_k \frac{x}{\log x} \left(\log \log x\right)^{\left\lfloor \frac{\log(k+4)}{\log 2} \right\rfloor - 3}.
$$
For small $k$

Theorem 1.5 (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)

Let $N(k; x)$ be the set of $k$-near-perfect numbers in $[1, x]$. For $k = 2, 3$,

$$
\#N(k; x) \ll x \exp \left( -\frac{1}{2} \sqrt{\log x \log \log x} \left( 1 + O \left( \frac{\log \log \log x}{\log \log x} \right) \right) \right).
$$

For $k = 4, 5, 6, 7, 8, 9$,

$$
\#N(k, x) \sim c_k \frac{x}{\log x},
$$

where

$$
c_4 = c_5 = \frac{1}{6}, \quad c_6 = \frac{17}{84}, \quad c_7 = c_8 = \frac{493}{1260}, \quad c_9 = \frac{179017}{360360}.
$$
Theorem 1.6 (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)

Let $k$ be positive increasing, $N(k)$ be the set of $k$-near-perfect numbers and $N(k; x) = N(k) \cap [1, x]$.

- If $k(y) < (\log y)^{\log 2 - \epsilon}$, then $N(k)$ has density 0:

$$\#N(k; x) \ll \epsilon \frac{x}{(\log x)r(\epsilon)},$$

where

$$r(\epsilon) := 1 - \frac{(\log 2 - \epsilon)(1 + \log_2 2 - \log(\log 2 - \epsilon))}{\log 2} \in (0, 1).$$

- If $k(y) > (\log y)^{\log 2 + \epsilon}$, then $N(k)$ has positive density.
Our Results — Within-Perfect

Denote by $W(\ell; k; x)$ the set of $(\ell; k)$-within-perfect numbers in $[1, x]$.

**Theorem 1.7 (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)**

Let $\epsilon \in (0, 1/3)$, $k(y) \leq y^\epsilon$ be positive, increasing, unbounded, and $\Sigma$ be the set $\{\frac{\sigma(m)}{m} : m \geq 1\}$.

- If $\ell \in \Sigma \subset \mathbb{Q}$, then
  
  $$\lim_{x \to \infty} \frac{\#W(\ell; k; x)}{x / \log x} = \sum_{\sigma(m) = \ell m} \frac{1}{m}.$$  

- If $\ell \in (\mathbb{Q} \cap [1, \infty)) \setminus \Sigma$, then
  
  $$\#W(\ell; k; x) = O_\ell(x^{\min\{3/4, \epsilon+2/3\} + o(1)}).$$
Near-Perfect Numbers
Sierpiński introduced the notion of **pseudoperfectness** in 1965.

A natural number is said to be pseudoperfect if it is a sum of some subset of its proper divisors.
A Bit of History

Erdös and Benkoski studied the asymptotic density for pseudoperfect numbers, as well as that of abundant numbers that are not pseudoperfect (i.e., weird numbers).
A Bit of History

In 2012, Pollack and Shevelev introduced the notion of $k$-near-perfectness defined before.
Theorem 2.1 (Pollack-Shevelev 2012)

Let $k \in \mathbb{N}$ and $N(k; x)$ denotes the set of all $k$-near-perfect numbers up to $x$. Then as $x \to \infty$,

$$\# N(k; x) \ll_k \frac{x}{\log x} \left(\log \log x\right)^{k-1}.$$ 

$$\# N(1; x) \ll x^{3/4+o(1)}$$

$$\# N(k; x) \ll x \exp\left(-(c_k + o(1)) \sqrt{\log x \log \log x}\right) (k = 2, 3),$$

where $c_2 = 1/\sqrt{6} \approx 0.4082$, $c_3 = \sqrt{2}/4 \approx 0.3536$. 
A Bit of Statistics — Normal Order

Definition 2.2 (normal order; Hardy-Ramanujan 1917)

For $f$ and $g$ positive arithmetic functions, $f$ has normal order $g$ if for any $\epsilon > 0$

$$\lim_{x \to \infty} \frac{1}{x} \# \{ n \leq x : (1 - \epsilon)g(n) < f(n) < (1 + \epsilon)g(n) \} = 1.$$
Hardy and Ramanujan showed that the prime-divisor-counting function $\omega(n)$ and $\Omega(n)$ both have normal order $\log \log n$.

