Optimal Test Functions for $n$–Level Densities and Applications to Central Point Vanishing

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Summary

Review of L-Functions

Outline of Maine Results

1-Level Case (ILS, F-M)

n-Level Case

References

Journal of Number Theory: Email sjm1@williams.edu

https://www.journals.elsevier.com/journal-of-number-theory
Summary

- Review of $L$-functions
- Applications: Bounding average rank, high vanishing
- Ideas of Proof: Functional Analysis, Reduction of Dimension
Non-Brilliant Moments: Worst Results of My Career

IF time permits, will give some explicit bounds at the end. Not optimized.

As order of vanishing increases, result gets better but initially bad.
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1. (Approximately) at most 250% of cuspidal newforms vanish to order 2 or more.

2. There are at least \( \log \log \log x \) primes at most \( x! \)
   - Uses PNT: \( \pi(x) \approx x/\log x! \).
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- There are at least $\log \log \log x$ primes at most $x!$
  - Uses PNT: $\pi(x) \approx x / \log x!$.
Review of $L$-Functions
**Example: Riemann Zeta Function**

**Riemann Zeta Function**

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \text{ for } \Re(s) > 1.
\]

**Functional Equation**

\[
\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \text{ for } s \in \mathbb{C} \setminus \{1\}.
\]

**Riemann Hypothesis**

All nontrivial zeros (not negative even integers) of \( \zeta \) are of the form \( \gamma = \frac{1}{2} + i\sigma \) with \( \sigma \in \mathbb{R} \).
General $L$-functions

- Euler product

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} \prod_{j=1}^{d} (1 - \alpha_{f,j}(p)p^{-s})^{-1},$$

- meromorphic continuation to $\mathbb{C}$, of finite order, at most finitely may poles (all on the line $\Re(s) = 1$),

- Functional equation: $\omega \in \mathbb{R}$, $G(s)$ product of $\Gamma$-fns:

$$e^{i\omega} G(s)L(s, f) = e^{-i\omega} \overline{G(1 - \bar{s})L(1 - \bar{s})}.$$
Random Matrix Theory (RMT)

- Ensembles of matrices (Real Symmetric, Hermitian) with entries drawn from probability distribution; Classical Compact Groups.

- Study distribution of normalized eigenvalues for given ensemble.

Applications of RMT

Behavior of zeros of $L$-functions and energy levels of heavy nuclei well-modeled by eigenvalues of random matrix ensembles.
Riemann hypothesis $\implies$ zeros of $L(s, f)$ are of the form
$$\rho_f = \frac{1}{2} + i\gamma_f \text{ with } \gamma_f \in \mathbb{R}.$$
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**1-level Density**

$$D(f; \phi) := \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi} \log(c_f)\right) \text{ where } \phi \geq 0 \text{ is even, Schwartz, Fourier transform } \hat{\phi} \text{ compactly supported, } \phi(0) > 0.$$

$c_f > 1$ is the analytic conductor.
Riemann hypothesis $\implies$ zeros of $L(s, f)$ are of the form $\rho_f = \frac{1}{2} + i\gamma_f$ with $\gamma_f \in \mathbb{R}$.

1-level Density

$D(f; \phi) := \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi} \log(c_f)\right)$ where $\phi \geq 0$ is even, Schwartz, Fourier transform $\hat{\phi}$ compactly supported, $\phi(0) > 0$. $c_f > 1$ is the analytic conductor.

Idea:
Varying $\phi$, $D(f; \phi)$ measures density of zeros of $L(s, f)$ near central point $s = 1/2$. 
1-level Density

Impossible to calculate $D(f; \phi)$ explicitly in practice...
1-level Density

Impossible to calculate $D(f; \phi)$ explicitly in practice... so take averages over finite subfamilies of $\mathcal{F}$:

$$\mathcal{F}(Q) := \{ f \in \mathcal{F} : c_f \leq Q \}$$

$$\mathbb{E}(\mathcal{F}(Q); \phi) := \frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}} D(f; \phi).$$

Then take a limit:

$$\lim_{Q \to \infty} \mathbb{E}(\mathcal{F}(Q); \phi) = \int_{-\infty}^{\infty} \phi(x) W(\mathcal{F})(x) \, dx$$

where $W(\mathcal{F})$ is a distribution depending on $\mathcal{F}$. 
1-level Density

**Katz-Sarnak Philosophy:** \( W(\mathcal{F}) \) is dependent on a symmetry group \( G = G(\mathcal{F}) \) of \( \mathcal{F} \), write \( W(\mathcal{F}) = W_{1,G} \).

