Painlevé VI and Tracy-Widom Distributions in Random Graphs, Random Matrix Theory and Number Theory

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Introduction

Goals of the Talk

- Painlevé VI in random matrix theory and number theory.
- Tracy-Widom distributions in d-regular graphs.

Random graphs: joint with Tim Novikoff, Anthony Sabelli.

RMT / Number Theory: joint with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith.

L-functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Measures of Spacings: *n*-Level Density and Families

Let g_i be even Schwartz functions whose Fourier Transform is compactly supported, $L(s,f)$ an L-function with zeros $\frac{1}{2} + \mathrm{i}\gamma_f$ and conductor Q_f:

$$D_{n,f}(g) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\gamma_{f,j_1}\frac{\log Q_f}{2\pi}\right)\cdots g_n\left(\gamma_{f,j_n}\frac{\log Q_f}{2\pi}\right)$$

- Properties of *n*-level density:
  - ⋄ Individual zeros contribute in limit
  - ⋄ Most of contribution is from low zeros
  - ⋄ Average over similar *L*-functions (family)

n-Level Density

n-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of L-functions ordered by conductors, g_k an even Schwartz function: $D_{n,\mathcal{F}}(g) =$

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right)\cdots g_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N\to\infty$, n-level density converges to

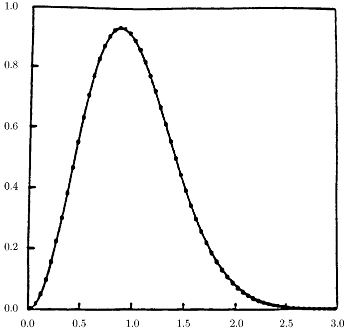
$$\int g(\overrightarrow{x})\rho_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{x})d\overrightarrow{x} \;=\; \int \widehat{g}(\overrightarrow{u})\widehat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\overrightarrow{u})d\overrightarrow{u}.$$

Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Theory and Models

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko) versus RMT prediction.

Orthogonal Random Matrix Models

RMT: $SO(2N)$: $2N$ eigenvalues in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \ \propto \ \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

Independent Model:

$$\mathcal{A}_{2N,2r} \ = \ \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interaction Model: Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$: $1 \ \leq \ j,k \ \leq \ N-r$:

$$d\varepsilon_{2r}(\theta) \ \propto \ \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Indep):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Comparing the RMT Models

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right)$$

$$= \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

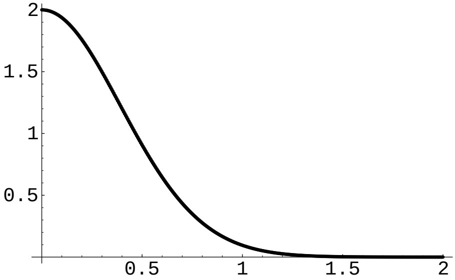
where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd.} \end{cases}$$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

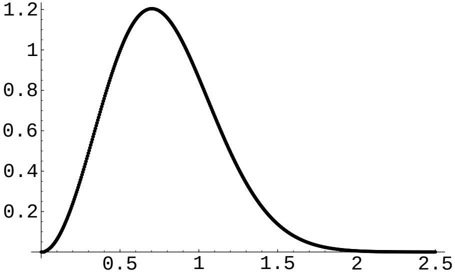
Data

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

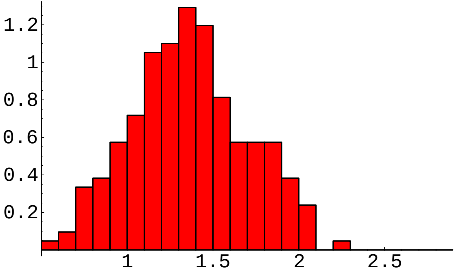


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

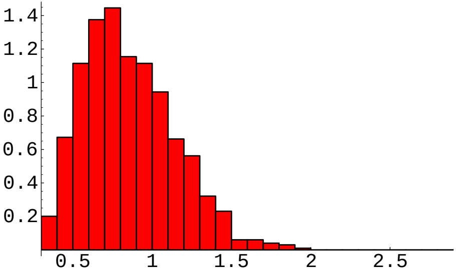


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	**23 Rank 4 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

