He's just going through a phase: Miller's SMALL students and Phase Transitions

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Introduction to Continued Fractions and Maclaurin's Inequality

Joint with Francesco Cellarosi (UIUC), Doug Hensley (Texas A & M), Jake Wellens (SMALL '13)

Plan of Part I: Continued Fractions and Maclaurin's Inequality

- Classical ergodic theory of continued fractions.
  - ◇ Almost surely geometric mean $\sqrt[n]{a_1 \cdots a_n} \to K_0$.
  - ◇ Almost surely arithmetic mean $(a_1 + \cdots + a_n)/n \to \infty$.

- Symmetric averages and Maclaurin's inequalities.
  - ◇ $S(x, n, k) := \binom{n}{k}^{-1} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$.
  - ◇ $\mathrm{AM} = S(x, n, 1)^{1/1} \ge S(x, n, 2)^{1/2} \ge \cdots \ge S(x, n, n)^{1/n} = \mathrm{GM}$.

- Results / conjectures on typical / periodic continued fraction averages.

- Elementary proofs of weak results, sketch of stronger results.

To appear in Exp. Math.:http://arxiv.org/abs/1402.0208.

Continued Fractions

- Every real number $\alpha \in (0,1)$ can be expressed as

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdots}}}} = [a_1, a_2, a_3, \dots], \ \ a_i \in \{1, 2, \dots\}.$$

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- The sequence $\{a_i\}_i$ is finite iff $\alpha \in \mathbb{Q}$.

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- $x = \frac{p}{q} \in \mathbb{Q}$ then a_i's the partial quotients of Euclidean Alg.

$$\frac{\mathbf{106}}{\mathbf{333}} = [{\color{red}3}, {\color{blue}7}, {\color{purple}15}]$$

$$\begin{aligned} \mathbf{333} &= {\color{red}3} \cdot \mathbf{106} + 15 \\ 106 &= {\color{blue}7} \cdot 15 + 1 \\ 15 &= {\color{purple}15} \cdot 1 + 0. \end{aligned}$$

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- $\{a_i\}_i$ preperiodic iff α a quadratic irrational;
ex: $\sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \dots]$.

Gauss Map: Definition

- The Gauss map $T : (0,1] \to (0,1]$, $T(x) = \{\frac{1}{x}\} = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ generates the continued fraction digits

$$a_1 \;=\; \lfloor 1/T^0(\alpha) \rfloor, \quad a_{i+1} \;=\; \lfloor 1/T^i(\alpha) \rfloor, \quad \dots$$

corresponding to the Markov partition

$$(0,1] \;=\; \bigsqcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k} \right].$$

- T preserves the measure $d\mu = \frac{1}{\log 2} \frac{1}{1+x} dx$ and it is mixing.

Gauss Map: Example: $\sqrt{3}-1 = [1,2,1,2,1,2,\dots]$

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$$\begin{aligned} T^1(\sqrt{3}-1) &= \frac{1}{\sqrt{3}-1} - \left\lfloor \frac{1}{\sqrt{3}-1} \right\rfloor = \frac{\sqrt{3}+1}{3-1} - 1 = \frac{\sqrt{3}-1}{2} \\ a_2 &= \left\lfloor \frac{2}{\sqrt{3}-1} \right\rfloor = 2. \end{aligned}$$

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Statistics of Continued Fraction Digits 1/3

- The digits a_i follow the Gauss-Kuzmin distribution:

$$\lim_{n\to\infty} \mathbb{P}(a_n = k) = \log_2\left(1 + \frac{1}{k(k+2)}\right)$$

(note the expectation is infinite).

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- Pointwise ergodic theorem (applied to f and $\log f$) reads

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = \infty \quad \text{almost surely}$$

$$\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} = e^{\int \log f \, d\mu} \quad \text{almost surely.}$$

Statistics of Continued Fraction Digits 2/3

- Geometric mean converges a.s. to Khinchin's constant:

$$\lim_{n\to\infty}(a_1a_2\cdots a_n)^{1/n} = \prod_{k=1}^{\infty}\left(1+\frac{1}{k(k+2)}\right)^{\log_2 k} = K_0 \approx 2.6854.$$

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- Hölder means: For $p < 1$, almost surely

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} a_i^p\right)^{1/p} = K_p = \left(\sum_{k=1}^{\infty} -k^p \log_2\left(1-\frac{1}{(k+1)^2}\right)\right)^{1/p}.$$

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- $\lim_{p\to 0} K_p = K_0$.

Statistics of Continued Fraction Digits 3/3

- Khinchin also proved: For $a_m' = a_m$ if $a_m < m(\log m)^{4/3}$ and 0 otherwise:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i'}{n \log n} = \frac{1}{\log 2} \quad \text{in measure.}$$

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- Diamond and Vaaler (1986) showed that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n a_i \ - \ \max_{1\le i\le n} a_i}{n \log n} \ = \ \frac{1}{\log 2} \quad \text{almost surely.}$$

Maclaurin Inequalities

Definitions and Maclaurin's Inequalities

- Both $\frac{1}{n}\sum_{i=1}^{n} x_i$ and $\left(\prod_{i=1}^{n} x_i\right)^{1/n}$ are defined in terms of elementary symmetric polynomials in $x_1, \dots, x_n$.
- Define **k^{th} elementary symmetric mean of $x_1, \dots, x_n$** by

$$S(x,n,k) \ := \ \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

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Maclaurin's Inequalities

For positive $x_1, \dots, x_n$ we have

$$\mathrm{AM} := S(x,n,1)^{1/1} \ge S(x,n,2)^{1/2} \ge \cdots \ge S(x,n,n)^{1/n} =: \mathrm{GM}$$

(and equalities hold iff $x_1 = \cdots = x_n$).