**Examples:**

\[
W_{1,0}(x) = 1 + \frac{1}{2} \delta(x)
\]

\[
W_{1,SO(\text{Even})}(x) = 1 + \frac{\sin(2\pi x)}{2\pi x}
\]

\[
W_{1,SO(\text{Odd})}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x} + \delta(x).
\]
1-level Density

**Quantity of interest**

\[
\lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q)), \text{ where AveRank}(\mathcal{F}(Q)) \text{ is average order of vanishing of the } L\text{-functions with } f \in \mathcal{F}(Q) \text{ at } s = 1/2.
\]

Trivially

\[
\lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q)) \leq \frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x) \, dx}{\phi(0)}.
\]
\[ D_n(f; \phi) := \sum_{\gamma_j, f} \phi \left( \frac{\gamma_j f}{2\pi} \log(c_f), \frac{\gamma_j f}{2\pi} \log(c_f), \ldots, \frac{\gamma_n f}{2\pi} \log(c_f) \right). \]
$n$-level Density

\[ D_n(f; \phi) := \sum_{\gamma_j, f} \phi \left( \frac{\gamma_1, f}{2\pi} \log(c_f), \frac{\gamma_2, f}{2\pi} \log(c_f), \ldots, \frac{\gamma_n, f}{2\pi} \log(c_f) \right). \]

Higher Dimensional Bound

\[
\lim_{Q \to \infty} \text{WeightedAveRank}(\mathcal{F}(Q)) \leq \frac{\int_{\mathbb{R}^n} \phi(x) W_{n,G}(x) \, dx_1 \cdots dx_n}{\phi(0)}.
\]
**n-level Density**

\[ D_n(f; \phi) := \sum_{\gamma_{j,f}} \phi \left( \frac{\gamma_{1,f}}{2\pi} \log(c_f), \frac{\gamma_{2,f}}{2\pi} \log(c_f), \ldots, \frac{\gamma_{n,f}}{2\pi} \log(c_f) \right). \]

**Higher Dimensional Bound**

\[ \lim_{Q \to \infty} \text{WeightedAveRank}(\mathcal{F}(Q)) \leq \frac{\int_{\mathbb{R}^n} \phi(x) W_{n,G}(x) \, dx_1 \cdots dx_n}{\phi(0)}. \]

**Goal**

Higher level densities give stronger bound. Minimize right-hand side over admissible \( \phi \) for \( n \) as large as possible.
Applications of $n$-level density

Average rank $\cdot \phi(0) \leq \int \phi(x) W_{G(\mathcal{F})}(x) dx$ if $\phi$ non-negative. Can also use to bound the percentage that vanish to order $r$ for any $r$.

**Theorem (Miller, Hughes-Miller)**

*Using $n$-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any $r$ there is a $c_r$ such that probability of at least $r$ zeros at the central point is at most $c_n r^{-n}$.*

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$. 
Katz-Sarnak Determinants

Set $K_\epsilon(x, y) := \frac{\sin(\pi(x-y))}{\pi(x-y)} + \epsilon \frac{\sin(\pi(x+y))}{\pi(x+y)}$, $\epsilon \in \{0, \pm 1\}$.

The $n$-level weights for classical compact groups are

\[
W_{n, \text{SO(Even)}}(x) = \det (K_1(x_i, x_j))_{i,j \leq n}
\]

\[
W_{n, \text{SO(Odd)}}(x) = \det (K_{-1}(x_i, x_j))_{i,j \leq n} + \sum_{k=1}^{n} \delta(x_k) \det (K_{-1}(x_i, x_j))_{i,j \neq k}
\]

\[
W_{n, O}(x) = \frac{1}{2} W_{n, \text{SO(Even)}}(x) + \frac{1}{2} W_{n, \text{SO(Odd)}}(x)
\]

\[
W_{n, U}(x) = \det (K_0(x_i, x_j))_{i,j \leq n}
\]

\[
W_{n, Sp}(x) = \det (K_{-1}(x_i, x_j))_{i,j \leq n}.
\]
Philosophy: Reduce dimension of number theory problem.