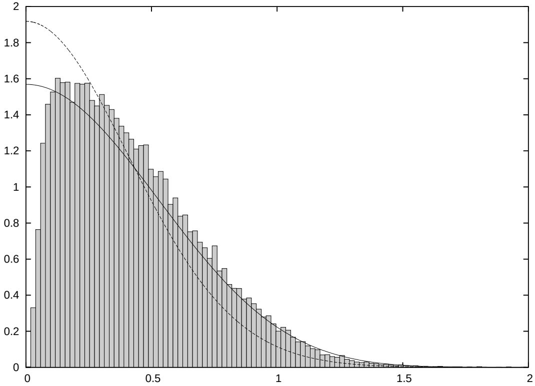
- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	64 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

New Model for Finite Conductors

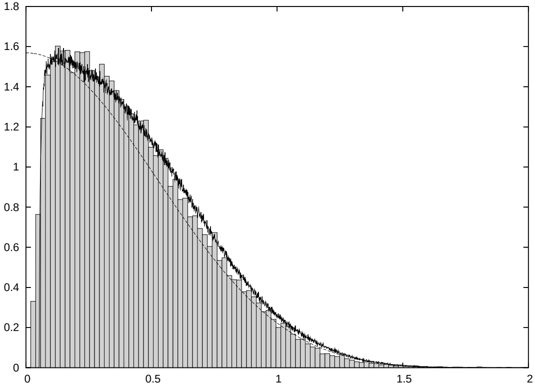
- Replace conductor N with $N_{\text{effective}}$.
  - ◇ Arithmetic info, predict with L-function Ratios Conj.
  - ◇ Do the number theory computation.
- Discretize Jacobi ensembles.
  - ◇ $L(1/2, E)$ discretized.
  - ◇ Study matrices in $\mathrm{SO}(2N_{\mathit{eff}})$ with $|\Lambda_A(1)| \geq ce^N$.
- Painlevé VI differential equation solver.
  - ◇ Use explicit formulas for densities of Jacobi ensembles.
  - ◇ Key input: Selberg-Aomoto integral for initial conditions.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



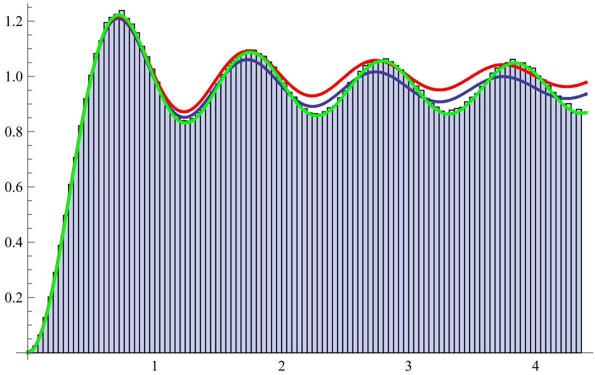
Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).

$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.

$\diamond$ Green: all lower order terms.

Conjectures for
d-Regular Graphs

Known and conjectured results for λ_2

- (Alon-Boppana, Burger, Serre) $\{G_m\}$ family of finite connected d-regular graphs, $\lim_{m\to\infty} |G_m| = \infty$:

$$\liminf_{m\to\infty} \lambda_2(G_m) \geq 2\sqrt{d-1}$$

- As $|G| \to \infty$, for $d \geq 3$ and any $\epsilon > 0$, “most" d-regular graphs G have

$$\lambda_2(G) \leq 2\sqrt{d-1} + \epsilon$$

(conjectured by Alon, proved for many families by Friedman).

Questions

For a family of d-regular graphs:

- What is the *distribution* of λ_2?
- What *percent* of the graphs are Ramanujan?

$\lambda(G) = \max\left(\lambda_+(G), \lambda_-(G)\right)$, where $\lambda_\pm(G)$ are largest non-trivial positive (negative) eigenvalues. If bipartite $\lambda_-(G) = -\lambda_+(G)$. If connected $\lambda_2(G) = \lambda_+(G)$.

Families Investigated (N even)

- $\mathcal{CI}_{N,d}$: d-regular connected graphs generated by choosing d perfect matchings.
- $\mathcal{SCI}_{N,d}$: subset of $\mathcal{CI}_{N,d}$ that are simple.
- $\mathcal{CB}_{N,d}$: d-regular connected bipartite graphs generated by choosing d permutations.
- $\mathcal{SCB}_{N,d}$: subset of $\mathcal{CB}_{N,d}$ that are simple.

