An alternative method for calculating Bessel integrals appearing in L-function zero statistics

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Why should we care about L-Functions?

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This was formalized by Riemann ζ-function, which is a well-known example of an *L*-function.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \Re(s) > 1,$$

Riemann Zeta

The Riemann ζ-function admits an analytic continuation by means of a functional equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

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Riemann Hypothesis: All non trivial zeros lie in the line $\Re(s) = 1/2$.

General L-functions

An L-function is

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \Re(s) > 1$$

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They have functional equation:

$$\Lambda(s,f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right) L(s,f) = \epsilon_f \Lambda(1-s,f)$$

where $\epsilon_f = \pm 1$

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- Goldfeld-Gross-Zagier: Gauss class number problem for imaginary quadratic fields

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- Montgomery and Dyson found that pair correlation of zeros of $\zeta(s)$ matched the pair correlation of eigenvalues of large unitary matrices in RMT.
- The **pair correlation conjecture** states that the pair correlation between pairs of zeros of $\zeta(s)$ is the same as the pair correlation function of random Hermitian matrices

Katz-Sarnak Philosophy

In the limit, statistics of L-functions match statistics for large random matrices from particular classical compact groups.

- $U(N)$
- $O(N)$
- $USp(2N)$

The GUE Hypothesis

- The **Gaussian Unitary Ensemble** is an ensemble of random $n \times n$ Hermitian matrices $(A = A^T)$ with upper triangular entries i.i.d.r.v. a certain probability measure independent of the upper triangular ones.

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- Key takeaway: Random matrix models are extremely useful for a comparisons of the behavior or zeros of L-Functions.

Some notation

- Ramanujan Sum: $R(n,q) = \sum_{(a,q)=1} \exp(an/q)$
- Principal Dirichlet Character χ_0:

$$\chi_0(n) \begin{cases} 1 \text{ if}(n,b) = 1 \\ 0 \text{ otherwise} \end{cases}$$

- $\phi(x)$ is a Schwartz even test function with support $[-\sigma, \sigma]$
- $\widehat{\phi}(x)$ is the Fourier Transform with support $[-1, 1]$

$$\widehat{\phi}(\xi) = \int_{-\infty}^{\infty} \phi(x) \exp(-2\pi i x \xi) \mathrm{d}x$$

- $J_{k-1}(x)$ is the Bessel function of the first kind.
- $R = k^2 N$ is the analytic conductor

Modular Form Preliminaries

- Define the Hecke congruence subgroup:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \; ad - bc = 1, \; c \equiv 0 \; (N) \right\}$$

- We say f is a weight k **holomorphic cuspform** of level N if

$$\forall \gamma \in \Gamma_0(N), \; f(\gamma z) = (cz + d)^k f(z).$$

- Denote $S_k(N)$ the space of all cusp forms of weight k for the Hecke congruence subgroup $\Gamma_0(N)$ of level N.
- $f \in S_k(N)$ **iff** f is holomorphic in the upper half plane satisfies

$$f\left(\frac{az + b}{cz + d}\right) \;=\; (cz + d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and vanishes at each cusp of $\Gamma_0(N)$.

Modular Form Preliminaries

- Let $f \in S_k(N)$ be of weight k and level N be a **cuspidal newform**. For our purposes, this implies that f is a cusp form of level N but not of level 1.
- With Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz),$$

  with f normalized so that $a_f(1) = 1$.
- Denote H_k^* to be the set of all cuspidal newforms $f \in S_k(N)$.

Hughes-Miller Results

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- nth centered moments agree with RMT for test functions whose Fourier transforms are supported in the extended range $(-\frac{2}{n}, \frac{2}{n})$, $2k \geq n$.

Hughes-Miller Results

- Calculated moments of a smooth counting function of the zeros near the central point of L-functions of weight k cuspidal newforms of prime level N.
- nth centered moments agree with RMT for test functions whose Fourier transforms are supported in the extended range $(-\frac{2}{n}, \frac{2}{n})$, $2k \geq n$.
- Provide additional support for Katz-Sarnak conjectures.

Methods for Proof and Key Idea

- Calculating multi-dimensional integrals in Bessel-Kloosterman expansion of Petersson formula.

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- This involves a change of variables (we lose support)
- Important!: Expanding support allows us to see more values of the function we are analyzing (and thus more zeros!)

Results from Iwaniac-Luo-Sarnak

$$\sum_{m \leq N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R\left(m^2, b\right) R(1, b)}{\varphi(b)}$$

$$\int_{y=0}^{\infty} J_{k-1}(y) \widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right) \frac{\mathrm{d}y}{\log R}$$

$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \Psi(x) \frac{\sin 2\pi x}{2\pi x}\, \mathrm{d}x - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k \log\log kN}{\log kN}\right)$$

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- HM changes variables to appeal to this result as a lemma, but loses support.
- To regain support, HM does complicated combinatorics

Methods for Proof and Key Idea

The Key Idea!

HM had difficulty in comparison with classical RMT. Instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$, we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.

The New Objective

Are there any other ways to achieve the same support without using combinatorial arguments? It is easier to show that it agrees with RMT?

