Optimal Test Functions for n–Level Densities and Applications to Central Point Vanishing

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Summary

- Review of *L*-functions
- Applications: Bounding average rank, high vanishing
- Ideas of Proof: Functional Analysis, Reduction of Dimension

Non-Brilliant Moments: Worst Results of My Career

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  - ◇ Uses PNT: $\pi(x) \approx x/\log x$!.
  - ◇ Can make better: https://arxiv.org/abs/0709.2184.

Review of L-Functions

Example: Riemann Zeta Function

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \text{ for } \Re(s) > 1.$$

Functional Equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \text{ for } s \in \mathbb{C} \setminus \{1\}.$$

Riemann Hypothesis

All nontrivial zeros (not negative even integers) of ζ are of the form $\gamma = \frac{1}{2} + i\sigma$ with $\sigma \in \mathbb{R}$.

General L-functions

- Euler product

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} \prod_{j=1}^{d} (1 - \alpha_{f,j}(p) p^{-s})^{-1},$$

- meromorphic continuation to $\mathbb{C}$, of finite order, at most finitely may poles (all on the line $\Re(s) = 1$),

- Functional equation: $\omega \in \mathbb{R}$, $G(s)$ product of Γ-fns:

$$e^{i\omega} G(s) L(s,f) = e^{-i\omega} \overline{G(1-\overline{s}) L(1-\overline{s})}.$$

Random Matrix Theory (RMT)

- Ensembles of matrices (Real Symmetric, Hermitian) with entries drawn from probability distribution; Classical Compact Groups.
- Study distribution of normalized eigenvalues for given ensemble.

Applications of RMT

Behavior of zeros of L-functions and energy levels of heavy nuclei well-modeled by eigenvalues of random matrix ensembles.

1-level Density

Riemann hypothesis $\Longrightarrow$ zeros of $L(s, f)$ are of the form $\rho_f = \frac{1}{2} + \mathrm{i}\gamma_f$ with $\gamma_f \in \mathbb{R}$.

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$D(f;\phi) := \sum_{\gamma_f} \phi(\frac{\gamma_f}{2\pi}\log(c_f))$ where $\phi \geq 0$ is even, Schwartz, Fourier transform $\hat{\phi}$ compactly supported, $\phi(0) > 0$.
$c_f > 1$ is the analytic conductor.

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Idea:

Varying ϕ, $D(f;\phi)$ measures density of zeros of $L(s, f)$ near central point $s = 1/2$.

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Impossible to calculate $D(f;\phi)$ explicitly in practice...

1-level Density

Impossible to calculate $D(f;\phi)$ explicitly in practice... so take averages over finite subfamilies of $\mathcal{F}$:

$$\mathcal{F}(Q) := \{f \in \mathcal{F} : c_f \leq Q\}$$

$$\mathbb{E}(\mathcal{F}(Q);\phi) := \frac{1}{|\mathcal{F}(Q)|}\sum_{f\in\mathcal{F}} D(f;\phi).$$

Then take a limit:

$$\lim_{Q\to\infty} \mathbb{E}(\mathcal{F}(Q);\phi) = \int_{-\infty}^{\infty} \phi(x) W(\mathcal{F})(x)\,\mathrm{d}x$$

where $W(\mathcal{F})$ is a distribution depending on $\mathcal{F}$.

1-level Density

Katz-Sarnak Philosophy: $W(\mathcal{F})$ is dependent on a symmetry group $G = G(\mathcal{F})$ of $\mathcal{F}$, write $W(\mathcal{F}) = W_{1,G}$.

Examples:

$$\begin{aligned} W_{1,\mathrm{O}}(x) &= 1 + \frac{1}{2}\delta(x) \\ W_{1,\mathrm{SO(Even)}}(x) &= 1 + \frac{\sin(2\pi x)}{2\pi x} \\ W_{1,\mathrm{SO(Odd)}}(x) &= 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta(x). \end{aligned}$$

1-level Density

Quantity of interest

$\lim_{Q\to\infty} \mathrm{AveRank}(\mathcal{F}(Q))$, where $\mathrm{AveRank}(\mathcal{F}(Q))$ is average order of vanishing of the L-functions with $f \in \mathcal{F}(Q)$ at $s = 1/2$.