Theorem (Iwaniec-Luo-Sarnak)

Let $\psi$ be an even Schwartz function with $\text{supp}(\hat{\psi}) \subset (-2, 2)$. Then

$$
\sum_{m \leq N^\epsilon} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y) \hat{\psi} \left( \frac{2 \log(by\sqrt{N}/4\pi m)}{\log R} \right) \frac{y}{\log R} 
$$

$$
= -\frac{1}{2} \left[ \int_{-\infty}^{\infty} \psi(x) \frac{\sin 2\pi x}{2\pi x} x - \frac{1}{2} \psi(0) \right] + O \left( \frac{k \log \log kN}{\log kN} \right),
$$

where $R = k^2N$, $\varphi$ is Euler’s totient function, and $R(n,q)$ is a Ramanujan sum.
2-Level Density

\[
\int_{x_1=2}^{R^\sigma} \int_{x_2=2}^{R^\sigma} \phi \left( \frac{\log x_1}{\log R} \right) \phi \left( \frac{\log x_2}{\log R} \right) J_{k-1} \left( 4\pi \sqrt{\frac{m^2 x_1 x_2 N}{c}} \right) \frac{dx_1}{\sqrt{x_1}} \frac{dx_2}{\sqrt{x_2}}
\]
2-Level Density

\[
\int_{R^\sigma} \int_{R^\sigma} \hat{\phi} \left( \frac{\log x_1}{\log R} \right) \hat{\phi} \left( \frac{\log x_2}{\log R} \right) J_{k-1} \left( 4\pi \frac{\sqrt{m^2 x_1 x_2 N}}{c} \right) \frac{dx_1 dx_2}{\sqrt{x_1 x_2}}
\]

Change of variables and Jacobian:

\[
\begin{align*}
U_2 &= x_1 x_2 \\
U_1 &= x_1 \\
x_2 &= \frac{u_2}{u_1} \\
x_1 &= U_1
\end{align*}
\]

\[
\left| \frac{\partial x}{\partial u} \right| = \left| \begin{array}{cc} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{array} \right| = \frac{1}{u_1}
\]
2-Level Density

\[
\int_{x_1=2}^{R^\sigma} \int_{x_2=2}^{R^\sigma} \hat{\phi} \left( \frac{\log x_1}{\log R} \right) \hat{\phi} \left( \frac{\log x_2}{\log R} \right) J_{k-1} \left( 4\pi \frac{\sqrt{m^2 x_1 x_2 N}}{c} \right) \frac{dx_1}{\sqrt{x_1 x_2}} dx_2
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\left| \frac{\partial x}{\partial u} \right| = \begin{vmatrix} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} \end{vmatrix} = \frac{1}{u_1} \quad \text{and}
\]

\[
\int \int \hat{\phi} \left( \frac{\log u_1}{\log R} \right) \hat{\phi} \left( \frac{\log \left( \frac{u_2}{u_1} \right)}{\log R} \right) \frac{1}{\sqrt{u_2}} J_{k-1} \left( 4\pi \frac{\sqrt{m^2 u_2 N}}{c} \right) \frac{du_1}{u_1} du_2
\]
2-Level Density

Change variables: \( w = \log u_1 / \log R \); \( u_1 \)-integral is

\[
\int_{w_1 = \log u_2 / \log R - \sigma}^{\sigma} \tilde{\phi}(w_1) \tilde{\phi} \left( \frac{\log u_2}{\log R} - w_1 \right) \, dw_1.
\]
2-Level Density

Change variables: $w = \log u_1 / \log R$; $u_1$-integral is

$$\int_{w_1 = \frac{\log u_2}{\log R} - \sigma}^{\sigma} \hat{\phi}(w_1) \hat{\phi} \left( \frac{\log u_2}{\log R} - w_1 \right) dw_1.$$

Support conditions imply

$$\psi_2 \left( \frac{\log u_2}{\log R} \right) := \int_{w_1 = -\infty}^{\infty} \hat{\phi}(w_1) \hat{\phi} \left( \frac{\log u_2}{\log R} - w_1 \right) dw_1.$$
2-Level Density

Change variables: \( w = \log u_1 / \log R; \) \( u_1 \)-integral is

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\]

Substituting gives

\[
\int_{u_2 = 0}^{\infty} J_{k-1} \left( 4\pi \frac{\sqrt{m^2 u_2 N}}{c} \right) \psi_2 \left( \frac{\log u_2}{\log R} \right) \frac{du_2}{\sqrt{u_2}}.
\]
Outline of Maine Results
Main Results

Main Idea

Restrict domain to only those $\phi$ which are products of single variable test functions: $\phi(x) = \phi_1(x_1) \cdots \phi_n(x_n)$ (equivalent to linear combinations of such products).
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Restrict domain to only those \( \phi \) which are products of single variable test functions: \( \phi(x) = \phi_1(x_1) \cdots \phi_n(x_n) \) (equivalent to linear combinations of such products).

