Number Theory and Probability Group
SMALL 2012

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http://www.williams.edu/Mathematics/sjmiller/
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Random Matrix Theory
(Luo and Triantafillou)
Goals

- We study distributions of structured ensembles. More structure and less averaging leads to new behavior.

- Generalize results on weighted structured ensembles.

- Look at transitions as the amount of structure changes.
Weighted Toeplitz Matrices

For fixed $n$, we consider $N \times N$ weighted Toeplitz matrices, whose entries are iidrv from a $\rho$ with mean 0, variance 1 and finite higher moments and randomly chosen $\epsilon_{ij} \in \{-1, 1\}$ with $\operatorname{Prob}(\epsilon_{ij} = 1) = p$. A weighted Toeplitz matrix is of the form

\[
\begin{pmatrix}
\epsilon_{11} b_0 & \epsilon_{12} b_1 & \cdots & \epsilon_{1(N-1)} b_{N-2} & \epsilon_{1N} b_{N-1} \\
\epsilon_{21} b_1 & \epsilon_{22} b_0 & \cdots & \epsilon_{2(N-1)} b_{N-3} & \epsilon_{2N} b_{N-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\epsilon_{(N-1)1} b_{N-2} & \epsilon_{(N-1)2} b_{N-3} & \cdots & \epsilon_{(N-1)(N-1)} b_0 & \epsilon_{(N-1)N} b_1 \\
\epsilon_{N1} b_{N-1} & \epsilon_{N2} b_{N-2} & \cdots & \epsilon_{N(N-1)} b_1 & \epsilon_{NN} b_0
\end{pmatrix}
\]
**Configurations**

By eigenvalue trace lemma, $k^{\text{th}}$ uncentered moment is

$$
\frac{1}{N^{1+k/2}} \mathbb{E} \left( \sum_{i_1, \ldots, i_k} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1} \right).
$$

In Toeplitz ensembles, all terms on a diagonal are the same, so we relabel $a_{i_j i_{j+1}}$ as $b_{|i_j - i_{j+1}|}$.

**Lemma**

The only terms that contribute to the $2k^{\text{th}}$ moment of the limiting spectral measure are terms where the $b$’s are matched in exactly pairs.
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Sketch of Proof: degree of freedom argument.

Idea: Count contribution from each pairing.

If $a_{ij}a_{ij+1} = a_{ik}a_{ik+1}$, then $|i_j - i_{j+1}| = -|i_k - i_{k+1}|$. 
Examples of Configurations

In top left configuration, \( a_{ij} = a_{jk} \), \( a_{li} = a_{kl} \)
\[ \Rightarrow |i - j| = -|j - k|, \ |l - i| = -|k - l|. \]
Past Weighted Toeplitz results

**Theorem: Beckwith-Miller-Shen (SMALL 2011)**

Consider the weighted ensemble where the \((i, j)\)th and \((j, i)\)th entries of these matrices are multiplied by a randomly chosen \(\epsilon_{ij} \in \{1, -1\}\), with \(\text{Prob}(\epsilon_{ij} = 1) = p\).

For \(p = 1/2\), the limiting spectral measure is the semi-circle. For all other \(p\), the limiting measure has unbounded support, converges to original ensemble’s limiting measure as \(p \to 1\) (weakly convergent, surely if density is even).
**New Results**

**Theorem: LMT ’12**

1. For palindromic Toeplitz matrices, the depression of the contribution depends only on the crossing number.

2. Given any matrix ensemble, when $p = 1/2$, the limiting spectral measure is the semicircle distribution (special dependencies allowed between matrix elements).

3. Any distribution that had unbounded or bounded support before weighting still has unbounded or bounded support after weighting.
1. Depression Dependency on Number of Crossings

**Theorem: Dependency on Number of Crossings**

- For palindromic Toeplitz matrices, depression of contribution depends only on crossing number.