Trivially

$$\lim_{Q\to\infty} \mathrm{AveRank}(\mathcal{F}(Q)) \ \leq \ \frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x)\, \mathrm{d}x}{\phi(0)}.$$

n-level Density

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$$D_n(f;\phi) := \sum_{\substack{\gamma_{j,f} \\ |j| \text{ distinct}}} \phi\left(\frac{\gamma_{1,f}}{2\pi}\log(c_f), \frac{\gamma_{2,f}}{2\pi}\log(c_f), \dots, \frac{\gamma_{n,f}}{2\pi}\log(c_f)\right).$$

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Higher Dimensional Bound

$$\lim_{Q\to\infty} \mathrm{WeightedAveRank}(\mathcal{F}(Q)) \le \frac{\int_{\mathbb{R}^n} \phi(x) W_{n,G}(x)\, dx_1 \cdots dx_n}{\phi(0)}.$$

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Goal

Higher level densities give stronger bound. Minimize right-hand side over admissible ϕ for n as large as possible.

Applications of n-level density

Average rank $\cdot\ \phi(0) \ \leq\ \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if ϕ non-negative.
Can also use to bound the percentage that vanish to order r for any r.

Theorem (Miller, Hughes-Miller)

Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_r such that probability of at least r zeros at the central point is at most $c_n r^{-n}$.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$.

Katz-Sarnak Determinants

Set $K_\epsilon(x,y) := \frac{\sin(\pi(x-y))}{\pi(x-y)} + \epsilon\frac{\sin(\pi(x+y))}{\pi(x+y)}$, $\epsilon \in \{0, \pm 1\}$.
The n-level weights for classical compact groups are

$$W_{n,\mathrm{SO(Even)}}(x) = \det\left(K_1(x_i,x_j)\right)_{i,j\le n}$$

$$W_{n,\mathrm{SO(Odd)}}(x) = \det\left(K_{-1}(x_i,x_j)\right)_{i,j\le n} + \sum_{k=1}^{n}\delta(x_k)\det\left(K_{-1}(x_i,x_j)\right)_{i,j\neq k}$$

$$W_{n,\mathrm{O}}(x) = \frac{1}{2}W_{n,\mathrm{SO(Even)}}(x) + \frac{1}{2}W_{n,\mathrm{SO(Odd)}}(x)$$

$$W_{n,\mathrm{U}}(x) = \det\left(K_0(x_i,x_j)\right)_{i,j\le n}$$

$$W_{n,\mathrm{Sp}}(x) = \det\left(K_{-1}(x_i,x_j)\right)_{i,j\le n}.$$

Philosophy: Reduce dimension of number theory problem.

Theorem (Iwaniec-Luo-Sarnak)

Let Ψ be an even Schwartz function with $\mathrm{supp}(\widehat{\Psi}) \subset (-2,2)$. Then

$$\sum_{m \le N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{dy}{\log R}$$

$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \Psi(x)\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right),$$

where $R = k^2N$, φ is Euler's totient function, and $R(n,q)$ is a Ramanujan sum.

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

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Change of variables and Jacobian:

$$\begin{aligned} u_2 &= x_1x_2 & \qquad x_2 &= \frac{u_2}{u_1} \\ u_1 &= x_1 & \qquad x_1 &= u_1 \end{aligned}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{vmatrix} = \frac{1}{u_1}$$

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$$\int\int\widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\frac{du_1du_2}{u_1}$$

2-Level Density

Change variables: $w = \log u_1 / \log R$; u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R} - w_1\right) dw_1.$$

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Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) \;:=\; \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\, \widehat{\phi}\left(\frac{\log u_2}{\log R} - w_1\right) dw_1.$$

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Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2 u_2 N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}.$$

Outline of Maine Reults

Main Results

Main Idea

Restrict domain to only those ϕ which are products of single variable test functions: $\phi(x) = \phi_1(x_1) \cdots \phi_n(x_n)$ (equivalent to linear combinations of such products).