By observing that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)}$, $\log \tau(n)$ has normal order $(\log 2) \log \log n$. 
Application

- Explains why there is phase change in density at \((\log y)^{\log 2}\).
- Push the bound of Pollack-Shevelev from \(\frac{x}{\log x} (\log \log x)^{k-1}\) to \(\frac{x}{\log x} (\log \log x)^{\left\lfloor \frac{\log k}{\log 2} \right\rfloor}\).
- How to do beyond this?
Sketch of Pollack-Shevelev’s Argument

- Let $y > 0$. We say a natural number $n$ is $y$-smooth if $n$ has all prime factors of $n$ is $\leq y$.

- Denote by $P^+(n)$ the largest prime factor of $n$.

- The counting of $N(k; x)$ is done by a partition:

  \[ N_1(k; x) := \{ n \in N(k; x) : n \text{ is } y\text{-smooth} \} \]
  \[ N_2(k; x) := \{ n \in N(k; x) : P^+(n) > y \text{ and } P^+(n)^2 | n \} \]

- Negligible contributions from $N_1(k; x)$ and $N_2(k; x)$ ($= O(x/(\log x)^2)$).
Sketch of Pollack-Shevelev’s Argument

- \( N_3(k; x) := \{ n \in N(k; x) : P^+(n) > y \text{ and } P^+(n) \parallel n \}. \)
Sketch of Pollack-Shevelev’s Argument

- \( N_3(k; x) := \{ n \in N(k; x) : P^+(n) > y \text{ and } P^+(n) \parallel n \} \).

- Major contribution from \( N_3(k; x) \). \( N_3(k; x) \) is further partitioned into \( N'_3(k; x) \) and \( N''_3(k; x) \).
Sketch of Pollack-Shevelev’s Argument

- \( N_3(k; x) := \{ n \in N(k; x) : P^+(n) > y \text{ and } P^+(n) \mid\mid n \} \).

- Major contribution from \( N_3(k; x) \). \( N_3(k; x) \) is further partitioned into \( N_3'(k; x) \) and \( N_3''(k; x) \),
  - where \( N_3'(k; x) \) consists of \( n \in N_3(k; x) \) such that \( \tau(n) \leq k \) and \( N_3''(k; x) := N_3(k; x) \setminus N_3'(k; x) \).
Sketch of Pollack-Shevelev’s Argument

- \( N_3(k; x) := \{ n \in N(k; x) : P^+(n) > y \text{ and } P^+(n) \| n \} \).

- Major contribution from \( N_3(k; x) \). \( N_3(k; x) \) is further partitioned into \( N'_3(k; x) \) and \( N''_3(k; x) \),
  - where \( N'_3(k; x) \) consists of \( n \in N_3(k; x) \) such that \( \tau(n) \leq k \) and \( N''_3(k; x) := N_3(k; x) \setminus N'_3(k; x) \).

- By Landau’s Theorem, \( \#N'_3(k; x) \) is clearly \( O(\frac{x}{\log x} (\log \log x)^{k-1}) \).
Sketch of Pollack-Shevelev’s Argument

- \( N_3(k; x) := \{ n \in N(k; x) : P^+(n) > y \text{ and } P^+(n) \mid n \} \).

- Major contribution from \( N_3(k; x) \). \( N_3(k; x) \) is further partitioned into \( N'_3(k; x) \) and \( N''_3(k; x) \),
  - where \( N'_3(k; x) \) consists of \( n \in N_3(k; x) \) such that \( \tau(n) \leq k \) and \( N''_3(k; x) := N_3(k; x) \setminus N'_3(k; x) \).

