Linear Recurrences from Schreier Multisets

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Schreier sets

A finite set $F \subset \mathbb{N}$ is said to be Schreier if $F = \emptyset$ or $\min F \geq |F|$.

Schreier: $\{2, 3\}, \{4, 5, 10\}$

Not Schreier: $\{3, 7, 10, 13\}$

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$$\mathcal{A}_n \;=\; \{F \subset \{1, 2, \ldots, n\} \;:\; n \in F \text{ and } F \text{ is Schreier}\}$$

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$$\mathcal{A}_4 = \{\{2, 4\}, \{3, 4\}, \{4\}\} \implies |\mathcal{A}_4| = 3$$

$$\mathcal{A}_5 = \{\{2, 5\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}, \{5\}\} \implies |\mathcal{A}_5| = 5$$

Schreier sets and the Fibonacci sequence

A. Bird (2012)

For $n \in \mathbb{N}$,

$$|\{F \subseteq \{1, 2, \ldots, n\} : n \in F \text{ and } F \text{ is Schreier}\}| = F_n,$$

where $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

A general problem

For $(s, p, q, n) \in \mathbb{N}^4$,

$$\mathcal{A}_{p,q,n}^{(s_1,\ldots,s_n)} := \{F \subset \{\underbrace{1,\ldots,1}_{s_1},\ldots,\underbrace{n-1,\ldots,n-1}_{s_{n-1}},\underbrace{n,\ldots,n}_{s_n}\} : n \in F \text{ and } q\min F \geq p|F|\}$$

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What is the sequence $(|\mathcal{A}_{p,q,n}^{(s_1,\dots,s_n)}|)_{n=1}^{\infty}$?

The special case $q = 1$, $s_1 = \cdots = s_{n-1} = s$, and $s_n = 1$

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$\{2, 2\}, \{3, 3, 6\} \in \mathcal{A}_{1,7}^{(3)}$

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$\{3, 3, 6, 7\} \notin \mathcal{A}_{1,7}^{(3)}$

Count uncolored Schreier multisets

For $(s, p) \in \mathbb{N}^2$, what is the sequence $(|\mathcal{A}_{p,n}^{(s)}|)_{n=1}^{\infty}$?

Data

$$|\mathcal{A}^{(2)}_{1,n}| : \quad 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, \ldots$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

$$|\mathcal{A}^{(2)}_{2,n}| : \quad 0, 1, 1, 1, 2, 3, 5, 8, 12, 19, 30, 47, 74, 116, \ldots$$

$$a_n = a_{n-1} + a_{n-3} + a_{n-5}$$

$$|\mathcal{A}^{(3)}_{1,n}| : \quad 1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, \ldots$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$$

$$|\mathcal{A}^{(3)}_{2,n}| : \quad 0, 1, 1, 1, 2, 3, 5, 8, 13, 21, 33, 53, 85, \ldots$$

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Data

$|\mathcal{A}_{1,n}^{(2)}|$: $1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, \dots$

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Number of terms: $\boxed{s+1}$; **gap btwn consecutive indices:** $\boxed{p}$

Theorem

Polymath Jr. 25 Schreier Group

For $(s, p) \in \mathbb{N}^2$ and $n \geq sp + 2$,

$$|\mathcal{A}_{p,n}^{(s)}| \; = \; |\mathcal{A}_{p,n-1}^{(s)}| + |\mathcal{A}_{p,n-1-p}^{(s)}| + |\mathcal{A}_{p,n-1-2p}^{(s)}| + \cdots + |\mathcal{A}_{p,n-1-sp}^{(s)}|.$$

Proof

$$I_s(n) := \{\underbrace{1, \ldots, 1}_{s}, \ldots, \underbrace{n-1, \ldots, n-1}_{s}, n\}$$

$$c(F, k) := \text{number of copies of } k \text{ in } F$$

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$$\mathcal{A}_{p,n}^{(s)}$$

$$= \{F \subset I_s(n) : n \in F, c(F,n-1) = 0, \min F \geq p|F|\} \cup$$
$$\{F \subset I_s(n) : n \in F, c(F,n-1) = 1, \min F \geq p|F|\} \cup$$
$$\{F \subset I_s(n) : n \in F, c(F,n-1) = 2, \min F \geq p|F|\} \cup$$
$$\ldots$$
$$\{F \subset I_s(n) : n \in F, c(F,n-1) = s, \min F \geq p|F|\}$$

