Within-perfect & near-perfect numbers

Kevin Kwan[1], ck2854@ math.columbia.edu
Steven Miller [2], sjm1@williams.edu
(Joint work with Peter Cohen, Katherine Cordwell, Alyssa Epstein and Adam Lott)

Department of Mathematics, Columbia University [1]

Department of Mathematics & Statistics, Williams College [2]

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Background

- A natural number n is **perfect** if $\sigma(n) = 2n$, where $\sigma(n) := \sum_{d|n} d$.
- Are there infinitely many perfect numbers? $(6, 28, 496, 8128, \ldots ?)$
- Does odd perfect number exists?

Analytic Progress

Let $V(x)$ be the number of perfect numbers up to x. As $x \to \infty$,

$$
\begin{aligned}
\text{(Volkmann 1955)} \quad V(x) &= O(x^{5/6}) \\
\text{(Hornfeck 1955)} \quad V(x) &= O(x^{1/2}) \\
\text{(Kanold 1956)} \quad V(x) &= o(x^{1/2}) \\
\text{(Erdős 1956)} \quad V(x) &= O(x^{1/2-\delta}) \\
\text{(Kanold 1957)} \quad V(x) &= O(x^{1/4} \frac{\log x}{\log \log x}) \\
\text{(Hornfeck \& Wirsing 1957)} \quad V(x) &= O(x^{\epsilon}) \\
\text{(Wirsing 1959)} \quad V(x) &\leq x^{W/\log \log x}
\end{aligned}
$$

Conjecture 1.1

As $x \to \infty$,

$$V(x) \sim \frac{e^{\gamma}}{\log 2} \log \log x.$$

Definitions

Let $k : [1, \infty) \to [0, \infty)$ be an increasing function.

Definition 1.2 (k-near-perfect)

*We say n is k-**near-perfect** if*

$$\sigma(n) = 2n + \sum_{d \in D_n} d,$$

where D_n is a set of proper divisors of n and $\#D_n \leq k(n)$.

Definition 1.3 ($(\ell; k)$-within-perfect)

*Let $\ell > 1$. Then we say n is $(\ell; k)$-**within-perfect** if*

$$|\sigma(n) - \ell n| < k(n).$$

Notations

- $N(k)$: set of k-near-perfect numbers
- $N(k;x) = N(k) \cap [1,x]$
- $W(\ell;k)$: set of $(\ell;k)$-within-perfect numbers
- $W(\ell;k;x) := W(\ell;k) \cap [1,x]$

Our Results — Near-Perfect

Theorem 1.4 (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)

Let $N(k;x)$ be the set of k-near-perfect numbers in $[1,x]$. For $k \geq 4$ and $k \neq 2^{s+2}-6, 2^{s+2}-5$ $(s \geq 2)$,

$$\#N(k;x) \asymp_k \frac{x}{\log x}(\log\log x)^{\lfloor \frac{\log(k+4)}{\log 2} \rfloor - 3}.$$

For small k

Theorem 1.5 (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)

Let $N(k;x)$ be the set of k-near-perfect numbers in $[1,x]$.
For $k=2,3$,

$$\#N(k;x) \ll x \exp\left(-\frac{1}{2}\sqrt{\log x \log\log x}\left(1+O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right).$$

For $k=4,5,6,7,8,9$,

$$\#N(k,x) \sim c_k \frac{x}{\log x},$$

where

$$c_4 = c_5 = \frac{1}{6},\ c_6 = \frac{17}{84},\ c_7 = c_8 = \frac{493}{1260},\ c_9 = \frac{179017}{360360}.$$

For Increasing k

Theorem 1.6 (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)

Let k be positive increasing, $N(k)$ be the set of k-near-perfect numbers and $N(k;x) = N(k) \cap [1,x]$.

- *If $k(y) < (\log y)^{\log 2 - \epsilon}$, then $N(k)$ has density 0:*

$$\#N(k;x) \ \ll_\epsilon \ \frac{x}{(\log x)^{r(\epsilon)}},$$

where

$$r(\epsilon) \ := \ 1 - \frac{(\log 2 - \epsilon)(1 + \log_2 2 - \log(\log 2 - \epsilon))}{\log 2} \in (0,1).$$

- *If $k(y) > (\log y)^{\log 2 + \epsilon}$, then $N(k)$ has positive density.*

Our Results — Within-Perfect

Denote by $W(\ell;k;x)$ the set of $(\ell;k)$-within-perfect numbers in $[1,x]$.