Maclaurin's work

IV. *A ſecond Letter from* Mr. Colin M^c Laurin, *Profeſſor of Mathematicks in the Univerſity of* Edinburgh *and F. R. S. to* Martin Folkes, *Eſq*; *concerning the Roots of Equations, with the Demonſtration of other Rules in Algebra; being the Continuation of the Letter publiſhed in the* Philoſophical Tranſactions, N° 394.

Edinburgh, April 19th, 1729.

S I R,

IN the Y ar 1725, I wrote to you that I had a Method of demonſtrating Sir *Iſaac Newton*'s Rule concerning the impoſſible Roots of Equations, deduced from this obvious Principle, that the Squares of the Differences of real Quantities muſt always be poſitive; and ſome time after, I ſent you the firſt Principles of that Method, which were publiſhed in the *Philoſophical Tranſactions* for the Month of *May*, 1726. The

This laſt is the Theorem publiſhed by the learned Mr. *Bernouilli* in the *Acta Lipſiæ* 1694. It is now high Time to conclude this long Letter; I beg you may accept of it as a Proof of that Reſpect and Eſteem with which

I am,

S I R,

Your moſt Obedient,

Moſt Humble Servant,

Colin Mac Laurin.

Proof

Standard proof through Newton's inequalities.

Define the k^{th} **elementary symmetric function** by

$$s_k(x) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

and the k^{th} **elementary symmetric mean** by

$$E_k(x) = s_k(x) \Big/ \binom{n}{k}.$$

Newton's inequality: $E_k(x)^2 \ge E_{k-1}(x) E_{k+1}(x)$.

New proof by Iddo Ben-Ari and Keith Conrad:

http://homepages.uconn.edu/benari/pdf/maclaurinMathMagFinal.pdf.

Sketch of Ben-Ari and Conrad's Proof

Bernoulli's inequality: $t > -1$: $(1+t)^n \geq 1 + nt$ or $1 + \frac{1}{n}x \geq (1+x)^{1/n}$.

Generalized Bernoulli: $x > -1$:

$$1 + \frac{1}{n}x \geq \left(1 + \frac{2}{n}x\right)^{1/2} \geq \left(1 + \frac{3}{n}x\right)^{1/3} \geq \cdots \geq \left(1 + \frac{n}{n}x\right)^{1/n}.$$

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Proof: Equivalent to $\frac{1}{k}\log\left(1 + \frac{k}{n}x\right) \geq \frac{1}{k+1}\log\left(1 + \frac{k+1}{n}x\right)$,
which follows by $\log t$ is strictly concave:

$$\lambda = \frac{1}{k+1},\ 1 + \frac{k}{n}x = \lambda \cdot 1 + (1-\lambda)\cdot\left(1 + \frac{k+1}{n}x\right).$$

Sketch of Ben-Ari and Conrad's Proof

Proof of Maclaurin's Inequalities:

Trivial for $n \in \{1, 2\}$, wlog assume $x_1 \leq x_2 \leq \cdots \leq x_n$.

Set $E_k := s_k(x)/\binom{n}{k}$, $\epsilon_k := E_k(x_1, \ldots, x_{n-1})$.

Have
$E_k(x_1, \ldots, x_n) = \left(1 - \frac{k}{n}\right) E_k(x_1, \ldots, x_{n-1}) + \frac{k}{n} E_k(x_1, \ldots, x_{n-1}) x_n$.

Proceed by induction in number of variables, use Generalized Bernoulli.

Main Results
(Elementary Techniques)

Symmetric Averages and Maclaurin's Inequalities

- Recall: $S(x,n,k) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$
  and $S(x,n,1)^{1/1} \ge S(x,n,2)^{1/2} \ge \cdots \ge S(x,n,n)^{1/n}$.

Symmetric Averages and Maclaurin's Inequalities

- Recall: $S(x,n,k) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$ and $S(x,n,1)^{1/1} \geq S(x,n,2)^{1/2} \geq \cdots \geq S(x,n,n)^{1/n}$.

- Khinchin's results: almost surely as $n \to \infty$

$$S(\alpha,1,1)^{1/1} \to \infty \quad \text{and} \quad S(\alpha,n,n)^{1/n} \to K_0.$$

- We study the intermediate means $S(\alpha,n,k)^{1/k}$ as $n \to \infty$ when $k = k(n)$, with

$$S(\alpha,n,k(n))^{1/k(n)} = S(\alpha,n,\lceil k(n) \rceil)^{1/\lceil k(n) \rceil}.$$

Our results on typical continued fraction averages

Recall: $S(\alpha, n, k) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1} \cdots a_{i_k}$

and $S(\alpha, n, 1)^{1/1} \ge S(\alpha, n, 2)^{1/2} \ge \cdots \ge S(\alpha, n, n)^{1/n}$.

Theorem 1

Let $f(n) = o(\log \log n)$ as $n \to \infty$. Then, almost surely,

$$\lim_{n \to \infty} S(\alpha, n, f(n))^{1/f(n)} \ = \ \infty.$$

Theorem 2

Let $f(n) = o(n)$ as $n \to \infty$. Then, almost surely,

$$\lim_{n \to \infty} S(\alpha, n, n - f(n))^{1/(n - f(n))} \ = \ K_0.$$

Note: Theorems do not cover the case $f(n) = cn$ for $0 < c < 1$.