Main Result

1. Choosing first \( n - 1 \) factors \( \phi_1, \ldots, \phi_{n-1} \) carefully, can integrate first \( n - 1 \) variables to obtain new weight function of a form similar to 1-dimensional weights.
2. 1-level case already solved, so choose \( \phi_n \) optimally for new weight.
1-Level Case (ILS, F-M)
1-level Case

Two Steps.

◊ Reduce problem to different optimization problem.

◊ Use functional analysis to solve reduced problem.
Step 1: Reduce Problem

Assume \( \text{supp}(\hat{\phi}) \subset [-1, 1] \). Plancherel on numerator, taking then inverting Fourier transform in denominator:

\[
\frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x) \, dx}{\phi(0)} = \frac{\int_{-1}^{1} \hat{\phi}(\xi) \widehat{W_{1,G}}(\xi) \, d\xi}{\int_{-1}^{1} \hat{\phi}(\xi) \, d\xi}.
\]
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\frac{\int_{-\infty}^{\infty} \phi(x) W_{1,G}(x) \, dx}{\phi(0)} = \frac{\int_{-1}^{1} \hat{\phi}(\xi) \overline{W_{1,G}(\xi)} \, d\xi}{\int_{-1}^{1} \hat{\phi}(\xi) \, d\xi}.
$$

Ahiezer’s Theorem and the Paley-Wiener Theorem show $\phi$ admissible $\iff \hat{\phi}(\xi) = (g \ast \check{g})(\xi)$ for some $g \in L^2[-\frac{1}{2}, \frac{1}{2}]$, where $\check{g}(\xi) = g(-\xi)$. 
**Step 1: Reduce Problem**

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Calculations show for classical compact group, $\overline{W_{1,G}(\xi)} = \delta(\xi) + m(\xi)$ on $[-1, 1]$, with $m(\xi)$ real, piecewise continuous, even.
Step 1: Reduce Problem

Some functional analysis: define compact, self-adjoint linear operator $K : L^2\left[-\frac{1}{2}, \frac{1}{2}\right] \to L^2\left[-\frac{1}{2}, \frac{1}{2}\right]$

$$ (Kg)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y)g(y) \, dy. $$
Step 1: Reduce Problem

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\[
(Kg)(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y)g(y) \, dy.
\]

Some manipulations (\( 1 \) is the characteristic function of a set):

\[
\int_{-1}^{1} \frac{\hat{\phi}(\xi) \hat{W}_{1,G}(\xi) \, d\xi}{\int_{-1}^{1} \hat{\phi}(\xi) \, d\xi} = \frac{\int_{-1}^{1} (g * \check{g})(\xi)(\delta(\xi) + m(\xi)) \, d\xi}{\int_{-1}^{1} (g * \check{g})(\xi) \, d\xi}
\]
Step 1: Reduce Problem

\[
\int_{-1/2}^{1/2} \int_{-1}^{1} \left( \delta(\xi) g(\xi + y) g(y) + m(\xi) g(\xi + y) g(y) \right) \, d\xi \, dy
\]

\[
= \frac{\int_{-1/2}^{1/2} \int_{-1}^{1} g(\xi + y) g(y) \, d\xi \, dy}{\int_{-1/2}^{1/2} \int_{-1}^{1} g(\xi + y) g(y) \, d\xi \, dy}
\]

\[
\langle g, g \rangle_{L^2} + \int_{-1}^{1} \int_{-1/2}^{1/2 + \xi} m(\xi) g(y) g(-\xi + y) \, dy \, d\xi
\]

\[
= \frac{\int_{-1/2}^{1/2} \int_{-1}^{1} g(\xi + y) \, d\xi \, g(y) \, dy}{\int_{-1/2}^{1/2} \int_{-1}^{1} g(\xi + y) \, d\xi \, g(y) \, dy}
\]

