Sum of Consecutive Terms of Pell and Related Sequences

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Motivation

Definition

Let r be a non-negative integer. Consider a sequence $\{f(n)\}$ of non-negative integers recursively defined by

$$f(n) \; := \; rf(n-1) + f(n-2)$$

with initial conditions so that it is not identically zero (we call this a non-degenerate sequence).

Not only is this a natural generalization of the Fibonacci numbers (which are just the $r = 1$ case), but similar to how the Fibonacci numbers count various objects, this sequence as well has a combinatorial interpretation. In [DHW] the authors show that $f(n)$ is the number of k-regular words over $\{1, 2, \ldots, n\}$ avoiding the patterns 122 and 213 (this means we cannot form a sub-word with three objects with this relative ordering).

Motivation

Additionally, $\{f(n)\}$ also makes a surprising appearance in elliptic curve research. Recent work in [PiWa] shows under certain circumstances, there exists a bijection between the set of integral points on elliptic curves of the form $y^2 = (r^2+4)x^4 - 4$ and the set of squares in $\{f(n)\}$ with odd indices.

Pell Numbers

Definition

Pell numbers occur in the infinite sequence defined by the recurrence

$$P(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ 2P(n-1) + P(n-2) & n \geq 2. \end{cases} \quad (1)$$

We call the sequence defined by the above recursion *the Pell sequence*. The first few numbers of the Pell sequence are 0, 1, 2, 5, 12, 29, 70, 169, 408, and 985. They are also known as the 2− Fibonacci numbers.

Lucas Numbers

Definition

Lucas numbers occur in the infinite sequence defined by the recurrence

$$L(n) = \begin{cases} 2 & n = 0 \\ 1 & n = 1 \\ L(n-1) + L(n-2) & n \geq 2. \end{cases} \quad (2)$$

We call the sequence defined by the above recursion *the Lucas sequence*. The first few numbers of the Lucas sequence are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123$ etc.

Closed Form of Fibonacci Numbers

The Fibonacci numbers have a closed form, given by *Binet's formula*:

$$F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{3}$$

where φ is the golden ratio and ψ is its negative inverse. These can be written explicitly as

$$\varphi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \psi = \frac{1-\sqrt{5}}{2} = -\frac{1}{\varphi}.$$

Binet Formula

Definition

If terms of a recursively defined infinite sequence can be given in a closed form, we call the closed form *a Binet-like formula* or simply *a Binet formula*.

Example

Example: Let

$$a = 1 + \sqrt{2} \quad \text{and} \quad b = 1 - \sqrt{2} = -\frac{1}{a}. \tag{4}$$

Then the n^{th} Pell Number has a Binet formula given by

$$P(n) = \frac{a^n - b^n}{2\sqrt{2}}. \tag{5}$$

An Observation

Theorem

The sum of any four consecutive Pell Numbers is 4 times the third term of the consecutive terms. In other words, the equality

$$\sum_{i=0}^{3} P(n+i) = 4P(n+2) \tag{6}$$

for all n.

A Stronger Observation

Theorem

For any $N \in \mathbb{N}$, the sum of any $4N$ consecutive Pell numbers is equal to a constant (dependent on N) multiplied by the $(2N+1)$st term of the consecutive terms. In particular,

$$\sum_{i=0}^{4N-1} P(n+i) = \frac{(a^{2N} - b^{2N})}{\sqrt{2}} P(n+2N) \tag{7}$$

where is is easy to see that $\frac{(a^{2N} - b^{2N})}{\sqrt{2}}$ *is an integer.*

This result can be proven algebraically using the Binet formula for Pell numbers and the fact that $ab = -1$.

Question

For which numbers $N \in \mathbb{N}$, does the sum of N consecutive Pell numbers equal a fixed constant, depending solely on N, times another Pell number?

4N+2 consecutive Pell numbers

Definition

The *Pell-Lucas sequence* or the *Companion Pell sequence* is a sequence of natural numbers defined by

$$Q(n) = \begin{cases} 2 & n = 0, 1 \\ 2Q(n-1) + Q(n-2) & n \geq 2. \end{cases} \tag{8}$$

Theorem

Fix any integer N. Then, there is no integer $C(N)$ such that for every n there exists an integer index $j(n;N)$ such that the following equation holds:

$$\sum_{i=0}^{4N+1} P(n+i) = C(N)P(j(n;N)).$$

Proof Sketch

Lemma

$$P(n+k)+(-1)^k P(n-k) \;=\; Q(k)P(n), \qquad k \in \mathbb{N} \cup \{0\}$$

Lemma

$$\sum_{k=0}^{n} P(k) = \tfrac{1}{2}(P(n+1)+P(n)-1)$$

1. Prove the above lemmas.
2. Use these lemmas to conclude that the sum of any $4N+2$ consecutive Pell numbers is a fixed integer multiple of the sum of two consecutive Pell numbers.