- By Landau’s Theorem, \#\( N'_3(k; x) \) is clearly \( O(\frac{x}{\log x} (\log \log x)^{k-1}) \).

- By elementary argument, \( N''_3(k; x) \) is also negligible \( (= O(\frac{x}{y} (\log x)^{3k+1}) \)).
Modification

- Problem: The estimation of $N'_3$ is too rough and does not use the arithmetic information of near-perfectness.

- Instead, consider $N_3^{(1)}(k; x)$ consists of $n \in N_3(k; x)$ such that $n = pm$, $\tau(m) \leq k$, $m \in N(k - \tau(m))$.

- $N_3^{(2)}(k; x) = N_3(k; x) \setminus N_3^{(1)}(k; x)$.

- Inductive argument to $N_3^{(1)}$: $n = p_1 m_1$
Repeat the partitioning process to $m_1$, i.e., estimate the sizes of the sets:

$$\left\{ n \leq x : n = p_1 m_1, p_1 > \max\{y, P^+(m_1)\}, m_1 \in M \left( k - \tau(m_1), \frac{x}{y} \right) \right\},$$

where $M = N_1, N_2, N_3^{(1)}, N_3^{(2)}$. 
Repeat the partitioning process to $m_1$, i.e., estimate the sizes of the sets:

$$\left\{n \leq x : n = p_1 m_1, p_1 > \max\{y, P^+(m_1)\}, m_1 \in M \left( k - \tau(m_1), \frac{x}{y} \right) \right\},$$

where $M = N_1, N_2, N_3^{(1)}, N_3^{(2)}$.

When $M = N_1, N_2, N_3^{(2)}$, the size is $\ll \frac{x}{\log x} \log \log x$. 
Repeat the partitioning process to \( m_1 \), i.e., estimate the sizes of the sets:

\[
\left\{ n \leq x : n = p_1 m_1, p_1 > \max\{y, P^+(m_1)\}, m_1 \in M \left( k - \tau(m_1), \frac{x}{y} \right) \right\},
\]

where \( M = N_1, N_2, N_3^{(1)}, N_3^{(2)} \).

- When \( M = N_1, N_2, N_3^{(2)} \), the size is \( \ll \frac{x}{\log x} \log \log x \).
- When \( M = N_3^{(1)} \), the set is equal to

\[
\left\{ n \leq x : n = p_1 p_2 m_2, p_1 > p_2 > \max\{y, P^+(m_2)\}, m_2 \in N \left( k - 3\tau(m_2), \frac{x}{y^2} \right) \right\}
\]
General: Repeating the process for $j - 1$ times

$$\#\left\{ n \leq x : n = p_1 \cdots p_{j-1} m_{j-1}, p_1 > \cdots > p_{j-1} > y_1, P^+(m_{j-1}), m_{j-1} \in M(k - (2^{j-1} - 1)\tau(m_{j-1})) \right\} \ll k \frac{x}{\log x} (\log \log x)^{j-1},$$

where $M = N_1, N_2, N_3^{(2)}$. 
\[
\# \left\{ n \leq x : n = p_1 \cdots p_{j-1} m_{j-1}, p_1 > \cdots > p_{j-1} > P^+(m_{j-1}) > y_1, \right. \\
m_{j-1} \in N_3^{(1)}(k - (2^{j-1} - 1)\tau(m_{j-1})) \} \\
= \# \left\{ n \leq x : n = p_1 \cdots p_{j-1} p_j m_j, p_1 > \cdots > p_{j-1} > p_j > y_1, P^+(m_j), \\
m_j \in N(k - (2^j - 1)\tau(m_j)) \right\}.
\]
When shall we stop?

- We stop when there are only finitely many $m_j$ such that $(2^j - 1)\tau(m_j) \leq k$ and $m_j \in N(k - (2^j - 1)\tau(m_j))$. Then we will have
  \[
  \#N(k; x) \ll_k \frac{x}{\log x} (\log \log x)^{j-1}.
  \]
When shall we stop?