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Main Result

1. Choosing first $n-1$ factors $\phi_1, \dots, \phi_{n-1}$ carefully, can integrate first $n-1$ variables to obtain new weight function of a form similar to 1-dimensional weights.
2. 1-level case already solved, so choose ϕ_n optimally for new weight.

1-Level Case (ILS, F-M)

1-level Case

Two Steps.

- ◇ Reduce problem to different optimization problem.
- ◇ Use functional analysis to solve reduced problem.

Step 1: Reduce Problem

Assume $\text{supp}(\hat{\phi}) \subset [-1, 1]$. Plancherel on numerator, taking then inverting Fourier transform in denominator:

$$\frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x)\, \mathrm{d}x}{\phi(0)} = \frac{\int_{-1}^{1} \hat{\phi}(\xi) \widehat{W_{1,G}}(\xi)\, \mathrm{d}\xi}{\int_{-1}^{1} \hat{\phi}(\xi)\, \mathrm{d}\xi}.$$

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Ahiezer's Theorem and the Paley-Wiener Theorem show ϕ admissible $\iff$ $\hat{\phi}(\xi) = (g * \breve{g})(\xi)$ for some $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$, where $\breve{g}(\xi) = \overline{g(-\xi)}$.

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Calculations show for classical compact group, $\widehat{W_{1,G}}(\xi) = \delta(\xi) + m(\xi)$ on $[-1,1]$, with $m(\xi)$ real, piecewise continuous, even.

Step 1: Reduce Problem

Some functional analysis: define compact, self-adjoint linear operator $K : L^2[-\frac{1}{2}, \frac{1}{2}] \to L^2[-\frac{1}{2}, \frac{1}{2}]$

$$(Kg)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x-y)g(y)\,\mathrm{d}y.$$

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Some manipulations (**1** is the characteristic function of a set):

$$\frac{\int_{-1}^{1} \hat{\phi}(\xi)\widehat{W_{1,G}}(\xi)\,\mathrm{d}\xi}{\int_{-1}^{1} \hat{\phi}(\xi)\,\mathrm{d}\xi} = \frac{\int_{-1}^{1}(g * \breve{g})(\xi)(\delta(\xi) + m(\xi))\,\mathrm{d}\xi}{\int_{-1}^{1}(g * \breve{g})(\xi)\,\mathrm{d}\xi}$$

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$$= \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} \left(\delta(\xi) g(\xi+y)\overline{g(y)} + m(\xi) g(\xi+y)\overline{g(y)}\right) \,\mathrm{d}\xi\,\mathrm{d}y}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} g(\xi+y)\overline{g(y)}\,\mathrm{d}\xi\,\mathrm{d}y}$$

$$= \frac{\langle g, g\rangle_{L^2} + \int_{-1}^{1} \int_{-\frac{1}{2}+\xi}^{\frac{1}{2}+\xi} m(\xi) g(y)\overline{g(-\xi+y)}\,\mathrm{d}y\,\mathrm{d}\xi}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} g(\xi+y)\,\mathrm{d}\xi\,\overline{g(y)}\,\mathrm{d}y}$$

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$$= \frac{\langle g, g\rangle_{L^2} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} m(\xi - y) g(y)\, \mathrm{d}y \overline{g(\xi)}\, \mathrm{d}\xi}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} g(\xi + y)\, \mathrm{d}\xi\, \overline{g(y)}\, \mathrm{d}y}$$

$$= \frac{\langle g, g\rangle_{L^2} + \langle Kg, g\rangle_{L^2}}{\langle g, \mathbf{1}\rangle_{L^2}\langle \mathbf{1}, g\rangle_{L^2}}$$

$$= \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}.$$

New Problem

Defining $R : L^2[-\frac{1}{2}, \frac{1}{2}] \to L^2[-\frac{1}{2}, \frac{1}{2}]$ by $R(g) := \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}$, minimize R over subset of $L^2[-\frac{1}{2}, \frac{1}{2}]$ with denominator $\neq 0$.