Tracy-Widom Distribution

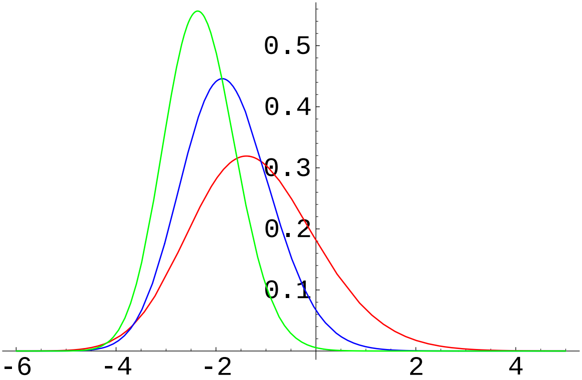
Limiting distribution of the normalized largest eigenvalues for ensembles of matrices: GOE ($\beta = 1$), GUE ($\beta = 2$), GSE ($\beta = 4$)

Applications

- Length of largest increasing subsequence of random permutations.
- Largest principle component of covariances matrices.
- Young tableaux, random tilings, queuing theory, superconductors....

Tracy-Widom Plots

Plots of the three Tracy-Widom distributions: $f_1(s)$ is red, $f_2(s)$ is blue and $f_4(s)$ is green.



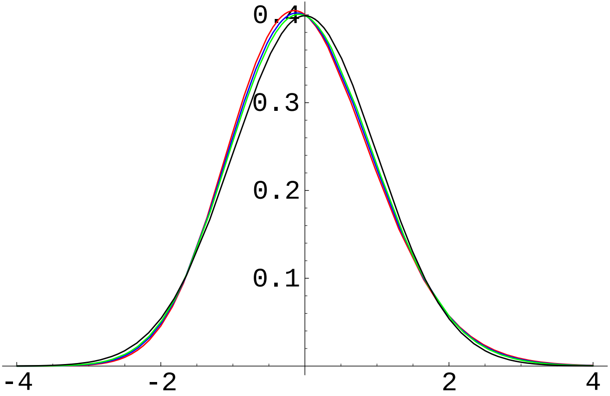
Tracy-Widom Distributions

Parameters for the Tracy-Widom distributions. F_β is the cumulative distribution function for f_β, and $F_\beta(\mu_\beta)$ is the mass of f_β to the left of its mean.

	Mean μ	Std Dev σ	$F_\beta(\mu_\beta)$
TW($\beta = 1$)	-1.21	1.268	0.5197
TW($\beta = 2$)	-1.77	0.902	0.5150
TW($\beta = 4$)	-2.31	0.720	0.5111
Std Normal	0.00	1.000	0.5000

Normalized Tracy-Widom Plots

Plots normalized to have mean 0 and variance 1: $f_1^{\text{norm}}(s)$ is red, $f_2^{\text{norm}}(s)$ is blue, $f_4^{\text{norm}}(s)$ is green, standard normal is black.



Conjectures

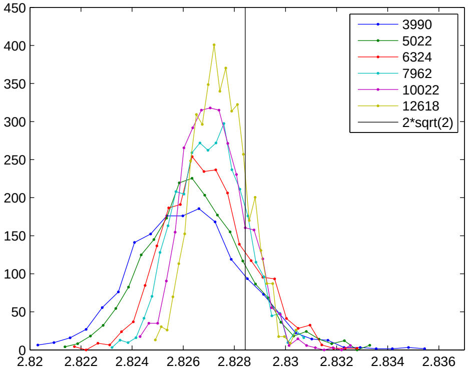
Conjectures

- The distribution of $\lambda_{\pm}(G)$ converges to the $\beta = 1$ Tracy-Widom distribution as $N \to \infty$ in all studied families.
- For non-bipartite families, $\lambda_{\pm}(G)$ are independent.
- The percent of the graphs that are Ramanujan approaches 52% as $N \to \infty$ (resp., 27%) in bipartite (resp., non-bipartite) families.

Evidence weaker for $\mathcal{CB}_{N,d}$ (d-regular connected bipartite graphs, not necessarily simple).

Distribution of $\lambda_+(G)$

Distribution of $\lambda_+(G)$ for 1000 graphs randomly chosen from $\mathcal{CI}_{N,3}$ for various N (vertical line is $2\sqrt{2}$).