Sketch of Proofs of Theorems 1 and 2

Theorem 1: For $f(n) = o(\log\log n)$ as $n \to \infty$:

$$\text{Almost surely} \quad \lim_{n\to\infty} S(\alpha, n, f(n))^{1/f(n)} = \infty.$$

Uses Niculescu's strengthening of Maclaurin (2000):

$$S(n, tj + (1-t)k) \geq S(n,j)^t \cdot S(n,k)^{1-t}.$$

Sketch of Proofs of Theorems 1 and 2

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Theorem 2: For $f(n) = o(n)$ as $n \to \infty$:

$$\text{Almost surely} \quad \lim_{n\to\infty} S(\alpha, n, n-f(n))^{1/(n-f(n))} \;=\; K_0.$$

Use (a.s.) $K_0 \leq \limsup_{n\to\infty} S(\alpha, n, cn)^{1/cn} \leq K_0^{1/c} < \infty, 0 < c < 1.$

Proof of Theorem 1: Preliminaries

Lemma

Let X be a sequence of positive real numbers. Suppose $\lim_{n\to\infty} S(X,n,k(n))^{1/k(n)}$ exists. Then, for any $f(n) = o(k(n))$ as $n \to \infty$, we have

$$\lim_{n\to\infty} S(X,n,k(n)+f(n))^{1/(k(n)+f(n))} \;=\; \lim_{n\to\infty} S(n,k(n))^{1/k(n)}.$$

Proof: Assume $f(n) \geq 0$ for large enough n, and for display purposes write k and f for $k(n)$ and $f(n)$.

From Newton's inequalities and Maclaurin's inequalities, we get

$$\left(S(X,n,k)^{1/k}\right)^{\frac{k}{k+f}} \;=\; S(X,n,k)^{1/(k+f)} \;\leq\; S(X,n,k+f)^{1/(k+f)} \;\leq\; S(X,n,k)^{1/k}.$$

Proof of Theorem 1: $f(n) = o(\log\log n)$

Each entry of α is at least 1.
Let $f(n) = o(\log\log n)$. Set $t = 1/2$ and $(j,k) = (1, 2f(n) - 1)$, so that $tj + (1-t)k = f(n)$. Niculescu's result yields

$$S(\alpha,n,f(n)) \ \geq \ \sqrt{S(\alpha,n,1)\cdot S(\alpha,n,2f(n)-1)} \ > \ \sqrt{S(\alpha,n,1)}.$$

Square both sides, raise to the power $1/f(n)$:

$$S(\alpha,n,f(n))^{2/f(n)} \ \geq \ S(\alpha,n,1)^{1/f(n)}.$$

From Khinchin almost surely if $g(n) = o(\log n)$

$$\lim_{n\to\infty} \frac{S(\alpha,n,1)}{g(n)} \ = \ \infty.$$

Let $g(n) = \log n/\log\log n$. Taking logs:

$$\log\left(S(\alpha,n,1)^{1/f(n)}\right) \ > \ \frac{\log g(n)}{f(n)} \ > \ \frac{\log\log n}{2f(n)}.$$

Proof of Theorem 2

Theorem 2: Let $f(n) = o(n)$ as $n \to \infty$. Then, almost surely,

$$\lim_{n\to\infty} S(\alpha, n, n - f(n))^{1/(n-f(n))} \ = \ K_0.$$

Proof: Follows immediately from:
For any constant $0 < c < 1$ and almost all α have

$$K_0 \ \le \ \limsup_{n\to\infty} S(\alpha, n, cn)^{1/cn} \ \le \ K_0^{1/c} \ < \ \infty.$$

To see this, note

$$S(\alpha, n, cn)^{1/cn} \ = \ \left(\prod_{i=1}^{n} a_i(\alpha)^{1/n}\right)^{n/cn} \left(\frac{\sum_{i_1 < \cdots < i_{(1-c)n} \le n} 1/(a_{i_1}(\alpha)\cdots a_{i_{(1-c)n}}(\alpha))}{\binom{n}{cn}}\right)^{1/cn}.$$

Limiting Behavior

Recall $S(\alpha, n, k) = \frac{1}{\binom{n}{k}} \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1} \cdots a_{i_k}$
and $S(\alpha, n, 1)^{1/1} \geq S(\alpha, n, 2)^{1/2} \geq \cdots \geq S(\alpha, n, n)^{1/n}$.

Proposition

For $0 < c < 1$ and for almost every α

$$K_0 \leq \limsup_{n \to \infty} S(\alpha, n, cn)^{1/cn} \leq K_0^{1/c} (K_{-1})^{1-1/c}.$$

Conjecture

Almost surely $F_+^\alpha(c) = F_-^\alpha(c) = F(c)$ for all $0 < c < 1$, with

$$F_+^\alpha(c) = \limsup_{n \to \infty} S(\alpha, n, cn)^{1/cn},$$
$$F_-^\alpha(c) = \liminf_{n \to \infty} S(\alpha, n, cn)^{1/cn}.$$

Limiting Behavior

Recall

$$F_+^{\alpha}(c) = \limsup_{n\to\infty} S(\alpha, n, cn)^{1/cn}$$
$$F_-^{\alpha}(c) = \liminf_{n\to\infty} S(\alpha, n, cn)^{1/cn},$$