- Dependency does not hold for doubly palindromic Toeplitz, consider 6\textsuperscript{th} moment.
2. Depression of at least \((2\rho - 1)^2\)

**Theorem: Depression at least \((2\rho - 1)^2\)**

- \(x(c)\) is original contribution from the specified configuration, \(2k^{\text{th}}\) moment, and \(e(c)\) number of vertices in crossing pairs.

- Consider "nice" ensembles, i.e., highly palindromic Toeplitz.

- Noncrossing: contrib. at most \((x(c) - 1)(2\rho - 1)^4 + 1\) and at least \((x(c) - 1)(2\rho - 1)^{2k} + 1\).

- Crossing: contribution at most \(x(c)(2\rho - 1)^{e(c)}\) and at least \(x(c)(2\rho - 1)^{2k}\).
3. Interpolation

**Theorem: Interpolation**

- Consider general real symmetric matrix ensemble.

- Noncrossing: contribution to the $2k^{th}$ moment reduced from $x(c)$ to at most $(2\rho - 1)^2(x(c) - 1) + 1$.

- Crossing: contribution to $2k^{th}$ moment reduced from $x(c)$ to at most $(2\rho - 1)^2 x(c)$.

- When $\rho = \frac{1}{2}$, obtain semicircle distribution.
Low-lying zeroes of Maass form $L$-functions (Alpoge)
Katz-Sarnak

- Birch-Swinnerton-Dyer: values of $L$-functions near a “central point” are ridiculously interesting.
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If we replace $\mathbb{Q}$ by $\mathbb{F}_q(t)$, we know the Riemann hypothesis (and more: Deligne). Deligne’s proof uses the action of a “monodromy group.”
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Katz-Sarnak: study the distribution of zeroes (of a family of $L$-functions) near this central point (via hitting them with neutrons), and you shall find out...
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Let’s find out how!
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Goldfeld-Kontorovich (famous number theorist and his former student): “we obtain the low-lying zero densities for... GL(3) Maass forms.”

“The methods presented here are capable of wide generalization... it should be possible to determine the symmetry types of families associated to... GL(n) for any $n \geq 2$. We hope to return to this topic in a future publication.”

**Theorem**

*Research is not for the faint of heart.*
My proof

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- **Main insights:**
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  - **Third:** use Poisson summation.
  - **Fourth:** use Fourier inversion (and Taylor expand copiously).
My proof (continued)

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Thanks!!!!
Gaps in Generalized Zeckendorf Decompositions (Bower, Insoft, Li and Tosteson)
Previous Results

Fibonacci Numbers: \[ F_{n+1} = F_n + F_{n-1}; \]
\[ F_1 = 1, \ F_2 = 2, \ F_3 = 3, \ F_4 = 5, \ldots . \]
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Example:
2012 = 1597 + 377 + 34 + 3 + 1 = \( F_{16} + F_{13} + F_{8} + F_{3} + F_{1} \).
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Lekkerkerkerkerker’s Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in \([F_n, F_{n+1})\) tends to

\[
\frac{n}{\varphi^2 + 1} \approx .276n, \text{ where } \varphi = \frac{1 + \sqrt{5}}{2} \text{ is the golden mean.}
\]
Previous Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

\[ H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \quad n \geq L \]

with \( H_1 = 1, \) \( H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \) \( n < L, \)

coefficients \( c_i \geq 0; \) \( c_1, c_L > 0 \) if \( L \geq 2; \) \( c_1 > 1 \) if \( L = 1. \)
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- Central Limit Type Theorem
Distribution of Gaps

For $H_{i_1} + H_{i_2} + \cdots + H_{i_n}$, the gaps are the differences:

$$i_n - i_{n-1}, \quad i_{n-1} - i_{n-2}, \ldots, \quad i_2 - i_1.$$
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Example: For $H_1 + H_8 + H_{18}$, the gaps are 7 and 10.
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**Definition**

Let $P_n(m)$ be the probability that a gap for a decomposition in $[H_n, H_{n+1})$ is of length $m$. 
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**Definition**

Let $P_n(m)$ be the probability that a gap for a decomposition in $[H_n, H_{n+1})$ is of length $m$.

