The Ordered Zeckendorf Game

Michael Lucas [1] and Cameron White [2]
in collaboration with Ivan Bortnovskyi, Iana Vranesko, Ren Watson
advised by Steven J. Miller

[1]University of Cambridge, ml2130@cam.ac.uk

[2]Williams College, cjw3@williams.edu

Young Mathematicians' Conference; August 1, 2025

Background

Introduction

Definition

The Fibonacci numbers F_i are defined by $F_1 = 1$, $F_2 = 2$ and $F_i = F_{i-1} + F_{i-2}$.

Introduction

Definition

The Fibonacci numbers F_i are defined by $F_1 = 1$, $F_2 = 2$ and $F_i = F_{i-1} + F_{i-2}$.

Theorem (Zeckendorf's Theorem)

Every positive integer has a unique ***Zeckendorf decomposition****, i.e., it can be uniquely written as a sum of non-adjacent Fibonacci numbers.*

Introduction

Definition

The Fibonacci numbers F_i are defined by $F_1 = 1$, $F_2 = 2$ and $F_i = F_{i-1} + F_{i-2}$.

Theorem (Zeckendorf's Theorem)

Every positive integer has a unique ***Zeckendorf decomposition****, i.e., it can be uniquely written as a sum of non-adjacent Fibonacci numbers.*

Example

$100 = 89 + 8 + 3 = F_{10} + F_5 + F_3.$

Classical Game

Classical Zeckendorf Game

Start with n lots of F_1. Move as follows:

Classical Zeckendorf Game

Start with n lots of F_1. Move as follows:

- Merge: $F_i, F_{i+1} \rightarrow F_{i+2}$

Classical Zeckendorf Game

Start with n lots of F_1. Move as follows:

- Merge: $F_i, F_{i+1} \rightarrow F_{i+2}$
- Split: $F_i, F_i \rightarrow F_{i-2}, F_{i+1}$

Classical Zeckendorf Game

Start with n lots of F_1. Move as follows:

- Merge: $F_i, F_{i+1} \to F_{i+2}$
- Split: $F_i, F_i \to F_{i-2}, F_{i+1}$
- Split twos: $F_2, F_2 \to F_1, F_3$

Classical Zeckendorf Game

Start with n lots of F_1. Move as follows:

- Merge: $F_i, F_{i+1} \rightarrow F_{i+2}$
- Split: $F_i, F_i \rightarrow F_{i-2}, F_{i+1}$
- Split twos: $F_2, F_2 \rightarrow F_1, F_3$
- Merge ones: $F_1, F_1 \rightarrow F_2$

Classical Zeckendorf Game

Start with n lots of F_1. Move as follows:

- Merge: $F_i, F_{i+1} \to F_{i+2}$
- Split: $F_i, F_i \to F_{i-2}, F_{i+1}$
- Split twos: $F_2, F_2 \to F_1, F_3$
- Merge ones: $F_1, F_1 \to F_2$

The last player to move wins.

Example Game

n=5

- $\{1^5\}$

Example Game

n=5

- $\{1^5\}$
- $\{1^3 \wedge 2^1\}$

Example Game

n=5

- $\{1^5\}$
- $\{1^3 \wedge 2^1\}$
- $\{1^2 \wedge 3^1\}$

Example Game

n=5

- $\{1^5\}$
- $\{1^3 \wedge 2^1\}$
- $\{1^2 \wedge 3^1\}$
- $\{2^1 \wedge 3^1\}$

Example Game

n=5

- $\{1^5\}$
- $\{1^3 \wedge 2^1\}$
- $\{1^2 \wedge 3^1\}$
- $\{2^1 \wedge 3^1\}$
- $\{5^1\}$

Example Game

n=5

- $\{1^5\}$
- $\{1^3 \wedge 2^1\}$
- $\{1^2 \wedge 3^1\}$
- $\{2^1 \wedge 3^1\}$
- $\{5^1\}$

Player Two won in 4 moves.

Game Lengths

Question: How long is the game?

Game Lengths

Question: How long is the game?

Theorem

The shortest game is of length $n - Z(n)$, where $Z(n)$ is the number of terms in the Zeckendorf decomposition of n.

Game Lengths

Question: How long is the game?