Statistical evidence for conjectures

- Well-modeled by Tracy-Widom with $\beta = 1$.
- Means approach $2\sqrt{d-1}$ according to power law.
- Variance approach 0 according to power law.
- Comparing the exponents of the power laws, see the number of standard deviations that $2\sqrt{d-1}$ falls to the right of the mean goes to 0 as $N \to \infty$.
- $\lambda_\pm(G)$ appear independent in non-bipartite families.
- As $N \to \infty$ the probability that a graph is Ramanujan is the mass of the Tracy-Widom distribution to the left of its mean (52%) if bipartite (27% otherwise).

Power law exponents of means and standard deviations

- Means: $\mu_{\mathcal{F}_{N,d}} \approx 2\sqrt{d-1} - c_{\mu,N,d} N^{m(\mathcal{F}_{N,d})}$
- Standard Deviations: $\sigma_{\mathcal{F}_{N,d}} \approx c_{\sigma,N,d} N^{s(\mathcal{F}_{N,d})}$
- Thus $2\sqrt{d-1} \approx \mu_{\mathcal{F}_{N,d}} + \frac{c_{\mu,N,d}}{c_{\sigma,N,d}} N^{m(\mathcal{F}_{N,d})-s(\mathcal{F}_{N,d})} \sigma_{\mathcal{F}_{N,d}}$

Ramanujan Threshold

As $N \to \infty$, if $m(\mathcal{F}_{N,d}) < s(\mathcal{F}_{N,d})$ then $2\sqrt{d-1}$ falls zero standard deviations to the right of the mean.

3-Regular Graphs

Experiments: Comparisons with Tracy-Widom Distribution

- Each set is 1000 random 3-regular graphs from $\mathcal{CI}_{N,3}$ normalized to have mean 0 and variance 1.
- 19 degrees of freedom, critical values 30.1435 ($\alpha = .05$) and 36.1908 ($\alpha = .01$).
- Only showing subset of data.

χ^2-Tets of $\lambda_+(G)$ for $\mathcal{CI}_{N,3}$ versus Tracy-Widom Distributions

Critical values: 30.1 ($\alpha = .05$), 36.2 ($\alpha = .01$).

N	$\mathrm{TW}_1^{\mathrm{norm}}$	$\mathrm{TW}_2^{\mathrm{norm}}$	$\mathrm{TW}_4^{\mathrm{norm}}$	$\mathrm{N}(0,1)$
26	52.4	43.7	36.8	30.3
100	72.1	41.3	28.9	13.2
796	3.7	4.9	7.0	19.3
3168	17.4	19.6	24.0	61.3
6324	20.8	19.8	21.4	28.6
12618	9.9	9.3	10.6	17.2
20000	37.4	41.1	41.4	71.2
mean (all)	32.5	27.2	24.9	49.1
median (all)	20.0	19.1	18.0	25.2
mean (last 10)	22.3	24.9	29.1	66.7
median (last 10)	21.2	21.8	22.2	64.5

χ^2-Tests of $\lambda_+(G)$ against $\beta = 1$ Tracy-Widom

Critical values: 30.1 ($\alpha = .05$), 36.2 ($\alpha = .01$).

N	$\mathcal{CI}_{N,3}$	$\mathcal{SCI}_{N,3}$	$\mathcal{CB}_{N,3}$	$\mathcal{SCB}_{N,3}$
26	52.4	111.6	142.7	14.3
100	72.1	19.8	23.4	18.5
796	3.7	14.9	20.9	19.6
3168	17.4	22.2	70.6	25.4
12618	9.9	13.1	36.9	13.7
20000	37.4	14.9	27.4	12.1
mean (all)	32	21	78	19
standard deviation (all)	42	18	180	7
mean (last 10)	22	17	44	17
standard deviation (last 10)	8	5	37	8
mean (last 5)	22	17	32	14
standard deviation (last 5)	10	4	23	1

Experiment: Mass to the left of the mean for $\lambda_+(G)$

- Each set of 1000 3-regular graphs.
- mass to the left of the mean of the Tracy-Widom distributions:
  - ◇ 0.519652 ($\beta = 1$)
  - ◇ 0.515016 ($\beta = 2$)
  - ◇ 0.511072 ($\beta = 4$)
  - ◇ 0.500000 (standard normal).
- two-sided z-test: critical thresholds: 1.96 (for $\alpha = .05$) and 2.575 (for $\alpha = .01$).