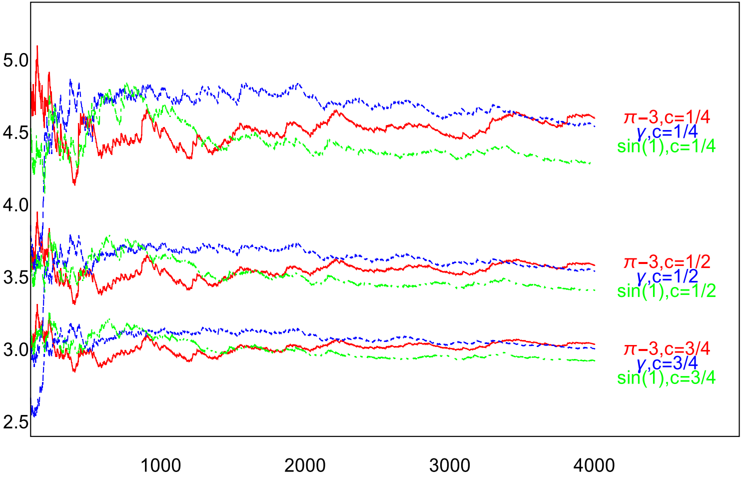
and we conjecture $F_+^{\alpha}(c) = F_-^{\alpha}(c) = F(c)$ a.s.

Assuming conjecture, can show that the function $c \mapsto F(c)$ is continuous.

Assuming conjecture is false, we can show that for every $0 < c < 1$ the set of limit points of the sequence $\{S(\alpha, n, cn)^{1/cn})\}_{n\in\mathbb{N}}$ is a non-empty interval inside $[K, K^{1/c}]$.

Evidence for Conjecture 1

- $n \mapsto S(\alpha, n, cn)^{1/cn}$ for $c = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and $\alpha = \pi - 3$, γ, $\sin(1)$.



Our results on periodic continued fraction averages 1/2

- For $\alpha = \sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \ldots]$,

$$\lim_{n\to\infty} S(\alpha, n, 1)^{1/1} = \frac{3}{2} \neq \infty$$

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- Let us look at $S(\alpha, n, cn)^{1/cn}$ for $c = 1/2$.

$$S(\alpha, n, \lceil \tfrac{n}{2} \rceil) = \begin{cases} S(\alpha, n, \frac{n}{2}) & \text{if } n \equiv 0 \bmod 2; \\ S(\alpha, n, \frac{n+1}{2}) & \text{if } n \equiv 1 \bmod 2. \end{cases}$$

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- We find the limit $\lim_{n\to\infty} S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil}$ in terms of x, y.

Our results on periodic continued fraction averages 2/2

Theorem 3

Let $\alpha = [\overline{x, y}]$. Then $S(\alpha, n, \lceil \frac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil}$ converges as $n \to \infty$ to the $\frac{1}{2}$-Hölder mean of x and y:

$$\lim_{n\to\infty} S(\alpha, n, \lceil \tfrac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left(\frac{x^{1/2} + y^{1/2}}{2} \right)^2.$$

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Suffices to show for $n \equiv 0 \bmod 2$, say $n = 2k$.

In this case we have that $S(\alpha, 2k, k)^{1/k} \to \left(\frac{x^{1/2}+y^{1/2}}{2} \right)^2$ monotonically as $k \to \infty$.

On the proof of Theorem 3, 1/2

$$\text{Goal}: \alpha = [\overline{x, y}] \Rightarrow \lim_{n\to\infty} S(\alpha, n, \lceil \tfrac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left(\frac{x^{1/2} + y^{1/2}}{2} \right)^2.$$

The proof uses an asymptotic formula for Legendre polynomials P_k (with $t = \frac{x}{y} < 1$ and $u = \frac{1+t}{1-t} > 1$):

$$P_k(u) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j}^2 (u-1)^{k-j}(u+1)^j$$

$$S(\alpha, 2k, k) = \frac{1}{\binom{2k}{k}} \sum_{j=0}^{k} \binom{k}{j}^2 x^j y^{k-j} = \frac{y^k}{\binom{2k}{k}} \sum_{j=0}^{k} \binom{k}{j}^2 t^j$$

$$= \frac{y^k}{\binom{2k}{k}} (1-t)^k P_k(u).$$

On the proof of Theorem 3, 2/2

$$\text{Goal}: \alpha = [\overline{x,y}] \Rightarrow \lim_{n\to\infty} S(\alpha, n, \lceil \tfrac{n}{2} \rceil)^{1/\lceil \frac{n}{2} \rceil} = \left(\frac{x^{1/2}+y^{1/2}}{2}\right)^2.$$

Using the *generalized Laplace-Heine asymptotic formula* for $P_k(u)$ for $u > 1$ and $t = \frac{x}{y} < 1$ and $u = \frac{1+t}{1-t} > 1$ gives

$$\begin{aligned} S(\alpha, 2k, k)^{1/k} &= y(1-t)\left(\frac{P_k(u)}{\binom{2k}{k}}\right)^{1/k} \\ &\longrightarrow y(1-t)\frac{u+\sqrt{u^2-1}}{4} = y\left(\frac{1+\sqrt{t}}{2}\right)^2 \\ &= \left(\frac{x^{1/2}+y^{1/2}}{2}\right)^2. \end{aligned}$$

A conjecture on periodic continued fraction averages 1/3

Expect the same result of Theorem 3 to hold for every quadratic irrational α and for every c.