Theorem

The shortest game is of length $n - Z(n)$, where $Z(n)$ is the number of terms in the Zeckendorf decomposition of n.

Theorem

The length of the longest game is bounded by $3n - Z(n) - IZ(n) + 1$, where $IZ(n)$ is the sum of the indices in the Zeckendorf decomposition of n.

Who Wins?

Question: Who wins the Zeckendorf game?

Who Wins?

Question: Who wins the Zeckendorf game?

Theorem

Player 2 has a winning strategy for $n > 2$.

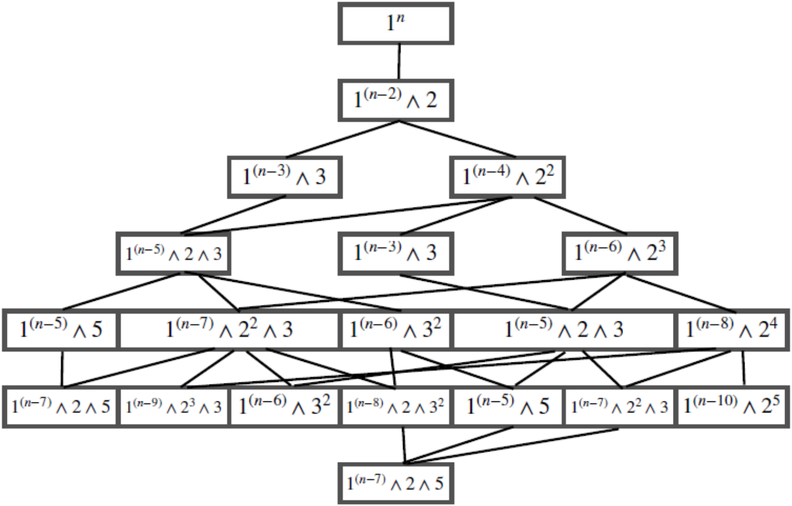
Proof that player 2 wins for $n > 2$

- Prove by contradiction using parity-steal argument.
- Assume player 1 has winning strategy, then show player 2 can steal it.

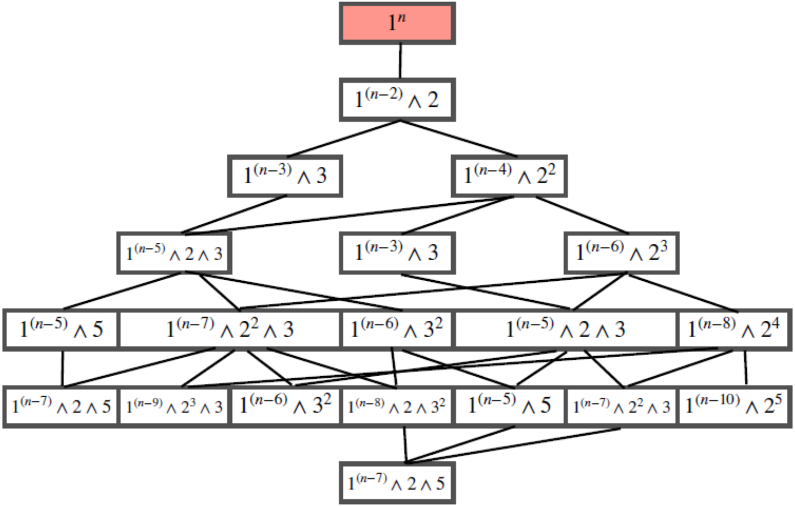
Proof that player 2 wins for $n > 2$

- Prove by contradiction using parity-steal argument.
- Assume player 1 has winning strategy, then show player 2 can steal it.
- Non-constructive; don't need to find winning strategy.