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\]
Step 1: Reduce Problem

\[
\langle g, g \rangle_{L^2} + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} m(\xi - y)g(y) \, dy \overline{g(\xi)} \, d\xi \\
= \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} g(\xi + y) \, d\xi \overline{g(y)} \, dy}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-1}^{1} m(\xi - y)g(y) \, dy \overline{g(\xi)} \, d\xi}
\]

\[
= \frac{\langle g, g \rangle_{L^2} + \langle Kg, g \rangle_{L^2}}{\langle g, 1 \rangle_{L^2} \langle 1, g \rangle_{L^2}} = \frac{\langle (I + K)g, g \rangle_{L^2}}{|\langle 1, g \rangle_{L^2}|^2}.
\]

New Problem

Defining \( R : L^2[-\frac{1}{2}, \frac{1}{2}] \rightarrow L^2[-\frac{1}{2}, \frac{1}{2}] \) by \( R(g) := \frac{\langle (I+K)g, g \rangle_{L^2}}{|\langle 1, g \rangle_{L^2}|^2} \),
minimize \( R \) over subset of \( L^2[-\frac{1}{2}, \frac{1}{2}] \) with denominator \( \neq 0 \).
Step 2: Minimization

Some observations:

- $R(g) \geq \lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q)) \geq 0$.
- Spectral Theorem $\iff$ orthonormal basis of eigenvectors of $K$, eigenvalues $\lambda_j$.
- $\lambda_j \geq -1$. 

\[ R(g) \geq \lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q)) \geq 0. \]

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- \( R(g) \geq \lim_{Q \to \infty} \text{AveRank}(\mathcal{F}(Q)) \geq 0 \).
- Spectral Theorem \( \implies \) orthonormal basis of eigenvectors of \( K \), eigenvalues \( \lambda_j \).
- \( \lambda_j \geq -1 \).

**Case 1: Eigenvalue \((-1)\)**

If have a \((-1)\)-eigenvector \( f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}] \) not orthogonal to 1, then
\[
R(f_0) = \frac{\langle (I+K)f_0,f_0 \rangle_{L^2}}{|\langle 1,f_0 \rangle_{L^2}|^2} = \frac{\langle f_0,f_0 \rangle_{L^2} - \langle f_0,f_0 \rangle_{L^2}}{|\langle 1,f_0 \rangle_{L^2}|^2} = 0.
\]
Step 2: Minimization

Case 2: $\lambda_j > -1$ for all $j$. More functional analysis!
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- $\ker(I + K) = \{0\}$ (all eigenvalues $> -1$).
- Fredholm Theory $\implies \exists f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying $(I + K)f_0 = 1$.
- $A := \langle 1, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$. 
Step 2: Minimization

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- Fredholm Theory $\implies \exists f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying $(I + K)f_0 = 1$.
- $A := \langle 1, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$.

For $g = f_0 + h \in L^2[-\frac{1}{2}, \frac{1}{2}]$ with $\langle 1, g \rangle_{L^2} \neq 0$, WLOG $\langle 1, g \rangle_{L^2} = A$. Then $\langle 1, h \rangle_{L^2} = 0$, so

$$R(g) = \frac{\langle 1, f_0 \rangle_{L^2} + \langle (I + K)h, h \rangle_{L^2} + \langle 1, h \rangle_{L^2} + \langle h, 1 \rangle_{L^2}}{|A|^2}$$

$$= \frac{A + \langle (I + K)h, h \rangle_{L^2} + 0 + 0}{|A|^2} \geq \frac{1}{A} = R(f_0).$$
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\( n \)-Level Case
Challenges:

1. \( \mathcal{W}_{n,G} \) more complicated.
2. Higher dimensional integral operators not as well-understood.
Challenges:

1. \( \hat{W}_{n,G} \) more complicated.

2. Higher dimensional integral operators not as well-understood.

A Solution

Restrict to minimizing over \( \phi(x) = \phi_1(x_1) \cdots \phi_n(x_n) \) with \( \phi_j \) as in 1-level case (equivalent to minimizing over finite sums).
An Approach

Outline:

◊ Choose $\phi_2, \ldots, \phi_n$ and integrate last $n - 1$ variables to obtain new weight function similar to 1-level weights.