**Big Question:** What is $P(m) = \lim_{n \to \infty} P_n(m)$?

**Big Question:** What is the distribution of the longest gap?
Theorem

Let \( H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L} \) be a Positive Linear Recurrence Sequence, then, if \( j \geq L \),

\[
P(j) = (\lambda_1 - 1)^2 \left( \frac{a_1}{C_{Lek}} \right) \lambda_1^{-j},
\]

where \( \lambda_1 \) is the largest root of the characteristic polynomial of the recurrence.
Positive Linear Recurrences of Any Length

**Theorem**

Let $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$ be a Positive Linear Recurrence Sequence, then, if $j \geq L$,

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where $\lambda_1$ is the largest root of the characteristic polynomial of the recurrence.

What can we say about the distribution of gaps $< L$ for any PLRS?
Positive Linear Recurrences of Any Length

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Let \( H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L} \) be a positive linear recurrence of length \( L \) where \( c_i \geq 1 \) for all \( 1 \leq i \leq L \). Then \( P(j) = \)

\[
\begin{cases} 
1 - \left( \frac{a_1}{C_{\text{Lek}}} \right) \left( \lambda_1^{-n+2} - \lambda_1^{-n+1} + 2\lambda_1^{-1} + a_1^{-1} - 3 \right) & j = 0 \\
\lambda_1^{-1} \left( \frac{1}{C_{\text{Lek}}} \right) \left( \lambda_1 (1 - 2a_1) + a_1 \right) & j = 1 \\
(\lambda_1 - 1)^2 \left( \frac{a_1}{C_{\text{Lek}}} \right) \lambda_1^{-j} & j \geq 2
\end{cases}
\]
Proof Set Up of case $j \geq 2$

**Theorem**

If $j \geq 2$, then $P(j) = (\lambda_1 - 1)^2 \left( \frac{a_1}{C_{\text{Lek}}} \right) \lambda_1^{-j}$.

Let $X_{i,i+j}(n) = \# \{ m \in [H_n, H_{n+1}) : \text{decomposition of } m \text{ includes } H_i, H_{i+j}, \text{ but not } H_q \text{ for } i < q < i+j \}$. 
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Let $Y(n) = \text{total number of gaps in decompositions for integers in } [H_n, H_{n+1})$. 
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Let $Y(n) = \text{total number of gaps in decompositions for integers in } [H_n, H_{n+1})$.

$$P(j) = \lim_{n \to \infty} \frac{1}{Y(n)} \sum_{i=1}^{n-j} X_{i,i+j}(n).$$
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Theorem

If $j \geq 2$, then $P(j) = (\lambda_1 - 1)^2 \left( \frac{a_1}{C_{\text{Lek}}} \right) \lambda_1^{-j}$.

Let $X_{i,i+j}(n) = \# \{ m \in [H_n, H_{n+1}) : \text{decomposition of } m \text{ includes } H_i, H_{i+j}, \text{ but not } H_q \text{ for } i < q < i + j \}$.

Let $Y(n) = \text{total number of gaps in decompositions for integers in } [H_n, H_{n+1})$.

$$P(j) = \lim_{n \to \infty} \frac{1}{Y(n)} \sum_{i=1}^{n-j} X_{i,i+j}(n).$$

Generalized Lekkerkerker:

$\Rightarrow Y(n) \sim (C_{\text{Lek}}n + d)(H_{n+1} - H_n)$. 

A Quick Counting Lesson: How do we count $X_{i,i+j}$?

We need to see the number of legal decompositions with a gap of length $j$.

Can count how many legal decompositions exist to the left and right of the gap.

**Lemma**

Let $H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L}$ be a Positive Linear Recurrence Sequence, then the number of legal decompositions which contain $H_m$ as the largest summand is $H_{m+1} - H_m$. 
Calculating $X_{i,i+j}$

**Theorem**

If $j \geq 2$, then $P(j) = (\lambda_1 - 1)^2 \left( \frac{a_1}{C_{lek}} \right) \lambda_1^{-j}$.