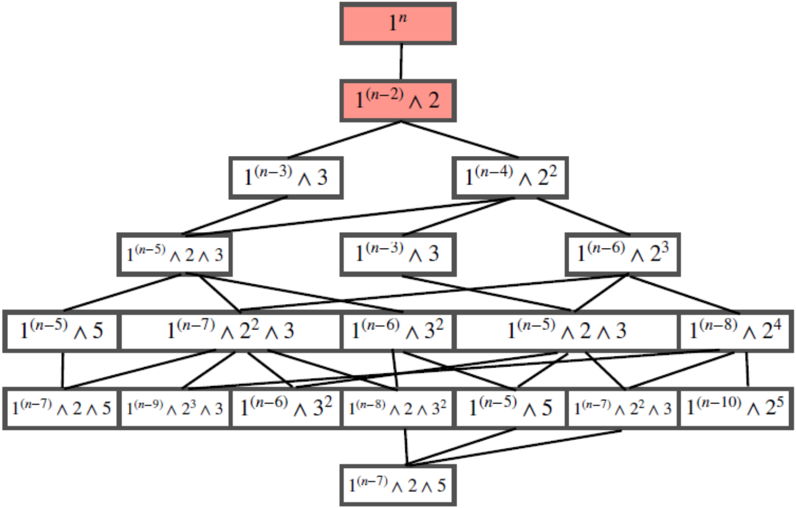
Sketch of Proof for Player Two's Winning Strategy



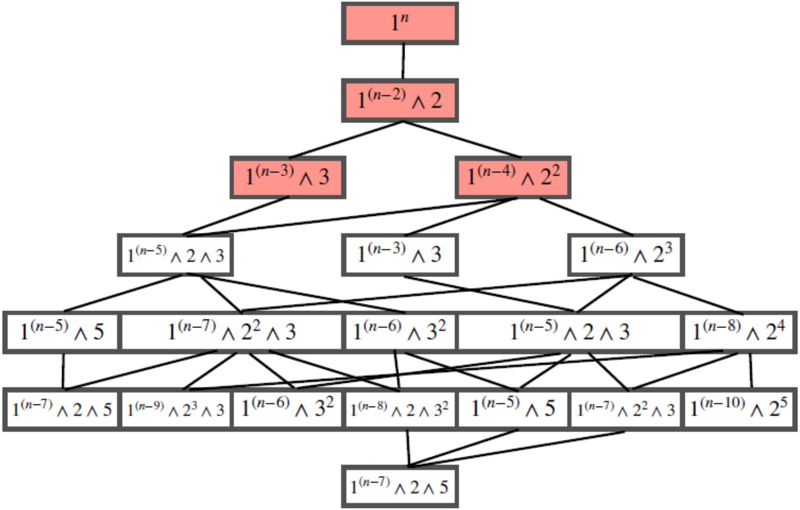
Sketch of Proof for Player Two's Winning Strategy



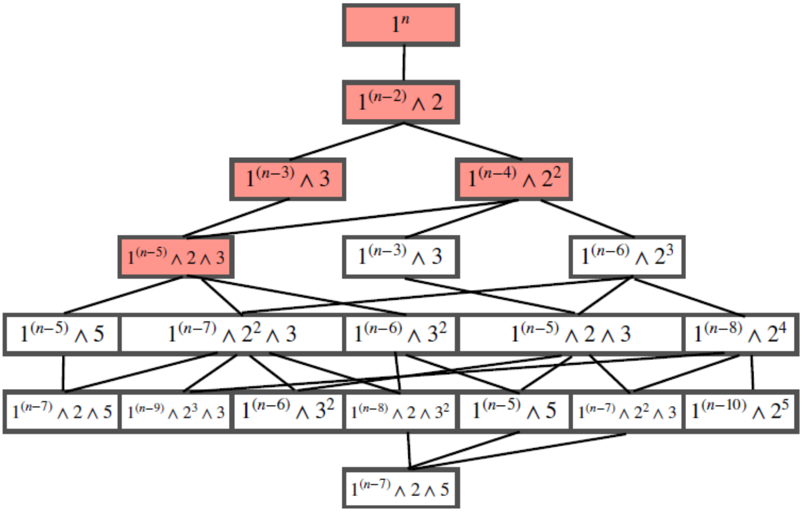
Sketch of Proof for Player Two's Winning Strategy