Step 2: Minimization

Some observations:

- $R(g) \geq \lim_{Q\to\infty} \text{AveRank}(\mathcal{F}(Q)) \geq 0$.
- Spectral Theorem $\implies$ orthonormal basis of eigenvectors of K, eigenvalues λ_j.
- $\lambda_j \geq -1$.

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Case 1: Eigenvalue (-1)

If have a (-1)-eigenvector $f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ not orthogonal to $\mathbf{1}$, then $R(f_0) = \frac{\langle (I+K)f_0, f_0\rangle_{L^2}}{|\langle \mathbf{1}, f_0\rangle_{L^2}|^2} = \frac{\langle f_0, f_0\rangle_{L^2} - \langle f_0, f_0\rangle_{L^2}}{|\langle \mathbf{1}, f_0\rangle_{L^2}|^2} = 0$.

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- Fredholm Theory $\Longrightarrow \exists f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying $(I + K)f_0 = \mathbf{1}$.
- $A := \langle \mathbf{1}, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$.

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For $g = f_0 + h \in L^2[-\frac{1}{2}, \frac{1}{2}]$ with $\langle \mathbf{1}, g \rangle_{L^2} \neq 0$, WLOG $\langle \mathbf{1}, g \rangle_{L^2} = A$. Then $\langle \mathbf{1}, h \rangle_{L^2} = 0$, so

$$\begin{aligned} R(g) &= \frac{\langle \mathbf{1}, f_0 \rangle_{L^2} + \langle (I+K)h, h \rangle_{L^2} + \langle \mathbf{1}, h \rangle_{L^2} + \langle h, \mathbf{1} \rangle_{L^2}}{|A|^2} \\ &= \frac{A + \langle (I+K)h, h \rangle_{L^2} + 0 + 0}{|A|^2} \ \geq \ \frac{1}{A} \ = \ R(f_0). \end{aligned}$$

n-Level Case

$n \geq 2$

Challenges:

1. $\widehat{W_{n,G}}$ more complicated.
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A Solution

Restrict to minimizing over $\phi(x) = \phi_1(x_1) \cdots \phi_n(x_n)$ with ϕ_j as in 1-level case (equivalent to minimizing over finite sums).

An Approach

Outline:

◇ Choose $\phi_2, \dots, \phi_n$ and integrate last $n-1$ variables to obtain new weight function similar to 1-level weights.

◇ Use 1-level approach to minimize choice of ϕ_1.

Example: $W_{2,\mathrm{U}}$

Problem

Minimize

$$\frac{\int_{\mathbb{R}^2} \phi_1(x_1)\phi_2(x_2) W_{2,\mathrm{U}}(x)\, \mathrm{d}x_1\, \mathrm{d}x_2}{\phi_1(0)\phi_2(0)} = \frac{\int_{[-1,1]^2} \hat{\phi}_1(\xi_1)\hat{\phi}_2(\xi_2)\widehat{W_{2,\mathrm{U}}}(\xi)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2}{\phi_1(0)\phi_2(0)}$$ over

ϕ_1, ϕ_2 even, Schwartz, $\phi_1(0), \phi_2(0) > 0$, and $\mathrm{supp}(\hat{\phi}_1), \mathrm{supp}(\hat{\phi}_2) \subset [-1, 1]$.

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$$\frac{\int_{\mathbb{R}^2} \phi_1(x_1)\phi_2(x_2) W_{2,\mathrm{U}}(x)\, dx_1\, dx_2}{\phi_1(0)\phi_2(0)} = \frac{\int_{[-1,1]^2} \hat{\phi}_1(\xi_1)\hat{\phi}_2(\xi_2)\widehat{W_{2,\mathrm{U}}}(\xi)\, d\xi_1\, d\xi_2}{\phi_1(0)\phi_2(0)}$$ over

ϕ_1, ϕ_2 even, Schwartz, $\phi_1(0), \phi_2(0) > 0$, and $\mathrm{supp}(\hat{\phi}_1), \mathrm{supp}(\hat{\phi}_2) \subset [-1, 1]$.