Conjecture 2

For every $\alpha = [\overline{x_1, \ldots, x_L}]$ and every $0 \leq c \leq 1$ the limit

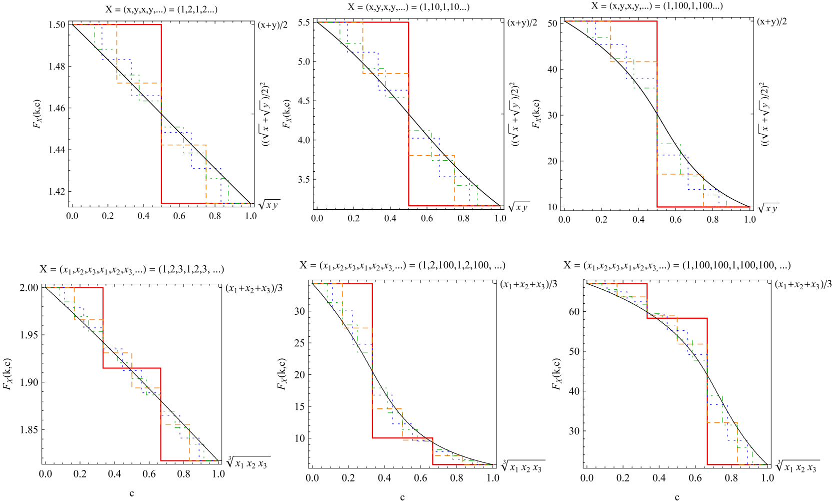
$$\lim_{n\to\infty} S(\alpha, n, \lceil cn \rceil)^{1/\lceil cn \rceil} =: F(\alpha, c)$$

exists and it is a continuous function of c.

Notice $c \mapsto F(\alpha, c)$ is automatically decreasing by Maclaurin's inequalities.

A conjecture on periodic continued fraction averages 2/3

Conjecture 2 for period 2 and period 3, $0 \le c \le 1$.



Main Results
(Sketch of More Technical Arguments)

Explicit Formula for $F(c)$

Result of Halász and Székely yields conjecture and $F(c)$.

Theorem 4

If $\lim_{n\to\infty} \frac{k}{n} = c \in (0,1]$, then for almost all $\alpha \in [0,1]$

$$\lim_{n\to\infty} S(\alpha,n,k)^{1/k} \ =: \ F(c)$$

exists, and $F(c)$ is continuous and given explicitly by

$$c(1-c)^{\frac{1-c}{c}} \exp\left\{\frac{1}{c}\left((c-1)\log r_c - \sum_{k=1}^{\infty} \log\left(r_c+k\right)\log_2\left(1-\frac{1}{(k+1)^2}\right)\right)\right\},$$

where r_c is the unique nonnegative solution of the equation

$$\sum_{k=1}^{\infty} \frac{r}{r+k} \log_2\left(1-\frac{1}{(k+1)^2}\right) \ = \ c-1.$$

Proof: Work of Halász and Székely

- Halász and Székely calculate asymptotic properties of iidrv $\xi_1, \dots, \xi_n$ when
  - $c = \lim_{n\to\infty} k/n \in [0,1]$.
  - ξ_j non-negative.
  - $\mathbb{E}[\log \xi_j] < \infty$ if $c = 1$.
  - $\mathbb{E}[\log(1+\xi_j) < \infty$ if $0 < c < 1$.
  - $\mathbb{E}[\xi_j] < \infty$ if $c = 0$.

- Prove $\lim_{n\to\infty} \sqrt[k]{S(\xi,n,k)/\binom{n}{k}}$ exists with probability 1 and determine it.

Proof: Work of Halász and Székely

Random variables $a_i(\alpha)$ not independent, but Halász and Székely only use independence to conclude sum of the form

$$\frac{1}{n}\sum_{k=1}^{n} f(T^k(\alpha))$$

(where T is the Gauss map and f is some function integrable with respect to the Gauss measure) converges a.e. to $\mathbb{E}f$ as $n \to \infty$.

Arrive at the same conclusion by appealing to the pointwise ergodic theorem.

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Introduction to MSTD

Joint with Peter Hegarty (Chambers), Oleg Lazarev (SMALL '12), Kevin O'Bryant (CUNY), ...

Statement

A finite set of integers, $|A|$ its size. Form

- Sumset: $A + A = \{a_i + a_j : a_j, a_j \in A\}$.
- Difference set: $A - A = \{a_i - a_j : a_j, a_j \in A\}$.

Arise in Goldbach's Problem, Twin Primes, Fermat's Last Theorem, ...

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Arise in Goldbach's Problem, Twin Primes, Fermat's Last Theorem, ...

Definition

We say A is difference dominated if $|A - A| > |A + A|$, balanced if $|A - A| = |A + A|$ and sum dominated (or an MSTD set) if $|A + A| > |A - A|$.

Questions

Expect generic set to be difference dominated:

- addition is commutative, subtraction isn't:
- Generic pair (x, y) gives 1 sum, 2 differences.

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Questions

- Do there exist sum-dominated sets?
- If yes, how many?

Examples

- Conway: $\{0, 2, 3, 4, 7, 11, 12, 14\}$.
- Marica (1969): $\{0, 1, 2, 4, 7, 8, 12, 14, 15\}$.
- Freiman and Pigarev (1973): $\{0, 1, 2, 4, 5, 9, 12, 13, 14, 16, 17, 21, 24, 25, 26, 28, 29\}$.
- Computer search of random subsets of $\{1, \dots, 100\}$: $\{2, 6, 7, 9, 13, 14, 16, 18, 19, 22, 23, 25, 30, 31, 33, 37, 39, 41, 42, 45, 46, 47, 48, 49, 51, 52, 54, 57, 58, 59, 61, 64, 65, 66, 67, 68, 72, 73, 74, 75, 81, 83, 84, 87, 88, 91, 93, 94, 95, 98, 100\}$.
- Recently infinite families (Hegarty, Nathanson).

Infinite Families

Key observation

If A is an arithmetic progression, $|A + A| = |A - A|$.

Infinite Families

Key observation

If A is an arithmetic progression, $|A+A| = |A-A|$.