We stop when there are only finitely many $m_j$ such that $(2^j - 1)\tau(m_j) \leq k$ and $m_j \in N(k - (2^j - 1)\tau(m_j))$. Then we will have

$$\#N(k; x) \ll_k \frac{x}{\log x} (\log \log x)^{j-1}.$$ 

These are satisfied when $j = \left\lfloor \frac{\log(k+4)}{\log 2} \right\rfloor - 2$. Also, $\tau(m_j)$ is small and can be handled directly. The result follows.
Within-Perfect Numbers
Recall

**Definition 3.1 ((ℓ; k)-within-perfect)**

Let \( \ell > 1 \), \( k \) be an increasing function. Then we say \( n \) is \((\ell; k)\)-within-perfect if

\[
|\sigma(n) - \ell n| < k(n).
\]
Davenport showed that $\sigma(n)/n$ has a distribution function in 1933. It follows that:

**Theorem 3.2**

Let $D(\cdot)$ denote the distribution function of $\sigma(n)/n$.

- If $k(n) = o(n)$, then the set of $(\ell; k)$-within-perfect numbers has density 0.
- If $k(n) \sim cn$ for some $c > 0$, then the set of $(\ell; k)$-within-perfect numbers has density $D(\ell + c) - D(\ell - c)$.
- If $n = o(k(n))$, then the set of $(\ell; k)$-within-perfect numbers has density 1.
Better Understanding?

- For the sublinear regime, from the above theorem we only know the density of \((\ell; k)\)-within-perfect numbers is 0.

- The next step is to find an explicit upper bound for the sublinear regime.
In 1975, Pomerance studied the distribution of \( S_{\ell,k} = \{ n \in \mathbb{N} : \sigma(n) = \ell n + k \} \), where \( \ell, k \in \mathbb{Z}, \ell \geq 2 \).

- \( S_{2,0} \) (Perfect numbers),
- \( S_{\ell,0} \) (\( \ell \)-multiply perfect numbers),
- \( S_{2,1} \) (Quasiperfect numbers),
- \( S_{2,-1} \) (Almost perfect numbers)
Theorem 3.3 (Pomerance 1975)

Denote $S_{\ell,k} \cap [1, x]$ by $S_{\ell,k}(x)$. As $x \to \infty$,

$$\#S_{\ell,k}(x) \ll_k \frac{x}{\log x}.$$
If $\sigma(n) = \ell n + k$, then $\sigma(n) \equiv k \pmod{n}$.

**Definition 3.4**

Say $n$ is a **regular solution** of $\sigma(n) \equiv k \pmod{n}$ if $n = pm$, with $p$ prime, $p \nmid m$, $m \mid \sigma(m)$, and $\sigma(m) = k$; otherwise $n$ is a **sporadic solution**.
Theorem 3.5 (Pomerance (1975))

The number of sporadic solutions of \( \sigma(n) \equiv k \mod n \) is \( O_k(x \exp(-\beta \log x \log \log x)^{1/2}) \) as \( x \to \infty \) for any \( \beta < 1/\sqrt{2} \).
### Theorem 3.5 (Pomerance (1975))

The number of sporadic solutions of $\sigma(n) \equiv k \pmod{n}$ is $O_k(x \exp(-\beta(\log x \log \log x)^{1/2}))$ as $x \to \infty$ for any $\beta < 1/\sqrt{2}$.

### Theorem 3.6 (Anavi-Pollack-Pomerance (2012))

Uniformly for $|k| \leq x^{1/4}$, the number of sporadic solutions of $\sigma(n) \equiv k \pmod{n}$ is at most $x^{1/2+o(1)}$, as $x \to \infty$. 
Theorem 3.5 (Pomerance (1975))

The number of sporadic solutions of \( \sigma(n) \equiv k \pmod{n} \) is \( O_k(x \exp(-\beta(\log x \log \log x)^{1/2})) \) as \( x \to \infty \) for any \( \beta < 1/\sqrt{2} \).