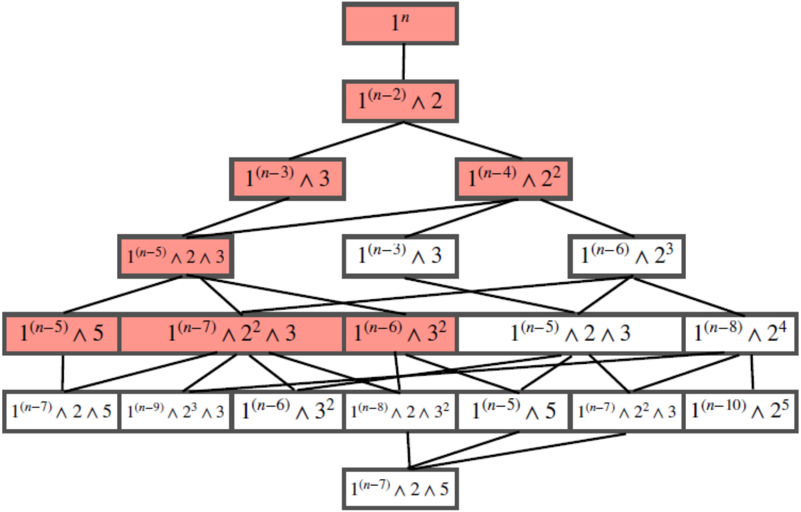
Sketch of Proof for Player Two's Winning Strategy



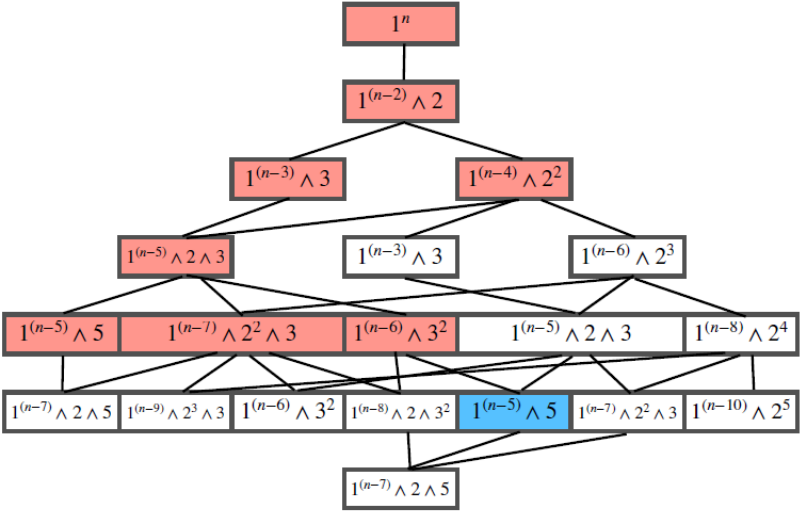
Sketch of Proof for Player Two's Winning Strategy



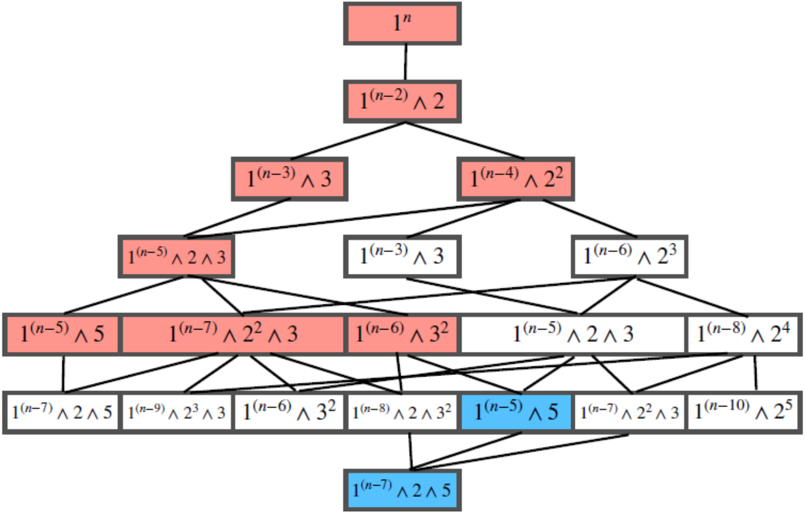
Sketch of Proof for Player Two's Winning Strategy