Theorem 1.7 (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)

Let $\epsilon \in (0,1/3)$, $k(y) \le y^\epsilon$ be positive, increasing, unbounded, and Σ be the set $\{\frac{\sigma(m)}{m} : m \ge 1\}$.

- *If $\ell \in \Sigma \subset \mathbb{Q}$, then*

$$\lim_{x\to\infty} \frac{\#W(\ell;k;x)}{x/\log x} = \sum_{\sigma(m) = \ell m} \frac{1}{m}.$$

- *If $\ell \in (\mathbb{Q} \cap [1,\infty)) \setminus \Sigma$, then*

$$\#W(\ell;k;x) = O_\ell(x^{\min\{3/4,\ \epsilon+2/3\}+o(1)}).$$

Near-Perfect Numbers

A Bit of History

Sierpiński introduced the notion of **pseudoperfectness** in 1965.

A natural number is said to be pseudoperfect if it is a sum of some subset of its proper divisors.

A Bit of History

Erdös and Benkoski studied the asymptotic density for pseudoperfect numbers, as well as that of abundant numbers that are not pseudoperfect (i.e., **weird numbers**).

A Bit of History

In 2012, Pollack and Shevelev introduced the notion of k-near-perfectness defined before.

Known result

Theorem 2.1 (Pollack-Shevelev 2012)

Let $k \in \mathbb{N}$ and $N(k;x)$ denotes the set of all k-near-perfect numbers up to x. Then as $x \to \infty$,

$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{k-1}.$$

$$\#N(1;x) \ll x^{3/4+o(1)}$$
$$\#N(k;x) \ll x\exp(-(c_k + o(1))\sqrt{\log x \log\log x}) \ (k = 2, 3),$$

where $c_2 = 1/\sqrt{6} \approx 0.4082$, $c_3 = \sqrt{2}/4 \approx 0.3536$.

A Bit of Statistics — Normal Order

Definition 2.2 (normal order; Hardy-Ramanujan 1917)

For f and g positive arithmetic functions, f has **normal order** *g if for any $\epsilon > 0$*

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \leq x : (1-\epsilon)g(n) < f(n) < (1+\epsilon)g(n)\} = 1.$$

Normal Order

Hardy and Ramanujan showed that the prime-divisor-counting function $\omega(n)$ and $\Omega(n)$ both have normal order $\log\log n$.

By observing that $2^{\omega(n)} \leq \tau(n) \leq 2^{\Omega(n)}$, $\log \tau(n)$ has normal order $(\log 2) \log\log n$.

Application

- Explains why there is phase change in density at $(\log y)^{\log 2}$.
- Push the bound of Pollack-Shevelev from $\frac{x}{\log x}(\log\log x)^{k-1}$ to $\frac{x}{\log x}(\log\log x)^{[\frac{\log k}{\log 2}]}$.
- How to do beyond this?

Sketch of Pollack-Shevelev's Argument

- Let $y > 0$. We say a natural number n is y-**smooth** if n has all prime factors of n is $\leq y$.

- Denote by $P^+(n)$ the largest prime factor of n.

- The counting of $N(k; x)$ is done by a partition:

$$N_1(k; x) := \{n \in N(k; x) : \ n \text{ is } y\text{-smooth}\}$$
$$N_2(k; x) := \{n \in N(k; x) : P^+(n) > y \text{ and } P^+(n)^2 | n\}$$

- Negligible contributions from $N_1(k; x)$ and $N_2(k; x)$ $(= O(x/(\log x)^2))$.

Sketch of Pollack-Shevelev's Argument

- $N_3(k;x) \;:=\; \{n \in N(k;x) : P^+(n) > y$ and $P^+(n) \,||\, n\}$.

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- $N_3(k;x) \ := \ \{n \in N(k;x) : P^+(n) > y \text{ and } P^+(n) \mid\mid n\}$.

- Major contribution from $N_3(k;x)$. $N_3(k;x)$ is further partitioned into $N_3'(k;x)$ and $N_3''(k;x)$,

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    - where $N_3'(k;x)$ consists of $n \in N_3(k;x)$ such that $\tau(n) \le k$ and $N_3''(k;x) := N_3(k;x) \setminus N_3'(k;x)$.

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- By Landau's Theorem, $\#N_3'(k;x)$ is clearly $O(\frac{x}{\log x}(\log\log x)^{k-1})$.