In the interval $[H_n, H_{n+1})$:  
How many decompositions contain a gap from $H_i$ to $H_{i+j}$?
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**Left:** For the indices less than $i$: $H_{i+1} - H_i$ choices.
Calculating $X_{i,j+i}$

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If $j \geq 2$, then $P(j) = (\lambda_1 - 1)^2 \left( \frac{a_1}{C_{lek}} \right) \lambda_1^{-j}$.

In the interval $[H_n, H_{n+1})$:
How many decompositions contain a gap from $H_i$ to $H_{i+j}$?

**Left:** For the indices less than $i$: $H_{i+1} - H_i$ choices.

**Right:** For the indices greater than $i + j$:
$H_{n-i-j+2} - H_{n-i-j+1} - (H_{n-i-j+1} - H_{n-i-j})$ choices.
Calculating $X_{i,j+j}$

**Theorem**

If $j \geq 2$, then $P(j) = (\lambda_1 - 1)^2 \left( \frac{a_1}{\mathcal{C}_{\text{Lek}}} \right) \lambda_1^{-j}$.

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**Right:** For the indices greater than $i + j$:
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So $X_{i,i+j}(n) = \text{Left} \times \text{Right} = (H_{i+1} - H_i)(H_{n-i-j+2} - H_{n-i-j+1} - (H_{n-i-j+1} - H_{n-i-j}))$. 

Say, 

\[ (H_{i+1} - H_i)(H_{n-i-j+2} - H_{n-i-j+1} - (H_{n-i-j+1} - H_{n-i-j})) \]
Final Steps of the Proof

For sufficiently large $n$, $H_n \approx a_1 \lambda_1^n$. 
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Then with some algebra...

$$P(j) = (\lambda_1 - 1)^2 \left( \frac{a_1}{C_{Lek}} \right) \lambda_1^{-j}.$$
Gaps in Generalized Zeckendorf Decompositions
(Bower, Insoft, Li and Tosteson)
Longest Gap:

**Big question:** Given a random number $x$ in the interval $[F_n, F_{n+1})$, what is the probability that $x$ has **longest gap** equal to $r$?
Our Method

What we do:
Our Method

What we do:

- Recast the problem through combinatorics.
Our Method

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- Recast the problem through combinatorics.
- Obtain generating functions!
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What we do:

- Recast the problem through combinatorics.
- Obtain generating functions!
- Get the important relationships.
- Analyze limiting behavior.
Cumulative Distribution Function

Pick $x$ randomly from the interval $[F_n, F_{n+1})$. We prove explicitly the cumulative distribution of $x$’s longest gap.
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**Theorem**

Let $r = \phi^2 / (\phi^2 + 1)$. Set $f(n) = \log rn / \log \phi + u$ for some fixed $u \in \mathbb{Z}$. As $n \to \infty$, the probability that $x \in [F_n, F_{n+1})$ has longest gap at most $f(n)$ converges to

$$
P(L(x) \leq f(n)) = e^{e(1-u) \log \phi + \{f(n)\}}.
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$$

**Immediate Corollary:** If $f(n, u)$ grows any slower or faster than $\log n / \log \phi$, then $P(L(x) \leq f(n))$ goes to 0 or 1 respectively.
Mean and Variance

We can use the CDF to determine the regular distribution function, and particularly the mean and variance.
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\[ P(u) = \mathbb{P}\left( L(x) \leq \frac{\log\left( \frac{\phi^2}{\phi^2+1} n \right)}{\log \phi} + u \right), \]
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then the distribution of the longest gap is approximately

\[ \frac{d}{du} P(u). \]

The mean is given by

\[ \mu = \int_{-\infty}^{\infty} u \frac{d}{du} P(u) du. \]

The variance follows similarly.
So the mean is about

\[ \mu = \frac{\log \left( \frac{\phi^2}{\phi^2 + 1} \right)}{\log \phi} + \int_{-\infty}^{\infty} e^{-e^{(1-u)\log \phi}} e^{(1-u)\log \phi} \log \phi \, du. \]
Mean and Variance