Experiment: Mass to the left of the mean for $\mathcal{CI}_{N,3}$

Critical values: 1.96 ($\alpha = .05$), 2.575 ($\alpha = .01$).

N	Obs mass	$z_{\mathrm{TW},1}$	$z_{\mathrm{TW},2}$	$z_{\mathrm{TW},4}$	z_{StdNorm}
26	0.483	-2.320	-2.026	-1.776	-1.075
100	0.489	-1.940	-1.646	-1.396	-0.696
796	0.521	0.085	0.379	0.628	1.328
6324	0.523	0.212	0.505	0.755	1.455
20000	0.526	0.402	0.695	0.944	1.644
μ (last 10)	0.518	0.473	0.531	0.655	1.202
$\widetilde{\mu}$ (last 10)	0.523	0.411	0.537	0.755	1.455
μ (last 5)	0.517	0.591	0.532	0.630	1.050
$\widetilde{\mu}$ (last 5)	0.515	0.421	0.695	0.700	0.949

Experiment: Mass left of mean: 3-Regular, sets of 100,000

Discarded: Matlab's algorithm didn't converge.
Critical values: 1.96 ($\alpha = .05$), 2.575 ($\alpha = .01$).

$\mathcal{CI}_{N,3}$	$z_{\mathrm{TW},1}$	$z_{\mathrm{TW},2}$	$z_{\mathrm{TW},4}$	z_{StdNorm}	Discarded
1002	0.239	3.173	5.667	12.668	0
2000	-0.128	2.806	5.300	12.301	0
5022	1.265	4.198	6.692	13.693	0
10022	0.391	3.324	5.819	12.820	0
40000	2.334	5.267	7.761	14.762	0

$\mathcal{SCI}_{N,3}$	$z_{\mathrm{TW},1}$	$z_{\mathrm{TW},2}$	$z_{\mathrm{TW},4}$	z_{StdNorm}	Discarded
1002	-1.451	1.483	3.978	10.979	0
2000	-0.457	2.477	4.971	11.972	0
5022	-0.042	2.891	5.386	12.387	1

Experiment: Mass left of mean: 3-Regular, sets of 100,000

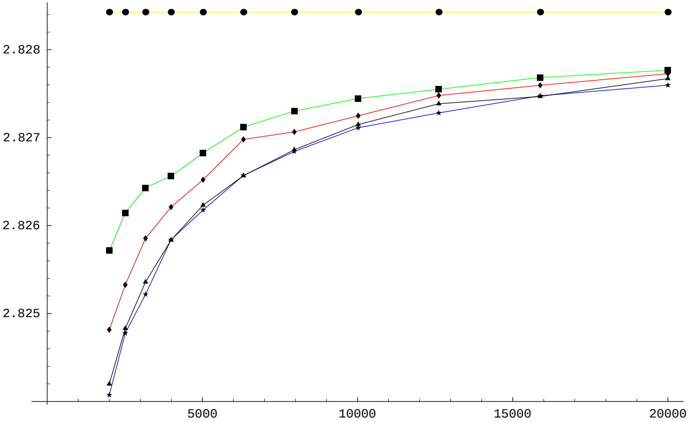
Critical values: 1.96 ($\alpha = .05$), 2.575 ($\alpha = .01$).

$\mathcal{CB}_{N,3}$	$z_{\text{TW},1}$	$z_{\text{TW},2}$	$z_{\text{TW},4}$	z_{StdNorm}	Discarded
1002	3.151	6.083	8.577	15.577	0
2000	3.787	6.719	9.213	16.213	1
5022	3.563	6.495	8.989	15.989	4
10022	2.049	4.982	7.476	14.477	0
12618	3.701	6.634	9.127	16.128	0
15886	2.999	5.931	8.425	15.426	0
20000	2.106	5.039	7.533	14.534	0
40000	1.853	4.786	7.280	14.281	0

$\mathcal{SCB}_{N,3}$	$z_{\text{TW},1}$	$z_{\text{TW},2}$	$z_{\text{TW},4}$	z_{StdNorm}	Discarded
1002	-1.963	0.971	3.465	10.467	0
2000	-0.767	2.167	4.661	11.663	2
5022	-0.064	2.869	5.364	12.365	4