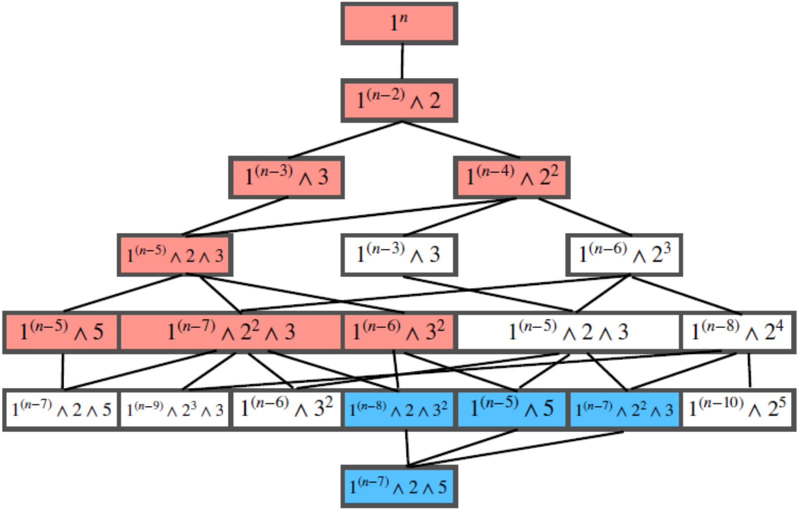
Sketch of Proof for Player Two's Winning Strategy



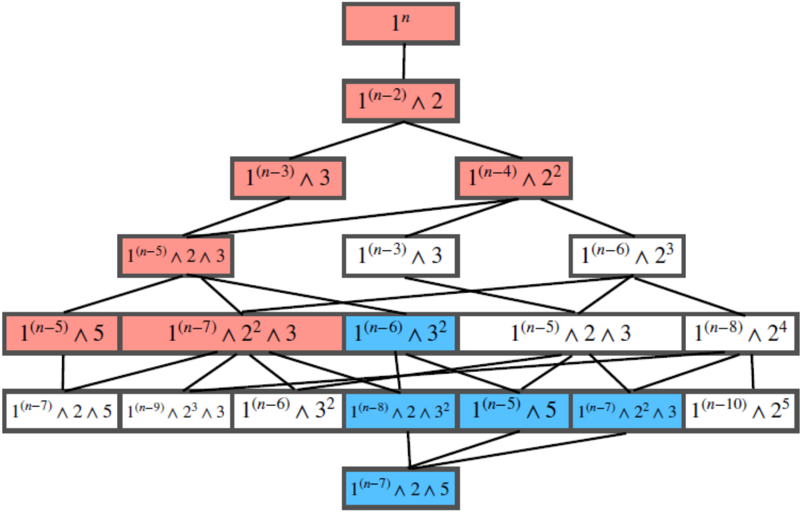
Sketch of Proof for Player Two's Winning Strategy



Sketch of Proof for Player Two's Winning Strategy



Sketch of Proof for Player Two's Winning Strategy



Ordered Game

Motivation

- What happens if we replace $\{F_1, F_1, \ldots F_1\}$ with $(F_1, F_1 \ldots, F_1)$...

Motivation

- What happens if we replace $\{F_1, F_1, \ldots F_1\}$ with $(F_1, F_1 \ldots, F_1)$**...**
- and only allow ourselves to merge/split elements that are **adjacent**?

Ordered Zeckendorf Game Rules

Start with an ordered list of n copies of F_1.

Ordered Zeckendorf Game Rules

Start with an ordered list of n copies of F_1.
Possible moves:

Ordered Zeckendorf Game Rules

Start with an ordered list of n copies of F_1.
Possible moves:

- **Merge:** $(F_i, F_{i+1}) \rightarrow F_{i+2}$
- **Split:** $(F_i, F_i) \rightarrow (F_{i-2}, F_{i+1})$ for $i > 2$
- **Split Twos:** $(F_2, F_2) \rightarrow (F_1, F_3)$
- **Merge Ones:** $(F_1, F_1) \rightarrow F_2$

Ordered Zeckendorf Game Rules

Start with an ordered list of n copies of F_1.
Possible moves:

- **Merge:** $(F_i, F_{i+1}) \to F_{i+2}$
- **Split:** $(F_i, F_i) \to (F_{i-2}, F_{i+1})$ for $i > 2$
- **Split Twos:** $(F_2, F_2) \to (F_1, F_3)$
- **Merge Ones:** $(F_1, F_1) \to F_2$
- **Swap:** $(F_i, F_j) \to (F_j, F_i)$ if $i > j$

Ordered Zeckendorf Game Rules

Start with an ordered list of n copies of F_1.
Possible moves:

- **Merge:** $(F_i, F_{i+1}) \to F_{i+2}$
- **Split:** $(F_i, F_i) \to (F_{i-2}, F_{i+1})$ for $i > 2$
- **Split Twos:** $(F_2, F_2) \to (F_1, F_3)$
- **Merge Ones:** $(F_1, F_1) \to F_2$
- **Swap:** $(F_i, F_j) \to (F_j, F_i)$ if $i > j$
- The last player to move wins.