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\[
\mu = \frac{\log \left( \frac{\phi^2}{\phi^2 + 1} \right)}{\log \phi} + \int_{-\infty}^{\infty} e^{-e^{(1-u)\log \phi}} e^{(1-u)\log \phi} \log \phi \, du.
\]

\[
= \frac{1}{\log \phi} \left( \log \left( \frac{\phi^2}{\phi^2 + 1} \right) - \int_{0}^{\infty} \log(w) \cdot e^{-w} \, dw \right).
\]
So the mean is about

\[ \mu = \log \left( \frac{\phi^2}{\phi^2 + 1} \right) + \int_{-\infty}^{\infty} e^{-e^{(1-u) \log \phi}} e^{(1-u) \log \phi} \log \phi \, du. \]

\[ = \frac{1}{\log \phi} \left( \log \left( \frac{\phi^2}{\phi^2 + 1} \right) - \int_{0}^{\infty} \log(w) \cdot e^{-w} \, dw \right). \]

**Theorem**

*In the continuous approximation, the mean is*

\[ \log \left( \frac{\phi^2 \, n}{\phi^2 + 1} \right) \frac{1}{\log \phi} - \gamma. \]
Positive Linear Recurrence Sequences

This method can be greatly generalized to **Positive Linear Recurrence Sequences** (linear recurrences with non-negative coefficients). WLOG:

\[ H_{n+1} = c_1 H_{n-(t_1=0)} + c_2 H_{n-t_2} + \cdots + c_L H_{n-t_L}. \]
Positive Linear Recurrence Sequences

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\[ H_{n+1} = c_1 H_{n-(t_1=0)} + c_2 H_{n-t_2} + \cdots + c_L H_{n-t_L}. \]

**Theorem (Zeckendorf’s Theorem for PLRS recurrences)**

Any \( b \in \mathbb{N} \) has a unique legal decomposition into sums of \( H_n \), \( b = a_1 H_{i_1} + \cdots + a_{i_k} H_{i_k} \).

Here legal reduces to non-adjacency of summands in the Fibonacci case.
Generating Function for PLRS

The number of \( b \in [H_n, H_{n+1}) \), with longest gap \(< f \) is the coefficient of \( x^{n-s} \) in the generating function:

\[
\sum_{k \geq 0} \left[ \left( (c_1 - 1)x^{t_1} + \cdots + (c_L - 1)x^{t_L} \right) \left( \frac{x^{s+1} - x^f}{1 - x} \right) + x^{t_1} \left( \frac{x^{s+t_2-t_1+1} - x^f}{1 - x} \right) + \cdots + x^{t_{L-1}} \left( \frac{x^{s+t_L-t_{L-1}} + 1 - x^f}{1 - x} \right) \right]^k \times \frac{1}{1 - x} \left( c_1 - 1 + c_2x^{t_2} + \cdots + c_Lx^{t_L} \right)
\]

A geometric series!
Phase Transitions
(Hogan)
Past Results

- **Martin and O’Bryant, 2006**: Positive percentage of sets are MSTD (more sum than difference) when sets chosen with uniform probability. Surprising: \[ x + y = y + x \] but \[ x - y \] usually not \[ y - x \].
Past Results

- **Martin and O’Bryant, 2006**: Positive percentage of sets are MSTD (more sum than difference) when sets chosen with uniform probability. Surprising: $x + y = y + x$ but $x - y$ usually not $y - x$.