Objective

We want to turn this:

$$S_2^{(2)} = -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1,p_2} \sum_{m \le N^\epsilon} \frac{1}{m} \sum_{b < N^{2006}} \frac{1}{b\varphi(b)}$$

$$\sum_{\chi (\text{ mod } b)} R(m^2,1)R(1,b)\bar{\chi}\left(p_1p_2\right) \times J_{k-1}\left(\frac{4\pi m\sqrt{p_1p_2}}{b\sqrt{N}}\right)$$

$$\prod_{j=1}^{2}\left(\widehat{\phi}\left(\frac{\log p_j}{\log R}\right)\frac{\log p_j}{\sqrt{p_j}\log R}\right) + O\left(N^{-\epsilon}\right)$$

Goal

Into this:

$$S_2^{(2)} = -\frac{1}{2}\left[\int_{-\infty}^{\infty}\Psi(x)\frac{\sin 2\pi x}{2\pi x}\,\mathrm{d}x - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right)$$

Fixing the prime

$$S_2^{(2)} = -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right) \frac{\log p_1}{\sqrt{p_1}\log R} \sum_{m \leq N^{\epsilon}} \frac{1}{m}$$

$$\sum_{b < N^{2006}} \sum_{\chi (\text{mod} b)} \frac{R\left(m^2, b\right) R(1, b) \chi(p_1)}{b\varphi(b)}$$

$$\times \sum_{p_2} \left(\widehat{\phi}\left(\frac{\log p_2}{\log R}\right) \frac{\chi\left(p_2\right) \log(p_2)}{\sqrt{p_2}\log R}\right) J_{k-1}\left(\frac{4\pi m\sqrt{p_2 p_1}}{b\sqrt{N}}\right)$$

$$+ O\left(N^{-\epsilon}\right)$$

Handling the χ

$$S_2^{(2)} = -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right) \frac{\log p_1}{\sqrt{p_1}\log R} \sum_{m \leq N^{\epsilon}} \frac{1}{m}$$

$$\sum_{b < N^{2006}} \frac{R\left(m^2, b\right) R(1,b) \chi_0(p_1)}{b\varphi(b)}$$

$$\times \sum_{p_2} \left(\widehat{\phi}\left(\frac{\log p_2}{\log R}\right) \frac{\chi_0\left(p_2\right)\log(p_2)}{\sqrt{p_2}\log R}\right) J_{k-1}\left(\frac{4\pi m\sqrt{p_2 p_1}}{b\sqrt{N}}\right)$$

$$+ O\left(N^{-\epsilon}\right)$$

From primes to integers

$$S_2^{(2)} = -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right) \frac{\log p_1}{\sqrt{p_1}\log R} \sum_{m \leq N^{\epsilon}} \frac{1}{m}$$

$$\sum_{b < N^{2006}} \frac{R\left(m^2, b\right) R(1, b)\chi_0(p_1)}{b\varphi(b)}$$

$$\times \sum_{n_1} \left(\widehat{\phi}\left(\frac{\log n_1}{\log R}\right) \frac{\chi_0\left(n_1\right)\Lambda\left(n_1\right)}{\sqrt{n_1}\log R}\right) J_{k-1}\left(\frac{4\pi m\sqrt{n_1 p_1}}{b\sqrt{N}}\right)$$

$$+ O\left(N^{-\epsilon}\right)$$

The Mellin Transform

$$\mathcal{M}J_{k-1}(s) = G_{k-1}(s) = \int_0^\infty x^{s-1} J_{k-1}(x)\mathrm{d}x$$

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$$J_{k-1}(s) = \frac{1}{2\pi i}\int_{\Re s=1} G_{k-1}(s)x^{-s}\mathrm{d}s$$

From sum to integral

$$S_2^{(2)} = -\frac{2^{n+1}\pi}{\sqrt{N}} \sum_{p_1} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right) \frac{\log p_1}{\sqrt{p_1}\log R} \sum_{m \leq N^{\epsilon}} \frac{1}{m}$$

$$\sum_{b < N^{2006}} \frac{R\left(m^2, b\right) R(1,b)\chi_0(p_1)}{b\varphi(b)} \frac{b\sqrt{N}}{2\pi m\sqrt{p_1}\log R}$$

$$\int_0^{\infty} J_{k-1}(x)\widehat{\phi}\left(\frac{2\log\left(\frac{bx\sqrt{N}}{4\pi m\sqrt{p_1}}\right)}{\log R}\right) \mathrm{d}x$$

Simplifying

$$S_2^{(2)} = -2^n \sum_{p_1} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right) \frac{\log p_1}{p_1 \log R} \sum_{m \le N^\epsilon} \frac{1}{m^2}$$

$$\sum_{b < N^{2006}} \frac{R\left(m^2, b\right) R(1, b)\chi_0(p_1)}{\varphi(b)} \frac{1}{\log R}$$

$$\int_0^\infty J_{k-1}(x)\widehat{\phi}\left(\frac{2\log\left(\frac{bx\sqrt{N}}{4\pi m\sqrt{\mathbf{p_1}}}\right)}{\log R}\right) \mathrm{d}x$$

Bessel and Gamma

$$S_2^{(2)} = -2^n \sum_{p_1} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right) \frac{\log p_1}{p_1 \log R} \sum_{m \leq N^\epsilon} \frac{1}{m^2}$$

$$\sum_{b < N^{2006}} \frac{R\left(m^2, b\right) R(1, b) \chi_0(p_1)}{\varphi(b)}$$

$$\int_{-\infty}^{\infty} \phi(x \log R) \left(\frac{2\pi mL}{b\sqrt{N}}\right)^{4\pi i x} \frac{\Gamma\left(\frac{k}{2} - 2\pi i x\right)}{\Gamma\left(\frac{k}{2} + 2\pi i x\right)} \mathrm{d}x$$

Current State

We have the following:

$$- 2^n \sum_{p_1} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right)\left(\frac{\varphi(p_1)\log p_1}{p_1^2 \log R}\right)$$

$$\left(\int_{-\infty}^{\infty} \phi(x) \frac{\sin\left(2\pi x \frac{\log A}{\log R}\right)}{2\pi x} \mathrm{d}x + \frac{1}{2}\phi(0)\right)$$

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