Example Game

$n = 5$

- $(F_1 \quad F_1 \quad F_1 \quad F_1 \quad F_1)$

Example Game

$n = 5$

- $(F_1 \quad F_1 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_1 \quad F_1 \quad F_1)$

Example Game

$n = 5$

- $(F_1 \quad F_1 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_2 \quad F_1)$

Example Game

$n = 5$

- $(F_1 \quad F_1 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_2 \quad F_1)$
- $(F_1 \quad F_3 \quad F_1)$

Example Game

$n = 5$

- $(F_1 \quad F_1 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_2 \quad F_1)$
- $(F_1 \quad F_3 \quad F_1)$
- $(F_1 \quad F_1 \quad F_3)$

Example Game

$n = 5$

- $(F_1 \quad F_1 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_2 \quad F_1)$
- $(F_1 \quad F_3 \quad F_1)$
- $(F_1 \quad F_1 \quad F_3)$
- $(F_2 \quad F_3)$

Example Game

$n = 5$

- $(F_1 \quad F_1 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_2 \quad F_1)$
- $(F_1 \quad F_3 \quad F_1)$
- $(F_1 \quad F_1 \quad F_3)$
- $(F_2 \quad F_3)$
- (F_4)

Example Game

$n = 5$

- $(F_1 \quad F_1 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_2 \quad F_1)$
- $(F_1 \quad F_3 \quad F_1)$
- $(F_1 \quad F_1 \quad F_3)$
- $(F_2 \quad F_3)$
- (F_4)

Player Two won in 6 moves.

Termination & Length

Game Termination

Question: Does the game always terminate?

Game Termination

Question: Does the game always terminate?

Theorem

The ordered Zeckendorf game always terminates. The final state is the Zeckendorf decomposition of n with the elements placed in ascending order.

Game Termination

Question: Does the game always terminate?

Theorem

The ordered Zeckendorf game always terminates. The final state is the Zeckendorf decomposition of n with the elements placed in ascending order.

- Proof of termination follows when we bound the game length.

Game Termination

Question: Does the game always terminate?

Theorem

The ordered Zeckendorf game always terminates. The final state is the Zeckendorf decomposition of n with the elements placed in ascending order.

- Proof of termination follows when we bound the game length.
- If not in ordered Zeckendorf decomposition, then we can make a move.

Shortest Game Length

Question: How short can the game be?

Shortest Game Length

Question: How short can the game be?

Theorem

The minimal length of the Ordered Zeckendorf Game is $n - Z(n)$, where $Z(n)$ is the number of terms in n's Zeckendorf decomposition.

Shortest Game Length Proof

Two steps:

1. Find a game of duration $n - Z(n)$.

Shortest Game Length Proof

Two steps:

1. Find a game of duration $n - Z(n)$.
2. Show that no other game is shorter than our game.

Shortest Game Length Proof

Two steps:

1. Find a game of duration $n - Z(n)$.
2. Show that no other game is shorter than our game.

- For step 1, if P1 and P2 both merge as far right as possible, then they merge every time.

Shortest Game Length Proof

Two steps:

1. Find a game of duration $n - Z(n)$.
2. Show that no other game is shorter than our game.

- For step 1, if P1 and P2 both merge as far right as possible, then they merge every time.
- For step 2, the **number of elements** the same or decreases by 1. No game can move from n to $Z(n)$ in fewer steps.

Longest game length

Question: What's the longest the game can last?

Theorem (Longest game)

Let $M(n)$ be the maximum game length. Then for $n \geq 2$

$$\frac{n^2}{4} \leq M(n) \leq \frac{n^2}{2}.$$

Proof (Upper Bound)

Claim: $M(n) \leq n^2/2$.