- **Iyer, Lazarev, Miller, Zhang, 2011**: Generalized results above to an arbitrary number of summands.
Phase Transition

**Theorem (Hegarty-Miller):** \( S = |A + A|, \ D = |A - A|, \)
g\( (x) := 2 \left( e^{-x} - (1-x) \right) \). Take \( k \in \{0, \ldots, N - 1\} \) with probability \( p(N) \to 0 \), then if

- \( p(N) = o(N^{-1/2}) \):
  \[
  S \sim \frac{(N \cdot p(N))^2}{2} \quad \text{and} \quad D \sim 2S \sim (N \cdot p(N))^2.
  \]

- \( p(N) = c \cdot N^{-1/2} \):
  \[
  S \sim g\left(\frac{c^2}{2}\right)N \quad \text{and} \quad D \sim g(c^2)N.
  \]

- \( N^{-1/2} = o(p(N)) \): Let \( S^c := (2N + 1) - S \), \( D^c := (2N + 1) - D \). Then
  \[
  S^c \sim 2 \cdot D^c \sim 4/p(N)^2.
  \]
Generalized Sumsets

Definition

For $s > d$, consider the Generalized Sumset $A_{s,d} = A + \cdots + A - A - \cdots - A$ where we have $s$ plus signs and $d$ minus signs. Let $h = s + d$. 
Generalized Sumsets

Definition

For $s > d$, consider the Generalized Sumset $A_{s,d} = A + \cdots + A - A - \cdots - A$ where we have $s$ plus signs and $d$ minus signs. Let $h = s + d$.

We want to study the size of this set as a function of $s, d$, and $\delta$ for $p(N) = cN^{-\delta}$.

Our goal: Extend the results of Hegarty-Miller to the case of Generalized Sumsets and determine where the phase transition occurs for $h > 2$. 
Cases for $\delta$

To answer, we must consider three different cases for $\delta$. 
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- **Fast Decay:** $\delta > \frac{h-1}{h}$.
- **Critical Decay:** $\delta = \frac{h-1}{h}$.
- **Slow Decay:** $\delta < \frac{h-1}{h}$. 
To answer, we must consider three different cases for $\delta$.

- **Fast Decay**: $\delta > \frac{h-1}{h}$.

- **Critical Decay**: $\delta = \frac{h-1}{h}$.

- **Slow Decay**: $\delta < \frac{h-1}{h}$.

These three cases correspond to the speed at which the probability of choosing elements decays to 0.
Fast Decay

- For $\delta > \frac{\delta - 1}{\delta}$, the set with more differences is larger 100% of the time.
Fast Decay

- For \( \delta > \frac{h^{-1}}{h} \), the set with more differences is larger 100% of the time.

- Compute the number of distinct \( h \)-tuples.
Fast Decay

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- Compute the number of distinct $h$-tuples.

- For $h$-tuples $a = (a_1, \cdots, a_h)$, $b = (b_1, \cdots, b_h)$, define indicator variable $Y_{a,b}$ to be 1 when $a$ and $b$ generate the same element.
Fast Decay

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- Bound the expected value and variance of the sum of these indicator variables.
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- Bound the expected value and variance of the sum of these indicator variables.

- Chebyshev’s Inequality: $\text{Prob}(|X - \mu_X| \geq k\sigma_X) \leq \frac{1}{k^2}$. 


Sumsets vs Sumdifferences (Vissuet)
In the Red Corner:

- Sumset := $S + S = \{xy : x, y \in S\}$
In the Red Corner:

- **Sumset**: \( S + S = \{ xy : x, y \in S \} \)
- **Underdog**
In the Red Corner:

- **Sumset**: $S + S = \{ xy : x, y \in S \}$

- **Underdog**

- **Weakness**: For abelian groups we have that $xy = yx$
In the Blue Corner:

- Sumdifference \( S - S = \{ xy^{-1} : x, y \in S \} \)
In the Blue Corner:

- Sumdifference \( := S - S = \{ xy^{-1} \mid x, y \in S \} \)

- Reigning Champion
in the Blue Corner:

- **Sumdifference**: \( S - S = \{ xy^{-1} : x, y \in S \} \)

- Reigning Champion

- Weakness: \( x \cdot x^{-1} \) is the identity \( \forall x \in S \).
Rules of the match

- The winner of each bout is the one who usually has the bigger cardinality.
Rules of the match

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- The match will consist of 3 different venues and will be best 2 out of 3.
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- Question that needs to be asked before we start:
Rules of the match

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- The match will consist of 3 different venues and will be best 2 out of 3.