Proof:

- WLOG, $A = \{0, 1, \dots, n\}$ as $A \to \alpha A + \beta$ doesn't change $|A+A|$, $|A-A|$.
- $A + A = \{0, \dots, 2n\}$, $A - A = \{-n, \dots, n\}$, both of size $2n+1$. $\square$

Previous Constructions

Most constructions perturb an arithmetic progression.

Example:

- MSTD set $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$.
- $A = \{0, 2\} \cup \{3, 7, 11\} \cup (14 - \{0, 2\}) \cup \{4\}$.

Example (Nathanson)

Theorem

$m, d, k \in \mathbb{N}$ with $m \geq 4$, $1 \leq d \leq m-1$, $d \neq m/2$, $k \geq 3$ if $d < m/2$ else $k \geq 4$. Let

- $B = [0, m-1] \backslash \{d\}$.
- $L = \{m-d, 2m-d, \dots, km-d\}$.
- $a^* = (k+1)m - 2d$.
- $A^* = B \cup L \cup (a^* - B)$.
- $A = A^* \cup \{m\}$.

Then A is an MSTD set.

Note: gives *exponentially* low density of MSTD sets.

New Explicit Constructions: Results and Notation

Previous best explicit sub-family of $\{1, \dots, n\}$ gave density of $C_1 n^d / 2^{n/2}$.

Our new family gives C_2/n^2, almost a positive percent.

Current record by Zhao: C_3/n.

Notation:

- $[a, b] = \{k \in \mathbb{Z} : a \leq k \leq b\}$.
- A is a P_n-set if its sumset and difference sets contain all but the first and last n possible elements (may or may not contain some of these fringe elements).

New Construction

Theorem (Miller-Orosz-Scheinerman ’09)

- *$A = L \cup R$ be a P_n, MSTD set where $L \subset [1, n]$, $R \subset [n+1, 2n]$, and $1, 2n \in A$.*
- *Fix a $k \geq n$ and let m be arbitrary.*
- *M any subset of $[n+k+1, n+k+m]$ st no run of more than k missing elements. Assume $n+k+1 \notin M$.*
- *Set $A(M) = L \cup O_1 \cup M \cup O_2 \cup R'$, where $O_1 = [n+1, n+k]$, $O_2 = [n+k+m+1, n+2k+m]$, and $R' = R + 2k + m$.*

Then $A(M)$ is an MSTD set, and $\exists C > 0$ st the percentage of subsets of $\{0, \dots, r\}$ that are in this family (and thus are MSTD sets) is at least C/r^2.

Phase Transition

Probability Review

X random variable with density $f(x)$ means

- $f(x) \geq 0$;
- $\int_{-\infty}^{\infty} f(x) = 1$;
- $\mathrm{Prob}(X \in [a, b]) = \int_a^b f(x)dx$.

Key quantities:

- Expected (Average) Value: $\mathbb{E}[X] = \int xf(x)dx$.
- Variance: $\sigma^2 = \int (x - \mathbb{E}[X])^2 f(x)dx$.

Binomial model

Binomial model, parameter $p(n)$

Each $k \in \{0, \dots, n\}$ is in A with probability $p(n)$.

Consider uniform model ($p(n) = 1/2$):

- Let $A \in \{0, \dots, n\}$. Most elements in $\{0, \dots, 2n\}$ in $A + A$ and in $\{-n, \dots, n\}$ in $A - A$.
- $\mathbb{E}[|A + A|] = 2n - 11$, $\mathbb{E}[|A - A|] = 2n - 7$.

Martin and O'Bryant '06

Theorem

Let A be chosen from $\{0, \ldots, N\}$ *according to the binomial model with constant parameter p (thus* $k \in A$ *with probability p). At least* $k_{\mathrm{SD};p}2^{N+1}$ *subsets are sum dominated.*

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- $k_{\mathrm{SD};1/2} \geq 10^{-7}$, expect about 10^{-3}.

- Proof ($p = 1/2$): Generically $|A| = \frac{N}{2} + O(\sqrt{N})$.
  - ◇ about $\frac{N}{4} - \frac{|N-k|}{4}$ ways write $k \in A + A$.
  - ◇ about $\frac{N}{4} - \frac{|k|}{4}$ ways write $k \in A - A$.
  - ◇ Almost all numbers that can be in $A \pm A$ are.
  - ◇ Win by controlling fringes.

Notation

- $X \sim f(N)$ means $\forall \epsilon_1, \epsilon_2 > 0$, $\exists N_{\epsilon_1,\epsilon_2}$ st $\forall N \geq N_{\epsilon_1,\epsilon_2}$

$$\text{Prob}\left(X \notin [(1-\epsilon_1)f(N), (1+\epsilon_1)f(N)]\right) \;<\; \epsilon_2.$$

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- $\mathcal{S} = |A+A|$, $\mathcal{D} = |A-A|$,
  $\mathcal{S}^c = 2N+1-\mathcal{S}$, $\mathcal{D}^c = 2N+1-\mathcal{D}$.

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  $\mathcal{S}^c = 2N + 1 - \mathcal{S}$, $\mathcal{D}^c = 2N + 1 - \mathcal{D}$.

New model: Binomial with parameter $p(N)$:

- $1/N = o(p(N))$ and $p(N) = o(1)$;
- $\text{Prob}(k \in A) = p(N)$.

Conjecture (Martin-O'Bryant)

As $N \to \infty$, A is a.s. difference dominated.

Main Result

Theorem (Hegarty-Miller)

$p(N)$ as above, $g(x) = 2\frac{e^{-x}-(1-x)}{x}$.