Theorem 3.6 (Anavi-Pollack-Pomerance (2012))

Uniformly for \( |k| \leq x^{1/4} \), the number of sporadic solutions of \( \sigma(n) \equiv k \pmod{n} \) is at most \( x^{1/2+o(1)} \), as \( x \to \infty \).

Theorem (Pollack-Pomerance-Thompson 2017)

Let \( \ell, k \) be integers with \( \ell > 0 \). Then the number of sporadic solutions \( n \leq x \) of \( \sigma(n) = \ell n + k \) is at most \( x^{3/5+o_\ell(1)} \) as \( x \to \infty \), uniformly in \( k \).
Our Results — Within-Perfect

Denote by $W(\ell; k; x)$ the set of $(\ell; k)$-within-perfect numbers in $[1, x]$.

**Theorem 3.7** (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)

Let $\epsilon \in (0, 1/3)$, $k(y) \leq y^{\epsilon}$ be positive, increasing, unbounded, and $\Sigma$ be the set $\{\frac{\sigma(m)}{m} : m \geq 1\}$.

1. If $\ell \in \Sigma \subset \mathbb{Q}$, then

   $$\lim_{x \to \infty} \frac{\#W(\ell; k; x)}{x / \log x} = \sum_{\sigma(m) = \ell m} \frac{1}{m}.$$ 

2. If $\ell \in (\mathbb{Q} \cap [1, \infty)) \setminus \Sigma$, then

   $$\#W(\ell; k; x) = O_{\ell}(x^{\min\{3/4, \epsilon/2\} + o(1)}).$$
Sketch of Proof

- Assume that an $\ell$-perfect numbers exist. Take $k(y) = y^\epsilon$ ($\epsilon \in (0, 1/3)$).

- Consider the collection of Diophantine equations

$$\sigma(n) - \ell n = k, \text{ where } k \in \mathbb{Z}, |k| < x^\epsilon.$$

- We can make the following assumptions one by one:
Assumptions

1. $n$ is regular (else by Pomerance’s Theorem the contribution from the complement is $\leq 2x^{2/3+\epsilon+o(1)}$).

2. $n = pm$ and $p > x^\epsilon$ (else by PNT & Hornfeck-Wirsing Theorem, the contribution from the complement is $\leq \frac{x^{\epsilon+o(1)}}{\log x}$).

3. $\sigma(m)/m \leq \ell$.

4. $\sigma(m)/m = \ell$ (else by Merten’s estimate, the number of $n$’s with $\sigma(m) = rm$ with $2 \leq r \leq \ell - 1$ and $p > x^\epsilon$ is $\ll (\ell - 2)x^\epsilon \log \log x$).
Thus, we only have to work with

\[ n = pm \text{ where } p \text{ is prime, } p \nmid m, \sigma(m) = \ell m \]

Once again by PNT and Hornfeck-Wirsing, for any \( c > 1 \),

\[
\#W(\ell; k; x) \leq \sum_{\substack{m \leq x^\varepsilon \\ \sigma(m) = \ell m}} \pi(x/m) \\
< c \sum_{\substack{m \leq x^\varepsilon \\ \sigma(m) = \ell m}} \frac{x/m}{\log(x/m)} \\
= c \frac{x}{\log x} \sum_{\substack{m \leq x^\varepsilon \\ \sigma(m) = \ell m}} \frac{1}{m} + O_\varepsilon \left( \frac{cx}{(\log x)^2} \sum_{\substack{m \leq x^\varepsilon \\ \sigma(m) = \ell m}} \frac{\log m}{m} \right) \\
< c \frac{x}{\log x} \sum_{\sigma(m) = \ell m} \frac{1}{m} + O_\varepsilon \left( \frac{cx}{(\log x)^2} \right). 
\]
Further Thoughts

- What happens if $\ell$ is irrational?