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  - where $N_3'(k;x)$ consists of $n \in N_3(k;x)$ such that $\tau(n) \leq k$ and $N_3''(k;x) := N_3(k;x) \setminus N_3'(k;x)$.
- By Landau's Theorem, $\#N_3'(k;x)$ is clearly $O(\frac{x}{\log x}(\log\log x)^{k-1})$.
- By elementary argument, $N_3''(k;x)$ is also negligible $(= O(\frac{x}{y}(\log x)^{3k+1}))$.

Modification

- Problem: The estimation of N_3' is too rough and does not use the arithmetic information of near-perfectness.
- Instead, consider $N_3^{(1)}(k;x)$ consists of $n \in N_3(k;x)$ such that $n = pm$, $\tau(m) \leq k$, $m \in N(k - \tau(m))$,
- $N_3^{(2)}(k;x) = N_3(k;x) \setminus N_3^{(1)}(k;x)$.
- Inductive argument to $N_3^{(1)}$: $n = p_1 m_1$

- Repeat the partitioning process to m_1, i.e., estimate the sizes of the sets:

$$\left\{ n \le x : n = p_1 m_1, p_1 > \max\{y, P^+(m_1)\}, m_1 \in M\left(k - \tau(m_1), \frac{x}{y} \right) \right\},$$

where $M = N_1, N_2, N_3^{(1)}, N_3^{(2)}$.

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where $M = N_1, N_2, N_3^{(1)}, N_3^{(2)}$.

- When $M = N_1, N_2, N_3^{(2)}$, the size is $\ll \frac{x}{\log x} \log\log x$.
- When $M = N_3^{(1)}$, the set is equal to

$$\left\{ n \le x : n = p_1 p_2 m_2, p_1 > p_2 > \max\{y, P^+(m_2)\}, m_2 \in N\left(k - 3\tau(m_2), \frac{x}{y^2} \right) \right\}$$

General: Repeating the process for $j-1$ times

$$\#\Big\{n \le x : n = p_1 \cdots p_{j-1} m_{j-1}, p_1 > \cdots > p_{j-1} > y_1, P^+(m_{j-1}),$$

$$m_{j-1} \in M(k - (2^{j-1} - 1)\tau(m_{j-1}))\Big\}$$

$$\ll_k \frac{x}{\log x} (\log \log x)^{j-1},$$

where $M = N_1, N_2, N_3^{(2)}$.

$$\#\Big\{ n \le x : n = p_1 \cdots p_{j-1} m_{j-1}, p_1 > \cdots > p_{j-1} > P^{+}(m_{j-1}) > y_1,$$
$$m_{j-1} \in N_3^{(1)}(k - (2^{j-1} - 1)\tau(m_{j-1}))\Big\}$$
$$= \#\Big\{ n \le x : n = p_1 \cdots p_{j-1} p_j m_j, p_1 > \cdots > p_{j-1} > p_j > y_1, P^{+}(m_j),$$
$$m_j \in N(k - (2^j - 1)\tau(m_j))\Big\}.$$

When shall we stop?

- We stop when there are only finitely many m_j such that $(2^j - 1)\tau(m_j) \leq k$ and $m_j \in N(k - (2^j - 1)\tau(m_j))$. Then we will have
$$\#N(k; x) \ \ll_k \ \frac{x}{\log x}(\log\log x)^{j-1}.$$

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$$\#N(k;x) \ll_k \frac{x}{\log x}(\log\log x)^{j-1}.$$

- These are satisfied when $j = \lfloor \frac{\log(k+4)}{\log 2} \rfloor - 2$. Also, $\tau(m_j)$ is small and can be handled directly. The result follows.

Within-Perfect Numbers

Recall

Definition 3.1 (($\ell; k$)-within-perfect)

*Let $\ell > 1$, k be an increasing function. Then we say n is ($\ell; k$)-**within-perfect** if*

$$|\sigma(n) - \ell n| < k(n).$$

Phase Transition of Density

Davenport showed that $\sigma(n)/n$ has a distribution function in 1933. It follows that:

Theorem 3.2

Let $D(\cdot)$ denote the distribution function of $\sigma(n)/n$.