Proof (Upper Bound)

Claim: $M(n) \leq n^2/2$.

- Define the **monovariant**: $f(S) := \sum_{j=1}^{k}(k+1-j)F_{i_j}$ for a game state $S = (F_{i_1}, \ldots, F_{i_k})$.

Proof (Upper Bound)

Claim: $M(n) \leq n^2/2$.

- Define the **monovariant**: $f(S) := \sum_{j=1}^{k}(k+1-j)F_{i_j}$ for a game state $S = (F_{i_1}, \ldots, F_{i_k})$.
- For example, if $S = (5, 3, 8)$, then $f(S) = 3 \cdot 5 + 2 \cdot 3 + 1 \cdot 8 = 29$.

Proof (Upper Bound)

Claim: $M(n) \leq n^2/2$.

- Define the **monovariant**: $f(S) := \sum_{j=1}^{k}(k+1-j)F_{i_j}$ for a game state $S = (F_{i_1}, \ldots, F_{i_k})$.
- For example, if $S = (5, 3, 8)$, then $f(S) = 3 \cdot 5 + 2 \cdot 3 + 1 \cdot 8 = 29$.
- Can check that each move **decreases** f by at least 1.

Proof (Upper Bound)

Claim: $M(n) \leq n^2/2$.

- Define the **monovariant**: $f(S) := \sum_{j=1}^{k}(k+1-j)F_{i_j}$ for a game state $S = (F_{i_1}, \ldots, F_{i_k})$.
- For example, if $S = (5, 3, 8)$, then $f(S) = 3 \cdot 5 + 2 \cdot 3 + 1 \cdot 8 = 29$.
- Can check that each move **decreases** f by at least 1.
- f begins at $n(n+1)/2$ and is bounded below by n, so $M(n) \leq n(n-1)/2$.

Proof (Lower Bound)

Claim: $M(n) \geq n^2/4$.

Proof (Lower Bound)

Claim: $M(n) \geq n^2/4$.

- Inductive strategy as follows:

State	Moves
$\underbrace{(1, 1, 1, \ldots, 1)}_{n}$	0

Proof (Lower Bound)

Claim: $M(n) \geq n^2/4$.

- Inductive strategy as follows:

State	Moves
$\underbrace{(1, 1, 1, \ldots, 1)}_{n}$	0
$(2, 1, \ldots, 1)$	1

Proof (Lower Bound)

Claim: $M(n) \geq n^2/4$.

- Inductive strategy as follows:

State	Moves
$\underbrace{(1,1,1,\ldots,1)}_{n}$	0
$(2,1,\ldots,1)$	1
$(1,\ldots,1,2)$	$n-1$

Proof (Lower Bound)

Claim: $M(n) \geq n^2/4$.

- Inductive strategy as follows:

State	Moves
$\underbrace{(1,1,1,\dots,1)}_{n}$	0
$(2,1,\dots,1)$	1
$(1,\dots,1,2)$	$n-1$
(Zeck decomp of $n-2,2)$	$M(n-2)+n-1$

Proof (Lower Bound)

Claim: $M(n) \geq n^2/4$.

- Inductive strategy as follows:

State	Moves
$\underbrace{(1, 1, 1, \dots, 1)}_{n}$	0
$(2, 1, \dots, 1)$	1
$(1, \dots, 1, 2)$	$n - 1$
(Zeck decomp of $n - 2, 2$)	$M(n - 2) + n - 1$
(Zeck decomp of n)	$\geq M(n - 2) + n - 1$

Proof (Lower Bound)

Claim: $M(n) \geq n^2/4$.

- Inductive strategy as follows:

State	Moves
$\underbrace{(1,1,1,\dots,1)}_{n}$	0
$(2,1,\dots,1)$	1
$(1,\dots,1,2)$	$n-1$
(Zeck decomp of $n-2,2$)	$M(n-2)+n-1$
(Zeck decomp of n)	$\geq M(n-2)+n-1$

- So $M(n) \geq M(n-2)+n-1$.

Proof (Lower Bound)

Claim: $M(n) \geq n^2/4$.