- $p(N) = o(N^{-1/2})$: $\mathcal{D} \sim 2\mathcal{S} \sim (Np(N))^2$;
- $p(N) = cN^{-1/2}$: $\mathcal{D} \sim g(c^2)N$, $\mathcal{S} \sim g\left(\frac{c^2}{2}\right)N$ ($c \to 0$, $\mathcal{D}/\mathcal{S} \to 2$; $c \to \infty$, $\mathcal{D}/\mathcal{S} \to 1$);
- $N^{-1/2} = o(p(N))$: $\mathcal{S}^c \sim 2\mathcal{D}^c \sim 4/p(N)^2$.

Can generalize to binary linear forms or arbitrarily many summands, still have critical threshold.

Inputs

Key input: recent strong concentration results of Kim and Vu (Applications: combinatorial number theory, random graphs, ...).

Need to allow dependent random variables.

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Need to allow dependent random variables.

Sketch of proofs: $\mathcal{X} \in \{\mathcal{S}, \mathcal{D}, \mathcal{S}^c, \mathcal{D}^c\}$.

1. Prove $\mathbb{E}[\mathcal{X}]$ behaves asymptotically as claimed;
2. Prove $\mathcal{X}$ is strongly concentrated about mean.

Setup

Only need new strong concentration for $N^{-1/2} = o(p(N))$.

Will assume $p(N) = o(N^{-1/2})$ as proofs are elementary (i.e., Chebyshev: $\mathrm{Prob}(|Y - \mathbb{E}[Y]| \geq k\sigma_Y) \leq 1/k^2$)).

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For convenience let $p(N) = N^{-\delta}$, $\delta \in (1/2, 1)$.

IID binary indicator variables:

$$X_{n;N} = \begin{cases} 1 & \text{with probability } N^{-\delta} \\ 0 & \text{with probability } 1 - N^{-\delta}. \end{cases}$$

$X = \sum_{i=1}^{N} X_{n;N}$, $\mathbb{E}[X] = N^{1-\delta}$.

Proof

Lemma

$P_1(N) = 4N^{-(1-\delta)}$, $\mathcal{O} = \#\{(m,n) : m < n \in \{1,\dots,N\} \bigcap A\}$.
With probability at least $1 - P_1(N)$ *have*

1. $X \in \left[\frac{1}{2}N^{1-\delta}, \frac{3}{2}N^{1-\delta}\right]$.
2. $\frac{\frac{1}{2}N^{1-\delta}(\frac{1}{2}N^{1-\delta}-1)}{2} \le \mathcal{O} \le \frac{\frac{3}{2}N^{1-\delta}(\frac{3}{2}N^{1-\delta}-1)}{2}$.

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Proof:

- (1) is Chebyshev: $\mathrm{Var}(X) = N\mathrm{Var}(X_{n;N}) \le N^{1-\delta}$.
- (2) follows from (1) and $\binom{r}{2}$ ways to choose 2 from r.

Concentration

Lemma

- $f(\delta) = \min\left(\frac{1}{2}, \frac{3\delta - 1}{2}\right)$, $g(\delta)$ *satisfies* $0 < g(\delta) < f(\delta)$.
- $p(N) = N^{-\delta}$, $\delta \in (1/2, 1)$, $P_1(N) = 4N^{-(1-\delta)}$, $P_2(N) = CN^{-(f(\delta) - g(\delta))}$.

With probability at least $1 - P_1(N) - P_2(N)$ *have* $\mathcal{D}/\mathcal{S} = 2 + O(N^{-g(\delta)})$.

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With probability at least $1 - P_1(N) - P_2(N)$ *have* $\mathcal{D}/\mathcal{S} = 2 + O(N^{-g(\delta)})$.

Proof: Show $\mathcal{D} \sim 2\mathcal{O} + O(N^{3-4\delta})$, $\mathcal{S} \sim \mathcal{O} + O(N^{3-4\delta})$.

As $\mathcal{O}$ is of size $N^{2-2\delta}$ with high probability, need $2 - 2\delta > 3 - 4\delta$ or $\delta > 1/2$.

Analysis of $\mathcal{D}$

Contribution from 'diagonal' terms lower order, ignore.

Difficulty: (m,n) and (m',n') could yield same differences.

Notation: $m < n$, $m' < n'$, $m \le m'$,

$$Y_{m,n,m',n'} = \begin{cases} 1 & \text{if } n - m = n' - m' \\ 0 & \text{otherwise.} \end{cases}$$

$\mathbb{E}[Y] \le N^3 \cdot N^{-4\delta} + N^2 \cdot N^{-3\delta} \le 2N^{3-4\delta}$. As $\delta > 1/2$, $\#\{\text{bad pairs}\} \lll \mathcal{O}$.

Claim: $\sigma_Y \le N^{r(\delta)}$with $r(\delta) = \frac{1}{2}\max(3-4\delta, 5-7\delta)$. This and Chebyshev conclude proof of theorem.

Proof of claim

Cannot use CLT as $Y_{m,n,m',n'}$ are not independent.

Use $\mathrm{Var}(U+V) \leq 2\mathrm{Var}(U) + 2\mathrm{Var}(V)$.

Write

$$\sum Y_{m,n,m',n'} \ = \ \sum U_{m,n,m',n'} + \sum V_{m,n,n'}$$

with all indices distinct (at most one in common, if so must be $n = m'$).

$$\mathrm{Var}(U) = \sum \mathrm{Var}(U_{m,n,m',n'}) + 2 \sum_{\substack{(m,n,m',n') \neq \\ (\widetilde{m},\widetilde{n},\widetilde{m}',\widetilde{n}')}} \mathrm{CoVar}(U_{m,n,m',n'}, U_{\widetilde{m},\widetilde{n},\widetilde{m}',\widetilde{n}'}).$$

Analyzing $\mathrm{Var}(U_{m,n,m',n'})$

At most N^3 tuples.