- *If $k(n) = o(n)$, then the set of $(\ell; k)$-within-perfect numbers has density 0.*
- *If $k(n) \sim cn$ for some $c > 0$, then the set of $(\ell; k)$-within-perfect numbers has density $D(\ell + c) - D(\ell - c)$.*
- *If $n = o(k(n))$, then the set of $(\ell; k)$-within-perfect numbers has density 1.*

Better Understanding?

- For the sublinear regime, from the above theorem we only know the density of $(\ell; k)$-within-perfect numbers is 0.
- The next step is to find an explicit upper bound for the sublinear regime.

Motivation — Fixing k

In 1975, Pomerance studied the distribution of $S_{\ell,k} = \{n \in \mathbb{N} : \sigma(n) = \ell n + k\}$, where $\ell, k \in \mathbb{Z}$, $\ell \geq 2$.

- $S_{2,0}$ (Perfect numbers),
- $S_{\ell,0}$ (ℓ-multiply perfect numbers),
- $S_{2,1}$ (Quasiperfect numbers),
- $S_{2,-1}$ (Almost perfect numbers)

Pomerance's Theorem

Theorem 3.3 (Pomerance 1975)

Denote $S_{\ell,k} \cap [1,x]$ *by* $S_{\ell,k}(x)$*. As* $x \to \infty$*,*

$$\#S_{\ell.k}(x) \ll_k \frac{x}{\log x}.$$

Ideas — Regular & Sporadic

If $\sigma(n) = \ell n + k$, then $\sigma(n) \equiv k \pmod{n}$.

Definition 3.4

Say n is a **regular solution** *of $\sigma(n) \equiv k \pmod{n}$ if $n = pm$, with p prime, $p \nmid m$, $m \mid \sigma(m)$, and $\sigma(m) = k$; otherwise n is a* **sporadic solution**.

Theorem 3.5 (Pomerance (1975))

The number of sporadic solutions of $\sigma(n) \equiv k \pmod{n}$ *is* $O_k(x \exp(-\beta(\log x \log\log x)^{1/2}))$ *as* $x \to \infty$ *for any* $\beta < 1/\sqrt{2}$.

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Theorem 3.6 (Anavi-Pollack-Pomerance (2012))

Uniformly for $|k| \leq x^{1/4}$, *the number of sporadic solutions of* $\sigma(n) \equiv k \pmod{n}$ *is at most* $x^{1/2+o(1)}$, *as* $x \to \infty$.

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Theorem (Pollack-Pomerance-Thompson 2017)

Let ℓ, k be integers with $\ell > 0$. Then the number of sporadic solutions $n \le x$ of $\sigma(n) = \ell n + k$ is at most $x^{3/5+o_\ell(1)}$ as $x \to \infty$, uniformly in k.

Our Results — Within-Perfect

Denote by $W(\ell;k;x)$ the set of $(\ell;k)$-within-perfect numbers in $[1,x]$.

Theorem 3.7 (Cohen-Cordwell-Epstein-K.-Lott-M. 2016)

Let $\epsilon \in (0,1/3)$, $k(y) \le y^{\epsilon}$ be positive, increasing, unbounded, and Σ be the set $\{\frac{\sigma(m)}{m} : m \geq 1\}$.

- *If $\ell \in \Sigma \subset \mathbb{Q}$, then*

$$\lim_{x\to\infty} \frac{\#W(\ell;k;x)}{x/\log x} = \sum_{\sigma(m)\,=\,\ell m} \frac{1}{m}.$$

- *If $\ell \in (\mathbb{Q}\cap[1,\infty))\setminus\Sigma$, then*

$$\#W(\ell;k;x) = O_\ell(x^{\min\{3/4,\ \epsilon+2/3\}+o(1)}).$$

Sketch of Proof

- Assume that an ℓ-perfect numbers exist. Take $k(y) = y^{\epsilon}$ ($\epsilon \in (0, 1/3)$).
- Consider the collection of Diophantine equations

$$\sigma(n) - \ell n = k, \text{ where } k \in \mathbb{Z}, |k| < x^{\epsilon}.$$

- We can make the following assumptions one by one:

Assumptions

1. n is regular (else by Pomerance's Theorem the contribution from the complement is $\leq 2x^{2/3+\epsilon+o(1)}$).

2. $n = pm$ and $p > x^{\epsilon}$ (else by PNT & Hornfeck-Wirsing Theorem, the contribution from the complement is $\leq \frac{x^{\epsilon+o(1)}}{\log x}$).

3. $\sigma(m)/m \leq \ell$.