◊ Use 1-level approach to minimize choice of $\phi_1$. 
Example: $W_{2,U}$

Problem

Minimize

\[
\int_{\mathbb{R}^2} \frac{\phi_1(x_1)\phi_2(x_2)W_{2,U}(x) \, dx_1 \, dx_2}{\phi_1(0)\phi_2(0)} = \int_{[-1,1]^2} \frac{\hat{\phi}_1(\xi_1)\hat{\phi}_2(\xi_2)\hat{W}_{2,U}(\xi) \, d\xi_1 \, d\xi_2}{\phi_1(0)\phi_2(0)}
\]

over

$\phi_1, \phi_2$ even, Schwartz, $\phi_1(0), \phi_2(0) > 0$, and

$\text{supp}(\hat{\phi}_1), \text{supp}(\hat{\phi}_2) \subset [-1, 1]$. 

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over

$\phi_1, \phi_2$ even, Schwartz, $\phi_1(0), \phi_2(0) > 0$, and $\text{supp}(\hat{\phi}_1), \text{supp}(\hat{\phi}_2) \subset [-1, 1]$.

1(x) characteristic function of appropriate set. A short computation:

$$W_{2,U}(x) = 1 - \frac{\sin^2(\pi(x_1 - x_2))}{\pi^2(x_1 - x_2)^2}$$

$$\hat{W}_{2,U}(\xi) = \delta(\xi_1)\delta(\xi_2) + \delta(\xi_1 + \xi_2)(|\xi_1| - 1)1(\xi_1).$$
Example: $\mathcal{W}_{2,U}$

For $\phi_2$ arbitrary,

$$
\frac{1}{\phi_2(0)} \int_{\xi_2 \in \mathbb{R}} \hat{\phi}_2(\xi_2) \mathcal{W}_{2,U}(\xi) \, d\xi_2 = \frac{\hat{\phi}_2(0)}{\phi_2(0)} \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\phi_2(0)} (|\xi_1| - 1) \mathbf{1}(\xi_1).
$$
Example: $W_{2,U}$

For $\phi_2$ arbitrary,

$$\frac{1}{\phi_2(0)} \int_{\xi_2 \in \mathbb{R}} \hat{\phi}_2(\xi_2) \overline{W}_{2,U}(\xi) \, d\xi_2 = \frac{\hat{\phi}_2(0)}{\phi_2(0)} \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\phi_2(0)} (|\xi_1| - 1) \mathbf{1}(\xi_1).$$

New Problem:

Normalizing by $\frac{\hat{\phi}_2(0)}{\phi_2(0)}$, minimize

$$\int_{\xi_1 \in \mathbb{R}} \frac{\hat{\phi}_1(\xi_1) \overline{W}(\xi_1)}{\phi_1(0)}$$

over $\phi_1$, where $\overline{W}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\phi_2(0)} (|\xi_1| - 1) \mathbf{1}(\xi_1)$. 
Example: $\mathcal{W}_{2,U}$

\[
\tilde{\mathcal{W}}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\hat{\phi}_2(0)}(|\xi_1| - 1)1(x)(\xi_1) = \delta(\xi_1) + m(\xi_1)
\]

- $\phi_2$ even $\implies m$ is even.
Example: $W_{2,U}$

\[ \widetilde{W}(\xi_1) = \delta(\xi_1) + \frac{\hat{\phi}_2(-\xi_1)}{\hat{\phi}_2(0)}(|\xi_1| - 1)1(x)(\xi_1) = \delta(\xi_1) + m(\xi_1) \]

- $\phi_2$ even $\implies$ $m$ is even.
- 1-level case $\implies$ optimal $\phi_1$ has $\hat{\phi}_1(\xi_1) = (g \ast \mathring{g})(\xi_1)$
  where $g \in L^2\left[-\frac{1}{2}, \frac{1}{2}\right]$ satisfying

\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y)g(y) \, dy. \]

Minimum value is $\frac{1}{\langle 1, g \rangle_{L^2}}$. 