$\mathbf{1}(x)$ characteristic function of appropriate set. A short computation:

$$W_{2,\mathrm{U}}(x) = 1 - \frac{\sin^2(\pi(x_1 - x_2))}{\pi^2(x_1 - x_2)^2}$$

$$\widehat{W_{2,\mathrm{U}}}(\xi) = \delta(\xi_1)\delta(\xi_2) + \delta(\xi_1 + \xi_2)(|\xi_1| - 1)\mathbf{1}(\xi_1).$$

Example: $W_{2,\mathrm{U}}$

For ϕ_2 arbitrary,

$$\frac{1}{\phi_2(0)}\int_{\xi_2\in\mathbb{R}} \hat{\phi}_2(\xi_2)\widehat{W_{2,\mathrm{U}}}(\xi)\,\mathrm{d}\xi_2 = \frac{\hat{\phi}_2(0)}{\phi_2(0)}\delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\phi_2(0)}(|\xi_1|-1)\mathbf{1}(\xi_1).$$

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New Problem:

Normalizing by $\frac{\hat{\phi}_2(0)}{\phi_2(0)}$, minimize

$$\frac{\int_{\xi_1\in\mathbb{R}} \hat{\phi}_1(\xi_1)\widetilde{W}(\xi_1)}{\phi_1(0)}$$

over ϕ_1, where $\widetilde{W}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\hat{\phi}_2(0)}(|\xi_1|-1)\mathbf{1}(\xi_1)$.

Example: $W_{2,\mathrm{U}}$

$$\widetilde{W}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\hat{\phi}_2(0)}(|\xi_1| - 1)\mathbf{1}(x)(\xi_1) = \delta(\xi_1) + m(\xi_1)$$

- ϕ_2 even $\implies$ m is even.

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- ϕ_2 even $\implies$ m is even.
- 1-level case $\implies$ optimal ϕ_1 has $\hat{\phi}_1(\xi_1) = (g * \breve{g})(\xi_1)$ where $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying

$$\mathbf{1}(x) = g(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x-y)g(y)\,\mathrm{d}y.$$

Minimum value is $\frac{1}{\langle \mathbf{1}, g\rangle_{L^2}}$.

Example: $W_{2,\mathrm{U}}$

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- $\diamond\ K(x,y) := -m(x-y)$.
  $\diamond\ K_n(x) := \int_{[-\frac{1}{2},\frac{1}{2}]^n} K(x,t_1)\dots K(t_{n-1},t_n)\,\mathrm{d}t_1\ \cdots\ \mathrm{d}t_n$.

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- $g(x) = \mathbf{1}(x) + \sum_{n=1}^{\infty} K_n(x)$.
- $\langle \mathbf{1}, g\rangle_{L^2} = 1 + \sum_{n=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x)\,\mathrm{d}x$.

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- Terms of series are nonnegative, so truncate after finitely many terms to get

$$\frac{\hat{\phi}_2(0)}{\phi_2(0)} \frac{1}{\langle \mathbf{1}, g\rangle_{L^2}} \leq \frac{\hat{\phi}_2(0)}{\phi_2(0)} \left(1 + \sum_{n=1}^{100} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x)\,\mathrm{d}x\right)^{-1} \approx 0.49386.$$

Numerical Data for $n = 2$

- Truncate at 100 terms with $\hat{\phi}_2(\xi_2) = (1 - |\xi_2|)\chi_{[-1,1]}(\xi_2)$.

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	Bound
$W_{2,\mathrm{O}}$	0.222483
$W_{2,\mathrm{SO(Even)}}$	0.252298
$W_{2,\mathrm{SO(Odd)}}$	0.130293
$W_{2,\mathrm{U}}$	0.493856
$W_{2,\mathrm{Sp}}$	0.130293

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$$\Pr(0) + \Pr(1) \ge \begin{cases} 0.777517 & W_{2,\mathrm{O}} \\ 0.506144 & W_{2,\mathrm{U}} \\ 0.869707 & W_{2,\mathrm{Sp}}. \end{cases}$$

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$$\begin{aligned} \Pr(0) &\geq 0.873851 \qquad & W_{2,SO(\text{Even})} \\ \Pr(1) &\geq 0.978285 \qquad & W_{2,SO(\text{Odd})} \end{aligned}$$

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