4. $\sigma(m)/m = \ell$ (else by Merten's estimate, the number of n's with $\sigma(m) = rm$ with $2 \leq r \leq \ell - 1$ and $p > x^{\epsilon}$ is $\ll (\ell - 2)x^{\epsilon} \log\log x$).

Thus, we only have to work with

$$n = pm \text{ where } p \text{ is prime, } p \nmid m,\ \sigma(m) = \ell m$$

Once again by PNT and Hornfeck-Wirsing, for any $c > 1$,

$$\begin{aligned} \#W(\ell;k;x) &\leq \sum_{\substack{m \leq x^{\epsilon} \\ \sigma(m)=\ell m}} \pi(x/m) \\ &< c \sum_{\substack{m \leq x^{\epsilon} \\ \sigma(m)=\ell m}} \frac{x/m}{\log(x/m)} \\ &= c\,\frac{x}{\log x} \sum_{\substack{m \leq x^{\epsilon} \\ \sigma(m)=\ell m}} \frac{1}{m} + O_{\epsilon}\left(\frac{cx}{(\log x)^2} \sum_{\substack{m \leq x^{\epsilon} \\ \sigma(m)=\ell m}} \frac{\log m}{m} \right) \\ &< c\,\frac{x}{\log x} \sum_{\sigma(m)=\ell m} \frac{1}{m} + O_{\epsilon}\left(\frac{cx}{(\log x)^2} \right). \end{aligned}$$

Further Thoughts

- What happens if ℓ is irrational?
- Wolke and Harman studied in terms of a Diophantine approximation and used the Prime Number Theorem in Short Interval.
- They showed that for any real $\ell \geq 1$ and for any $c \in (0.525, 1)$, there exists infinitely many natural numbers that are $(\ell; y^c)$-within-perfect. But the constructed set is quite sparse.
- Can we do better than this?

Further Thoughts

Conjecture (Pollack-Pomerance-Thompson 2017)

Let $x \geq 3$, ℓ, k be integers with $\ell > 0$ and $|k| \leq x$. Then the number of sporadic solutions $n \leq x$ of $\sigma(n) = \ell n + k$ is at most $x^{1/2+o(1)}$ as $x \to \infty$, uniformly in k, ℓ.

- If this were proven, then one can push our result to $k(x) < x^{1/2-\epsilon}$.
- How to do beyond this range in the sublinear regime?

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Within-Perfect Numbers

Sketch of Proof

- Assume ℓ-perfect numbers exist and $k(y) \leq y^{\epsilon}$ for $\epsilon \in (0, 1/3)$.

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- Assume ℓ-perfect numbers exist and $k(y) \leq y^{\epsilon}$ for $\epsilon \in (0, 1/3)$.
- Showing
$$\liminf_{x \to \infty} \frac{\#W(\ell; k; x)}{x/\log x} \geq \sum_{\sigma(m) = \ell m} \frac{1}{m}$$
is a direct consequence of the Prime Number Theorem.

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- Showing
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is a direct consequence of the Prime Number Theorem.

- Now we want to show
$$\limsup_{x\to\infty} \frac{\#W(\ell;k;x)}{x/\log x} \leq \sum_{\sigma(m)=\ell m} \frac{1}{m}$$

- It suffices to consider $k(y) = y^{\epsilon}$. Fix a large x and let $n \leq x$ satisfy $|\sigma(n) - \ell n| < x^{\epsilon}$.
- Rewrite this Diophantine inequality as a collection of Diophantine equations over certain range, i.e.,

$$\sigma(n) - \ell n \ = \ k, \text{ where } k \in \mathbb{Z}, \ |k| < x^{\epsilon}.$$

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- Rewrite this Diophantine inequality as a collection of Diophantine equations over certain range, i.e.,

$$\sigma(n) - \ell n \ = \ k, \text{ where } k \in \mathbb{Z}, \ |k| < x^{\epsilon}.$$

- In particular, we have a collection of congruences in the form of regular solutions:

$$\sigma(n) \ \equiv \ k \pmod{n}, \text{ where } k \in \mathbb{Z}, \ |k| < x^{\epsilon}.$$

Recall from Pomerance's Theorems

n is a regular solution if n is of the form

$$n = pm \text{ where } p \text{ is prime, } p \nmid m,\ m \mid \sigma(m), \text{ and } \sigma(m) = k \quad (*)$$

We can make the following assumptions one by one:

- n is in the form of (*).