- Question that needs to be asked before we start:

  Are You Ready To Rumbblllllee?
First Venue
The Match

- Sumsets terribly loses the first 13 of $\aleph_0$ rounds because there does not exist a subset of $[0, 14]$ such that $|S + S| > |S - S|$.
The Match

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- However, in Round 14, with the help of Conway, Sumset gets a jab in with the set: 
  \{0, 2, 3, 4, 7, 11, 12, 14\}. 
The Match

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- However, In Round 14, with the help of Conway, Sumset gets a jab in with the set: 
  \{0, 2, 3, 4, 7, 11, 12, 14\}.

- With the help of Coaches Martin and O’Bryant, Sumsets realizes that if he wants to win it has to concentrate on having a better "fringe."
Key Idea: In the $\mathbb{Z}$ case, *fringe matters most*, middle sums and differences are present with high probability.
**Sumset's tactics**

- **Key Idea:** In the $\mathbb{Z}$ case, **fringe matters most**, middle sums and differences are present with high probability.

- **If we choose the "fringe" of S cleverly, the middle of S will become largely irrelevant.** - Martin and O'Bryant's inspiring words.
First results

- So with the help of Martin and O’Bryant, Sumsets learns that there exists a positive percentage of subsets that are sum dominated.
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- Sadly the percentage of sum dominated sets is estimated to be .00045%. 
First results

- So with the help of Martin and O’Bryant, Sumsets learns that there exists a positive percentage of subsets that are sum dominated.

- Sadly the percentage of sum dominated sets is estimated to be $0.00045\%$.

- Sumset loses the first bout.
The Second Venue
The Second Venue
The Match Round 1

- Sumset faces some difficulties in the $\mathbb{Z}/n\mathbb{Z}$ venue because there is no fringe.
The Match Round 1

- Sumset faces some difficulties in the $\mathbb{Z}/n\mathbb{Z}$ venue because there is no fringe.

- Luckily for Sumset, because carousel go round and round there are know many ways to write each element.
The Match Round 1

- Sumset faces some difficulties in the $\mathbb{Z}/n\mathbb{Z}$ venue because there is no fringe.

- Luckily for Sumset, because carousel go round and round there are know many ways to write each element.

**Theorem**

If we let $S$ be a random subset of $\mathbb{Z}/n\mathbb{Z}$ (if $\alpha \in D_{2n}$ then $\mathbb{P}(\alpha \in S) = 1/2$) then

$$\lim_{n \to \infty} \mathbb{P}(|S \cdot S| = |S \cdot S^{-1}|) = 1.$$
Theorem

Similar results hold for Abelian Groups, Dihedral Groups, and Semi-direct Products of cyclic groups.
The Match Round 2

**Theorem**

*Similar results hold for Abelian Groups, Dihedral Groups, and Semi-direct Products of cyclic groups.*

Although for any finite $n$, there are more subsets $S$ of $D_{2n}$ such that $|S + S| > |S - S|$, the judges still decided to call the boat a draw due to limiting behavior.
The Third Venue
The Third Venue
The free group was Sumset’s strength, it is no longer in an abelian group.
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Not only that, but Sumdifference’s weakness is still there ($x \cdot x^{-1}$ is the identity for all $x \in S$).
The Match

- The free group was Sumset’s strength, it is no longer in an abelian group.

- Not only that, but Sumdifference’s weakness is still there \((x \cdot x^{-1} \text{ is the identity for all } x \in S)\).

- The match was very one sided.
Ping Pong

**Theorem (Free Group)**

If we let $\langle a, b \rangle_1$ be all words up to length $l$ and $S \subseteq \langle a, b \rangle_1$, then as $l$ goes to infinity we have that:

$$\mathbb{P}( |S \cdot S| \geq |S \cdot S^{-1}| ) = 1.$$
TAKE THAT SUM DIFFERENCE!