- Wolke and Harman studied in terms of a Diophantine approximation and used the Prime Number Theorem in Short Interval.

- They showed that for any real $\ell \geq 1$ and for any $c \in (0.525, 1)$, there exists infinitely many natural numbers that are $(\ell; y^c)$-within-perfect. But the constructed set is quite sparse.

- Can we do better than this?
Further Thoughts

**Conjecture (Pollack-Pomerance-Thompson 2017)**

Let $x \geq 3$, $\ell, k$ be integers with $\ell > 0$ and $|k| \leq x$. Then the number of sporadic solutions $n \leq x$ of $\sigma(n) = \ell n + k$ is at most $x^{1/2+o(1)}$ as $x \to \infty$, uniformly in $k, \ell$.

- If this were proven, then one can push our result to $k(x) < x^{1/2-\epsilon}$.

- How to do beyond this range in the sublinear regime?
Thank you

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Within-Perfect Numbers
Sketch of Proof

- Assume $\ell$-perfect numbers exist and $k(y) \leq y^\epsilon$ for $\epsilon \in (0, 1/3)$. 
Sketch of Proof

- Assume \( \ell \)-perfect numbers exist and \( k(y) \leq y^\epsilon \) for \( \epsilon \in (0, 1/3) \).

- Showing

\[
\liminf_{x \to \infty} \frac{\# W(\ell; k; x)}{x / \log x} \geq \sum_{\sigma(m) = \ell m} \frac{1}{m}
\]

is a direct consequence of the Prime Number Theorem.
Sketch of Proof

- Assume \( \ell \)-perfect numbers exist and \( k(y) \leq y^\epsilon \) for \( \epsilon \in (0, 1/3) \).

- Showing

\[
\liminf_{x \to \infty} \frac{\# W(\ell; k; x)}{x / \log x} \geq \sum_{\sigma(m) = \ell m} \frac{1}{m}
\]

is a direct consequence of the Prime Number Theorem.

- Now we want to show

\[
\limsup_{x \to \infty} \frac{\# W(\ell; k; x)}{x / \log x} \leq \sum_{\sigma(m) = \ell m} \frac{1}{m}
\]
It suffices to consider $k(y) = y^\epsilon$. Fix a large $x$ and let $n \leq x$ satisfy $|\sigma(n) - \ell n| < x^\epsilon$.

Rewrite this Diophantine inequality as a collection of Diophantine equations over certain range, i.e.,

$$\sigma(n) - \ell n = k,$$

where $k \in \mathbb{Z}, |k| < x^\epsilon$. 
It suffices to consider $k(y) = y^\epsilon$. Fix a large $x$ and let $n \leq x$ satisfy $|\sigma(n) - \ell n| < x^\epsilon$.

Rewrite this Diophantine inequality as a collection of Diophantine equations over certain range, i.e.,

$$\sigma(n) - \ell n = k, \text{ where } k \in \mathbb{Z}, \ |k| < x^\epsilon.$$

In particular, we have a collection of congruences in the form of regular solutions:

$$\sigma(n) \equiv k \pmod{n}, \text{ where } k \in \mathbb{Z}, \ |k| < x^\epsilon.$$
Recall from Pomerance’s Theorems

\[ n \text{ is a regular solution if } n \text{ is of the form} \]

\[ n = pm \text{ where } p \text{ is prime, } p \nmid m, \ m \mid \sigma(m), \text{ and } \sigma(m) = k \ (\ast) \]
We can make the following assumptions one by one:

- $n$ is in the form of (*).

By Pomerance’s theorem, the number of elements of $W(\ell; k; x)$ *NOT* of the form (*) is at most

$$2x^\epsilon x^{2/3+o(1)} = 2x^{2/3+\epsilon+o(1)},$$

which is negligible (compared with $x/\log x$).
Theorem 5.1 (Hornfeck-Wirsing)

The number of multiply perfect numbers less than or equal to $x$ is at most $x^{o(1)}$ as $x \to \infty$. 
$p > x^\epsilon$ in (*).