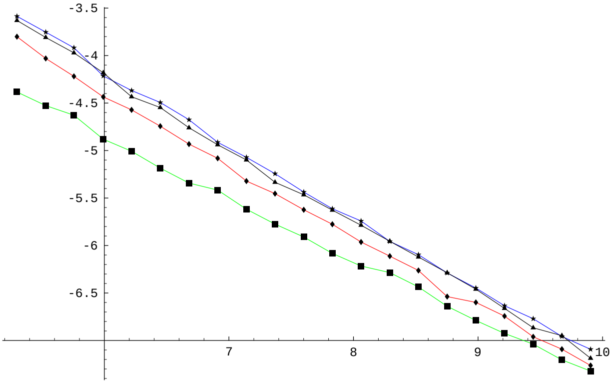
3-regular graphs: Sample means of $\lambda_+(G)$

Sets of 1000 random 3-regular graphs. $\mathcal{CI}_{N,3}$ is red, $\mathcal{SCI}_{N,3}$ is blue, $\mathcal{CB}_{N,3}$ is green, $\mathcal{SCB}_{N,3}$ is black; the solid yellow line is $2\sqrt{2} \approx 2.8284$.



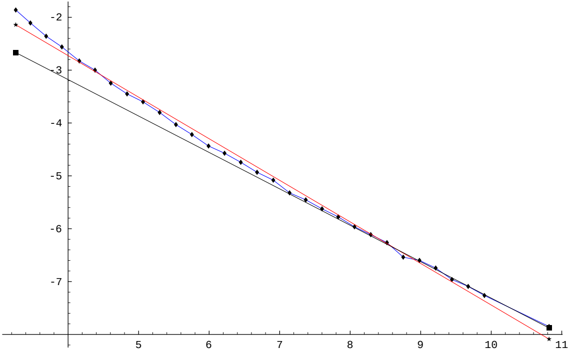
3-regular graphs: Sample means of $\lambda_+(G)$

Sets of 1000 random 3-regular graphs. $\mathcal{CI}_{N,3}$ is red, $\mathcal{SCI}_{N,3}$ is blue, $\mathcal{CB}_{N,3}$ is green, $\mathcal{SCB}_{N,3}$ is black.



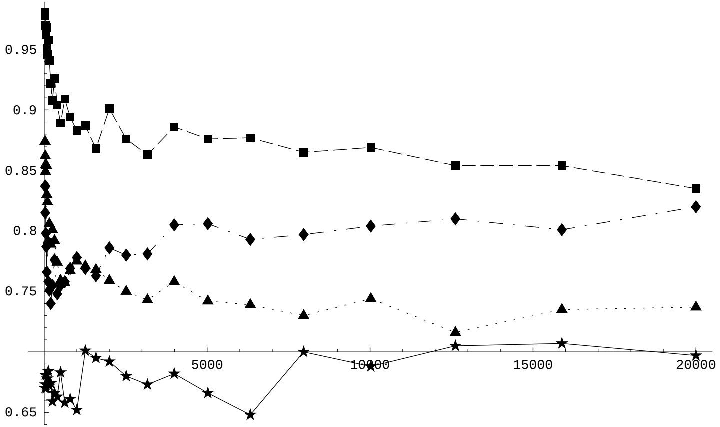
3-regular graphs: best fit means of $\lambda_+(G)$

Logarithm of the mean on $\log\left(c_{\mu,N,3} N^{m(\mathcal{CI}_{N,3})}\right)$ on N. Blue: data; red: best fit (all); black: best fit (last 10).



3-regular graphs: percent Ramanujan

Each set is 1000 random 3-regular graphs with N vertices. $\mathcal{CI}_{N,3}$ are stars, $\mathcal{SCI}_{N,3}$ are diamonds, $\mathcal{CB}_{N,3}$ are triangles, $\mathcal{SCB}_{N,3}$ are boxes.