$$\sum_{k=0}^{4N+1} P(n+k) = \frac{Q(2N+1)}{2}\left(P(n+2N+1)+P(n+2N)\right)$$

Proof Sketch

3. Define the *Pell sum sequence* $R(n) = P(n) + P(n+1)$.

4. Consider ratios of consecutive terms of the *Pell sequence* and the *Pell sum sequence*.

5. Conclude that consecutive terms of these sequences are relatively prime.

6. Use the observations made in steps 4 and 5 to prove that the sum of any $4N+2$ consecutive Pell numbers is not a fixed constant times another Pell number.

2N+1 consecutive Pell numbers

Theorem

Fix any integer $N > 0$. Then, there is no integer $C(N)$ such that for every n there exists an integer index $j(n; N)$ such that the following equation holds:

$$\sum_{i=0}^{2N} P(n+i) = C(N)P(\,j(n; N)\,).$$

4N+2 consecutive terms

Theorem

Consider sequences of the form $f(n) = rf(n-1) + f(n-2)$ *with* $r \in \mathbb{N}$, $r \geq 2$, $f(0) = 0$ *and* $f(1) = 1$. *Then, there is no integer* $C(N)$ *such that for every* n *there exists an integer index* $j(n; N)$ *such that the following equation holds:*

$$\sum_{i=0}^{4N+1} P(n+i) = C(N)P(j(n; N)).$$

Proof.

Similar to the proof shown for Pell numbers. Our companion sequence is $g(n) = r \cdot g(n-1) + g(n-2)$ with $g(0) = 2$ and $g(1) = r$. The following theorem works for $r = 1$. □

4N+2 consecutive terms

Theorem

The sum of $4N+2$ consecutive Fibonacci numbers is an integer multiple of the $(2N+3)^{rd}$ term in the sum, where the fixed constant is $L(2N+1)$, the $(2N+1)^{st}$ Lucas number.

Proof.

First, note that $\sum_{i=0}^{n} F(i) = F(n+2) - 1$. A straightforward induction yields $F(n+k) + (-1)^k F(n-k) = L(k)F(n)$. A straightforward manipulation now yields

$$\sum_{i=0}^{4N+1} F(n+i) = F(n+4N+3) - F(n+1) = L(2N+1)F(n+2N+2), \quad (9)$$

which completes the proof. □

2N+1 consecutive terms

Theorem

Consider a sequence of the form $\mathbf{f(n) = rf(n-1) + f(n-2)}$ *with* $r \in \mathbb{N}$, $r \geq 3$, $f(0) = 0$, *and* $f(1) = 1$. *Then the sum of any* $2N+1$ *consecutive terms of the sequence is a constant integer multiple of a term in the sequence if and only if* $N = 0$.

In this case, the "generating matrix" for the sequence is $\begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix}$.

From here, we can proceed with the same proof used for sums of odd numbers of consecutive Pell and Fibonacci numbers to prove this theorem.

Other Results

Theorem

Let $F(n)$ denote the nth Fibonacci number. Fix any integer $N > 0$. Then, there is no integer $C(N)$ such that for every n there exists an integer index $j(n;N)$ such that the following equation holds:

$$\sum_{i=0}^{2N} F(n+i) = C(N)F(\,j(n;N)\,).$$

We note that a trivial equality holds for $N = 0$.

Other Results

Define the order-k generalized Fibonacci sequence by

$$f_k(n) := \sum_{i=1}^{k} f_k(n-i) \tag{10}$$

with $f_k(1) = f_k(2) = \cdots = f_k(k-1) = 0$ and $f_k(k) = 1$.

Theorem

Let $F_k(n)$ denote the n^{th} order-k Fibonacci number, let $j(n;N), C(N) \in \mathbb{Z}$ be defined as above. Then,

$$\sum_{i=0}^{2N} F_k(n+i) = C(N) \cdot F_k(j(n;N))$$

has no solution which holds for all n, when $2 \mid k$ and $N > 2k+1$.

Download the paper and give it a read!

1. Dr. Miller's website :
https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/PellAndGeneralizations_ConsecSum_Polymath2023v20.pdf
2. ResearchGate:
https://www.researchgate.net/publication/381996155_Sum_of_Consecutive_Terms_of_Pell_and_Related_Sequences

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