Each has variance $N^{-4\delta} - N^{-8\delta} \leq N^{-4\delta}$.

Thus $\sum \mathrm{Var}(U_{m,n,m',n'}) \leq N^{3-4\delta}$.

Analyzing $\mathrm{CoVar}(U_{m,n,m',n'}, U_{\widetilde{m},\widetilde{n},\widetilde{m}',\widetilde{n}'})$

- All 8 indices distinct: independent, covariance of 0.

- 7 indices distinct: At most N^3 choices for first tuple, at most N^2 for second, get

$$\mathbb{E}[U_{(1)}U_{(2)}] - \mathbb{E}[U_{(1)}]\mathbb{E}[U_{(2)}] = N^{-7\delta} - N^{-4\delta}N^{-4\delta} \leq N^{-7\delta}.$$

- Argue similarly for rest, get $\ll N^{5-7\delta} + N^{3-4\delta}$.

Ongoing Research

Current and Open Problems

- Similar results for arbitrary finite groups (with Kevin Vissuet).
- Generalize phase transition results for more summands (SMALL '13 hopefully).
- Generalize to subsets of $\mathbb{Z}^+ \times \mathbb{Z}^+$ (SMALL '13 hopefully).
- Study the dependence of the divot on $p(N)$.

Divot: Lazarev - Miller - O'Bryant

Let $m(k)$ be the probability a uniformly drawn subset A of $[0, n]$ has $A + A$ missing exactly k summands as $n \to \infty$.

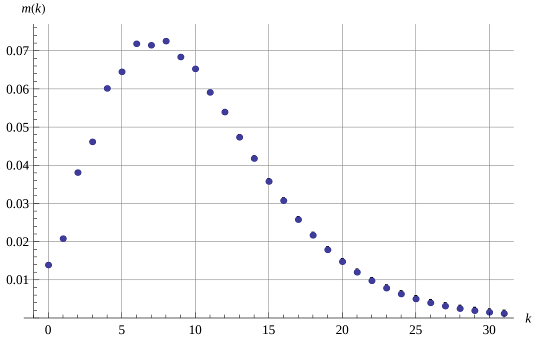

Figure: Experimental values of $m(k)$, vertical bars error (often smaller than dot!).

What happens if draw A from binomial with parameter $p(N)$?

Generalization of main result

Theorem (Hegarty-M): Binomial model with parameter $p(N)$ as before, u, v be non-zero integers with $u \geq |v|$, $\gcd(u, v) = 1$ and $(u, v) \neq (1, 1)$. Put $f(x, y) := ux + vy$ and let $\mathcal{D}_f$ denote the random variable $|f(A)|$. Then the following three situations arise:

1. $p(N) = o(N^{-1/2})$: Then
$$\mathcal{D}_f \sim (N \cdot p(N))^2.$$
2. $p(N) = c \cdot N^{-1/2}$ for some $c \in (0, \infty)$: Define the function $g_{u,v} : (0, \infty) \to (0, u + |v|)$ by
$$g_{u,v}(x) := (u + |v|) - 2|v| \left(\frac{1 - e^{-x}}{x} \right) - (u - |v|)e^{-x}.$$
Then
$$\mathcal{D}_f \sim g_{u,v}\left(\frac{c^2}{u} \right) N.$$
3. $N^{-1/2} = o(p(N))$: Let $\mathcal{D}_f^c := (u + |v|)N - \mathcal{D}_f$. Then $\mathcal{D}_f^c \sim \frac{2u|v|}{p(N)^2}$.

Generalization of Hegarty-Miller

Let f, g be two binary linear forms. Say f dominates g for the parameter $p(N)$ if, as $N \to \infty$, $|f(A)| > |g(A)|$ almost surely when A is a random subset (binomial model with parameter $p(N)$).

Theorem (Hegarty-M): $f(x, y) = u_1 x + u_2 y$ and $g(x, y) = u_2 x + g_2 y$, where $u_i \geq |v_i| > 0$, $\gcd(u_i, v_i) = 1$ and $(u_2, v_2) \neq (u_1, \pm v_1)$. Let

$$\alpha(u, v) := \frac{1}{u^2}\left(\frac{|v|}{3} + \frac{u - |v|}{2}\right) = \frac{3u - |v|}{6u^2}.$$

The following two situations can be distinguished :

- $u_1 + |v_1| \geq u_2 + |v_2|$ and $\alpha(u_1, v_1) < \alpha(u_2, v_2)$. Then f dominates g for all p such that $N^{-3/5} = o(p(N))$ and $p(N) = o(1)$. In particular, every other difference form dominates the form $x - y$ in this range.
- $u_1 + |v_1| > u_2 + |v_2|$ and $\alpha(u_1, v_1) > \alpha(u_2, v_2)$. Then there exists $c_{f,g} > 0$ such that one form dominates for $p(N) < cN^{-1/2}$ $(c < c_{f,g})$ and other dominates for $p(N) > cN^{-1/2}$ $(c > c_{f,g})$.

Open Problems

- One unresolved matter is the comparison of arbitrary difference forms in the range where $N^{-3/4} = O(p)$ and $p = O(N^{-3/5})$. Note that the property of one binary form dominating another is not monotone, or even convex.

- A very tantalizing problem is to investigate what happens while crossing a sharp threshold.

- One can ask if the various concentration estimates can be improved (i.e., made explicit).

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