  By Pomerance's theorem, the number of elements of $W(\ell; k; x)$ *NOT* of the form (*) is at most

$$2x^{\epsilon}x^{2/3+o(1)} = 2x^{2/3+\epsilon+o(1)},$$

  which is negligible (compared with $x/\log x$).

Theorem 5.1 (Hornfeck-Wirsing)

The number of multiply perfect numbers less than or equal to x is at most $x^{o(1)}$ as $x \to \infty$.

- $p > x^{\epsilon}$ in (*).

  By the Prime Number Theorem and Hornfeck-Wirsing theorem, the number of $n \leq x$ of the form (*) with $p \leq x^{\epsilon}$ is at most

  $$\frac{x^{\epsilon}}{\log x^{\epsilon}} x^{o(1)} \ll_{\epsilon} \frac{x^{\epsilon + o(1)}}{\log x},$$

  which is again negligible.

- $\sigma(m)/m \le \ell$ in (*).

  If $\sigma(m) = rm$ for some $r \ge \ell + 1$, then

$$
\begin{aligned}
\sigma(n) - \ell n &= \sigma(p)\sigma(m) - \ell pm \\
&= (1+p)(rm) - \ell pm \\
&= m(r + p(r - \ell)) \\
&\ge p > x^{\epsilon}.
\end{aligned}
$$

  Contradiction!

- $\sigma(m)/m = \ell$ in (*).

- $\sigma(m)/m = \ell$ in (*).

- Consider the case where $\sigma(m) = rm$ with $2 \leq r \leq \ell - 1$ and $p > x^{\epsilon}$. By Merten's estimate, the number of such n is

$$\begin{aligned}
&\leq \sum_{2\leq r\leq \ell-1} \sum_{x^{\epsilon}<p\leq x} \frac{x^{\epsilon}}{(\ell-r)p-r} \\
&\leq (\ell-2)x^{\epsilon} \sum_{x^{\epsilon}<p\leq x} \frac{1}{p-\ell+1} \\
&\leq 2(\ell-2)x^{\epsilon} \sum_{x^{\epsilon}<p\leq x} \frac{1}{p} \\
&\ll (\ell-2)x^{\epsilon}\log\log x.
\end{aligned}$$

Thus, we only have to work with

$$n = pm \text{ where } p \text{ is prime, } p \nmid m,\ \sigma(m) = \ell m \quad (**)$$

Next we estimate the contribution from (**).

By partial summation and Hornfeck-Wirsing Theorem, we have

$$\sum_{\sigma(m)\,=\,\ell m} \frac{\log m}{m}, \quad \sum_{\sigma(m)\,=\,\ell m} \frac{1}{m}$$

converge.

Let c be any constant greater than 1. By the Prime Number Theorem, there exists $x_0 = x_0(c) > 0$ such that for $x \geq x_0$, we have

$$\pi(x) < c\ \frac{x}{\log x}.$$

Then for $x \geq \max\{x_0^{1/(1-\epsilon)}, \ell^2\}$, we have

$$\begin{aligned}
\#W(\ell; k; x) &\leq \sum_{\substack{m \leq x^\epsilon \\ \sigma(m) = \ell m}} \pi(x/m) \\
&< c \sum_{\substack{m \leq x^\epsilon \\ \sigma(m) = \ell m}} \frac{x/m}{\log(x/m)} \\
&= c\ \frac{x}{\log x} \sum_{\substack{m \leq x^\epsilon \\ \sigma(m) = \ell m}} \frac{1}{m} + O_\epsilon\left(\frac{cx}{(\log x)^2} \sum_{\substack{m \leq x^\epsilon \\ \sigma(m) = \ell m}} \frac{\log m}{m}\right) \\
&< c\ \frac{x}{\log x} \sum_{\sigma(m) = \ell m} \frac{1}{m} + O_\epsilon\left(\frac{cx}{(\log x)^2}\right).
\end{aligned}$$

Therefore,

$$\limsup_{x\to\infty} \frac{\#W(\ell; k; x)}{x/\log x} \ \le \ c \sum_{\sigma(m)=\ell m} \frac{1}{m}.$$

Since the choice of constant $c > 1$ is arbitrary, we have

$$\limsup_{x\to\infty} \frac{\#W(\ell; k; x)}{x/\log x} \ \le \ \sum_{\sigma(m)=\ell m} \frac{1}{m}.$$

This completes the proof.