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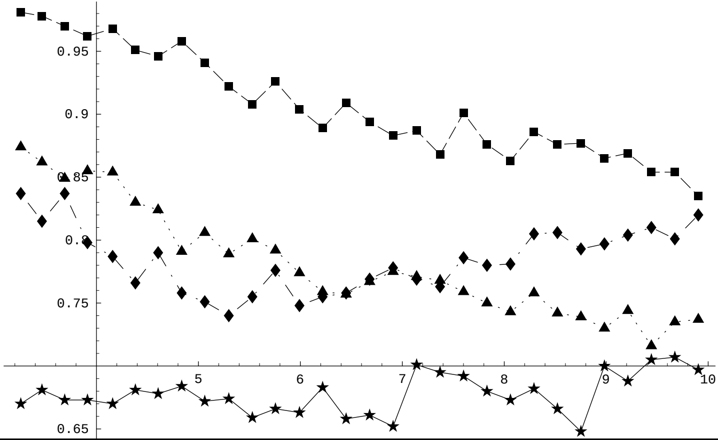

Best-fit exponents ($d = 3$) for $\lambda_+(G)$

First table means $m(\mathcal{F})$, second std devs $s(\mathcal{F})$.
Bold entries: $m(\mathcal{F}) > s(\mathcal{F})$.

N	$\mathcal{CI}_{N,3}$	$\mathcal{SCI}_{N,3}$	$\mathcal{CB}_{N,3}$	$\mathcal{SCB}_{N,3}$
$\{26, \dots, 20000\}$	-0.795	-0.828	-0.723	- 0.833
$\{80, \dots, 20000\}$	-0.761	-0.790	-0.671	-0.789
$\{252, \dots, 20000\}$	-0.735	-0.762	-0.638	-0.761
$\{26, \dots, 64\}$	- 1.058	-1.105	-1.065	-1.151
$\{80, \dots, 200\}$	-0.854	-0.949	-0.982	-0.968
$\{232, \dots, 632\}$	-0.773	-0.840	-0.737	-0.842
$\{796, \dots, 2000\}$	- 0.762	-0.805	-0.649	-0.785
$\{2516, \dots, 6324\}$	-0.791	-0.741	-0.579	-0.718
$\{7962, \dots, 20000\}$	- 0.728	-0.701	-0.584	-0.757

N	$\mathcal{CI}_{N,3}$	$\mathcal{SCI}_{N,3}$	$\mathcal{CB}_{N,3}$	$\mathcal{SCB}_{N,3}$
$\{26, \dots, 20000\}$	- 0.713	-0.725	-0.709	-0.729
$\{80, \dots, 20000\}$	-0.693	-0.703	**-0.697**	-0.706
$\{252, \dots, 20000\}$	- 0.679	-0.691	**-0.688**	-0.696
$\{26, \dots, 64\}$	-0.863	-0.918	-0.794	-0.957
$\{80, \dots, 200\}$	-0.694	-0.752	- 0.719	-0.750
$\{232, \dots, 632\}$	-0.718	-0.716	-0.714	-0.734
$\{796, \dots, 2000\}$	-0.602	-0.648	**-0.705**	-0.763
$\{2516, \dots, 6324\}$	- 0.614	-0.668	**-0.770**	-0.688
$\{7962, \dots, 20000\}$	-0.543	**-0.716**	**-0.671**	- 0.648

Best-fit exponents ($d = 3$) for $\lambda_+(G)$

$$2\sqrt{d-1} \approx \mu_{\mathcal{F}_{N,d}} + \frac{c_{\mu,N,d}}{c_{\sigma,N,d}} N^{m(\mathcal{F}_{N,d}) - s(\mathcal{F}_{N,d})} \sigma_{\mathcal{F}_{N,d}}$$

$m(\mathcal{F}_{N,d}) - s(\mathcal{F}_{N,d})$, Bold entries $m(\mathcal{F}) > s(\mathcal{F})$.

N	$\mathcal{CI}_{N,3}$	$\mathcal{SCI}_{N,3}$	$\mathcal{CB}_{N,3}$	$\mathcal{SCB}_{N,3}$
$\{26, \dots, 20000\}$	-0.082	-0.103	-0.014	-0.104
$\{80, \dots, 20000\}$	-0.068	-0.087	**0.026**	-0.083
$\{252, \dots, 20000\}$	- 0.056	-0.071	**0.050**	-0.065
$\{26, \dots, 64\}$	-0.195	-0.187	-0.271	-0.194
$\{80, \dots, 200\}$	-0.160	-0.197	- 0.263	-0.218
$\{232, \dots, 632\}$	-0.055	-0.124	-0.023	-0.108
$\{796, \dots, 2000\}$	-0.160	-0.157	**0.056**	-0.022
$\{2516, \dots, 6324\}$	-0.177	-0.073	**0.191**	-0.030
$\{7962, \dots, 20000\}$	-0.185	**0.015**	**0.087**	-0.109