1-level case $\implies$ optimal $\phi_1$ has $\hat{\phi}_1(\xi_1) = (g \ast \mathring{g})(\xi_1)$
Example: \( W_{2,U} \)

\[
1(x) = g(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y)g(y) \, dy.
\]

Solution is found by iteration:
Example: \( W_{2, U} \)

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1(x) = g(x) + \int_{-1/2}^{1/2} m(x - y)g(y) \, dy.
\]

Solution is found by iteration:

- \( K(x, y) := -m(x - y) \).
- \( K_n(x) := \int_{[-1/2, 1/2]^n} K(x, t_1) \cdots K(t_{n-1}, t_n) \, dt_1 \cdots dt_n \).
Example: $W_{2,U}$

\[
1(x) = g(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m(x - y)g(y) \, dy.
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Solution is found by iteration:

- $K(x, y) := -m(x - y)$.
- $K_n(x) := \int_{[-\frac{1}{2}, \frac{1}{2}]^n} K(x, t_1) \cdots K(t_{n-1}, t_n) \, dt_1 \cdots \, dt_n$.
- $g(x) = 1(x) + \sum_{n=1}^{\infty} K_n(x)$.
- $\langle 1, g \rangle_{L^2} = 1 + \sum_{n=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x) \, dx$. 
Example: $\mathcal{W}_{2,U}$

\[
\langle 1, g \rangle_{L^2} = 1 + \sum_{n=1}^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x) \, dx
\]

- Numerical data $\rightarrow \hat{\phi}_2(\xi_2) = (1 - |\xi_2|)\chi_{[-1,1]}(\xi_2)$ is a good choice.
Example: \( \mathcal{W}_{2,U} \)

\[
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- Numerical data \( \to \hat{\phi}_2(\xi) = (1 - |\xi|)\chi_{[-1,1]}(\xi) \) is a good choice.
- Terms of series are nonnegative, so truncate after finitely many terms to get

\[
\frac{\hat{\phi}_2(0)}{\phi_2(0)} \frac{1}{\langle 1, g \rangle_{L^2}} \leq \frac{\hat{\phi}_2(0)}{\phi_2(0)} \left( 1 + \sum_{n=1}^{100} \int_{-\frac{1}{2}}^{\frac{1}{2}} K_n(x) \, dx \right)^{-1} \approx 0.49386.
\]
Numerical Data for \( n = 2 \)

- Truncate at 100 terms with
  \[
  \hat{\phi}_2(\xi_2) = (1 - |\xi_2|) \chi_{[-1,1]}(\xi_2).
  \]
Numerical Data for $n = 2$

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- Series converge uniformly, so integrate term-by-term.
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<table>
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<tr>
<th>Bound</th>
<th>Value</th>
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<tr>
<td>( W_{2,O} )</td>
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<tr>
<td>( W_{2,SO(Even)} )</td>
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<tr>
<td>( W_{2,SO(Odd)} )</td>
<td>0.130293</td>
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<tr>
<td>( W_{2,U} )</td>
<td>0.493856</td>
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<tr>
<td>( W_{2,Sp} )</td>
<td>0.130293</td>
</tr>
</tbody>
</table>
Applications to Order of Vanishing

Assume $\mathcal{F}$ finite, $\Pr(N) :=$ probability that $L(s, f)$ has zero of order $N$ at $s = 1/2$. 
Applications to Order of Vanishing

Assume $F$ finite, $\Pr(N) :=$ probability that $L(s, f)$ has zero of order $N$ at $s = 1/2$. If $F \leftrightarrow G$, then

$$\sum_{N=0}^{\infty} N(N - 1) \Pr(N) \leq \frac{\int_{\mathbb{R}^2} \phi(x, y) W_{2,G}(x, y) \, dx \, dy}{\phi(0, 0)}.$$
Assume $\mathcal{F}$ finite, $\text{Pr}(N) :=$ probability that $L(s, f)$ has zero of order $N$ at $s = 1/2$. If $\mathcal{F} \leftrightarrow \mathcal{G}$, then

$$\sum_{N=0}^{\infty} N(N - 1) \text{Pr}(N) \leq \frac{\int_{\mathbb{R}^2} \phi(x, y) W_{2,G}(x, y) \, dx \, dy}{\phi(0, 0)}.$$

$$\text{Pr}(0) + \text{Pr}(1) \geq \begin{cases} 0.777517 & W_{2,0} \\ 0.506144 & W_{2,U} \\ 0.869707 & W_{2,Sp} \end{cases}.$$
Better results if every $f \in \mathcal{F}$ has same parity functional equation.
Applications to Order of Vanishing

Better results if every $f \in \mathcal{F}$ has same parity functional equation.

\[
\begin{align*}
\Pr(0) & \geq 0.873851 & W_{2,\text{SO(Even)}} \\
\Pr(1) & \geq 0.978285 & W_{2,\text{SO(Odd)}}
\end{align*}
\]
References
References I