By the Prime Number Theorem and Hornfeck-Wirsing theorem, the number of $n \leq x$ of the form (*) with $p \leq x^\epsilon$ is at most

$$\frac{x^\epsilon}{\log x} x^{o(1)} \ll \epsilon \frac{x^{\epsilon+o(1)}}{\log x},$$

which is again negligible.
\[ \sigma(m)/m \leq \ell \text{ in (*)}. \]

If \( \sigma(m) = rm \) for some \( r \geq \ell + 1 \), then

\[
\begin{align*}
\sigma(n) - \ell n &= \sigma(p)\sigma(m) - \ell pm \\
                    &= (1 + p)(rm) - \ell pm \\
                    &= m(r + p(r - \ell)) \\
                    \geq p &> x^\epsilon.
\end{align*}
\]

Contradiction!
\[ \sigma(m)/m = \ell \text{ in (*)}. \]
• $\sigma(m)/m = \ell$ in (*)

• Consider the case where $\sigma(m) = rm$ with $2 \leq r \leq \ell - 1$ and $p > x^\varepsilon$. By Merten’s estimate, the number of such $n$ is

\[
\leq \sum_{2 \leq r \leq \ell - 1} \sum_{x^\varepsilon < p \leq x} \frac{x^\varepsilon}{(\ell - r)p - r}
\leq (\ell - 2)x^\varepsilon \sum_{x^\varepsilon < p \leq x} \frac{1}{p - \ell + 1}
\leq 2(\ell - 2)x^\varepsilon \sum_{x^\varepsilon < p \leq x} \frac{1}{p}
\ll (\ell - 2)x^\varepsilon \log \log x.
Thus, we only have to work with

\[ n = pm \text{ where } p \text{ is prime, } p \nmid m, \sigma(m) = \ell m \quad (**) \]

Next we estimate the contribution from (**).

By partial summation and Hornfeck-Wirsing Theorem, we have

\[
\sum_{\sigma(m) = \ell m} \frac{\log m}{m}, \quad \sum_{\sigma(m) = \ell m} \frac{1}{m}
\]

converge.
Let $c$ be any constant greater than 1. By the Prime Number Theorem, there exists $x_0 = x_0(c) > 0$ such that for $x \geq x_0$, we have

$$\pi(x) < c \frac{x}{\log x}.$$ 

Then for $x \geq \max\{x_0^{1/(1-\epsilon)}, \ell^2\}$, we have

$$\#W(\ell; k; x) \leq \sum_{m \leq x^\epsilon} \pi(x/m) \frac{x/m}{\log(x/m)} < c \sum_{m \leq x^\epsilon} \frac{1}{m} + O_\epsilon \left( \frac{c x}{(\log x)^2} \sum_{m \leq x^\epsilon} \frac{\log m}{m} \right)$$

$$= c \frac{x}{\log x} \sum_{m \leq x^\epsilon} \frac{1}{m} + O_\epsilon \left( \frac{c x}{(\log x)^2} \sum_{m \leq x^\epsilon} \frac{\log m}{m} \right)$$

$$< c \frac{x}{\log x} \sum_{m \leq x^\epsilon} \frac{1}{m} + O_\epsilon \left( \frac{c x}{(\log x)^2} \right).$$
Therefore,

\[
\limsup_{x \to \infty} \frac{\#W(\ell; k; x)}{x / \log x} \leq c \sum_{\sigma(m) = \ell m} \frac{1}{m}.
\]

Since the choice of constant \( c > 1 \) is arbitrary, we have

\[
\limsup_{x \to \infty} \frac{\#W(\ell; k; x)}{x / \log x} \leq \sum_{\sigma(m) = \ell m} \frac{1}{m}.
\]

This completes the proof.