- Inductive strategy as follows:

State	Moves
$\underbrace{(1,1,1,\dots,1)}_{n}$	0
$(2,1,\dots,1)$	1
$(1,\dots,1,2)$	$n-1$
(Zeck decomp of $n-2, 2$)	$M(n-2)+n-1$
(Zeck decomp of n)	$\geq M(n-2)+n-1$

- So $M(n) \geq M(n-2)+n-1$.
- Induction gives $M(n) \geq n^2/4$.

Improvements on bounds of $M(n)$

- Previous bound of $M(n) \geq n^2/4$ can be improved
- A more sophisticated argument gives us...

Improvements on bounds of $M(n)$

- Previous bound of $M(n) \geq n^2/4$ can be improved
- A more sophisticated argument gives us...

Theorem (Our best upper bound so far)

For n sufficiently large, $0.423n^2 \leq M(n) \leq 0.5n^2$.

Strategies

Strategies

- **Question: Who has a winning strategy in the ordered Zeckendorf game?**

Strategies

- **Question: Who has a winning strategy in the ordered Zeckendorf game?**
- For $n < 27$, P2 has a winning strategy iff $n = 2, 9, 10, 11, 13, 19, 26$.

Strategies

- **Question: Who has a winning strategy in the ordered Zeckendorf game?**
- For $n < 27$, P2 has a winning strategy iff $n = 2, 9, 10, 11, 13, 19, 26$.
- Can use parity stealing to make statements about the relationships between winning strategies.

Acknowledgements

Acknowledgements

With thanks to

- NSF Grant DMS2241623
- Finnerty Fund
- Dr. Herchel Smith Fellowship Fund
- Churchill Foundation
- Prof Steven J Miller (our wonderful advisor)

References

- P. Baird-Smith et al., *The Zeckendorf Game*, arXiv:1809.04881, 2018.
- E. Zeckendorf, *Représentation des nombres naturels...*, Bull. Soc. Roy. Sci. Liège, 1972.

Appendix

- Upper bound calculations

Upper Bound Calculations

When merging (F_i, F_{i+1}) to F_{i+2} in the jth position, the weights of all terms to the left of (F_i, F_{i+1}) in S are decreased by 1. The change in the function at (F_i, F_{i+1}) is

$$\begin{aligned} &(k-j)F_{i+2} - (k+1-j)F_i - (k-j)F_{i+1} \\ &= (k-j)(F_{i+2} - F_i - F_{i+1}) - F_i \\ &= -F_i \\ &< 0. \end{aligned}$$

Upper Bound Calculations

When splitting (F_i, F_i) to (F_{i-2}, F_{i+1}), the weights on all other terms stay the same. The change in the function is therefore

$$\begin{aligned}
&(k+1-j)F_{i-2} + (k-j)F_{i+1} - (k+1-j)F_i - (k-j)F_i \\
&= (k-j)(F_{i-2} + F_{i+1} - 2F_i) + F_{i-2} - F_i \\
&= -F_{i-1} \\
&< 0.
\end{aligned}$$

Upper Bound Calculations

When splitting (F_2, F_2) to (F_1, F_3), the weights on all other terms stay the same. The change in the function is therefore

$$
\begin{aligned}
&(k+1-j)F_1 + (k-j)F_3 - (k+1-j)F_2 - (k-j)F_2 \\
&= (k-j)(F_1 + F_3 - 2F_2) + F_1 - F_2 \\
&= -1.
\end{aligned}
$$

Upper Bound Calculations

When splitting ones, the weights of all summands to the left are decreased by 1, and the value of the function at the pair of ones decreases by 1, so the function decreases. When switching $(F_{i_j}, F_{i_{j+1}})$, given that $F_{i_j} > F_{i_{j+1}}$, the weights of all other summands stay the same, so the change in the function is

$$
\begin{aligned}
&(k+1-j)F_{i_{j+1}} + (k-j)F_{i_j} - (k+1-j)F_{i_j} - (k-j)F_{i_{j+1}} \\
&= F_{i_{j+1}} - F_{i_j} \\
&< 0.
\end{aligned}
$$

Upper Bound Calculations

Since f begins at $n(n+1)/2$, decreases by at least 1 per move, and ends at at least n, the number of moves is bounded above by $\binom{n}{2}$. The final configuration must be the ordered Zeckendorf decomposition because any configuration not satisfying the Zeckendorf condition allows further moves.