Proof

$$\mathcal{A}_{p,n}^{(s)} = \bigcup_{k=0}^{s} \{F \subset I_s(n) : n \in F, c(F, n-1) = k, \min F \geq p|F|\}$$

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$$\mathcal{A}_{p,n-1-kp}^{(s)} \to \{F \subset I_s(n) : n \in F, c(F, n-1) = k, \min F \geq p|F|\}$$

$$F \mapsto \begin{cases} (F \backslash \{n-1\}) \cup \{n\}, & \text{if } k = 0; \\ (F + kp) \cup \{\underbrace{n-1, \ldots, n-1}_{k-1}, n\}, & \text{if } k \geq 1. \end{cases}$$

Proof

$$\mathcal{A}^{(s)}_{p,n} = \bigcup_{k=0}^{s} \{F \subset I_s(n) : n \in F, c(F, n-1) = k, \min F \geq p|F|\}$$

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$$|\mathcal{A}^{(s)}_{p,n}| = |\mathcal{A}^{(s)}_{p,n-1}| + |\mathcal{A}^{(s)}_{p,n-1-p}| + |\mathcal{A}^{(s)}_{p,n-1-2p}| + \cdots + |\mathcal{A}^{(s)}_{p,n-1-sp}|$$

Colored Schreier multisets

Let k_i denote the integer k with color i.

$$B_n^{(s)} = \{1_1, 1_2, \ldots, 1_s, \ldots, (n-1)_1, \ldots, (n-1)_s, n\}$$

$$\mathcal{B}_{p,n}^{(s)} := \{F \subset B_n^{(s)} : n \in F \text{ and } \min F \geq p|F|\}.$$

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For $(s, p) \in \mathbb{N}^2$, what is the sequence $(|\mathcal{B}_{p,n}^{(s)}|)_{n=1}^{\infty}$?

Data

$\mathcal{B}^{(2)}_{1,n}$: 1, 1, 3, 6, 13, 28, 60, 129, 277, 595, 1278, 2745, 5896, . . .

$$a_n = a_{n-1} + 2a_{n-2} + a_{n-3}$$

$\mathcal{B}^{(2)}_{2,n}$: 0, 1, 1, 1, 3, 5, 8, 15, 26, 45, 80, 140, 245, 431, . . .

$$a_n = a_{n-1} + 2a_{n-3} + a_{n-5}$$

$\mathcal{B}^{(3)}_{1,n}$: 1, 1, 4, 10, 26, 69, 181, 476, 1252, 3292, 8657, 22765, 59864, . . .

$$a_n = a_{n-1} + 3a_{n-2} + 3a_{n-3} + a_{n-4}$$

$\mathcal{B}^{(3)}_{2,n}$: 0, 1, 1, 1, 4, 7, 13, 28, 53, 105, 211, 413, 819, 1624, 3206, . . .

$$a_n = a_{n-1} + 3a_{n-3} + 3a_{n-5} + a_{n-7}$$

Data

$\mathcal{B}_{1,n}^{(2)}$: 1, 1, 3, 6, 13, 28, 60, 129, 277, 595, 1278, 2745, 5896, ...

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Coefficients: $\boxed{\binom{s}{i}}$, $0 \le i \le s$; **gap btwn consecutive indices:** $\boxed{p}$

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For $(s, p) \in \mathbb{N}^2$ and $n \geq sp + 2$,

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Step 1: Find $(b^s_{p,n})_{n=1}^{\infty}$ such that $(|\mathcal{B}^s_{p,n}|)_{n=1}^{\infty}$ is an s-periodic subsequence of $(b^s_{p,n})_{n=1}^{\infty}$

$(b^s_{p,n})_{n=1}^{\infty}$ satisfies the polynomial $p(x)$

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$(b^s_{p,n})_{n=1}^{\infty}$ satisfies the polynomial $p(x)$

Step 2: To prove $(|\mathcal{B}^s_{p,n}|)_{n=1}^{\infty}$ satisfies a recurrence encoded by $q(x)$, it suffices to verify $p(x)$ divides $q(x^s)$

Sample case: $\mathcal{B}_{1,n}^{(2)}$

Step 1: Find $b_{1,n}^{(2)}$

$$\mathcal{B}_{1,n}^{(2)} : 1, 1, 3, 6, 13, 28, 60, 129, 277, 595, 1278, 2745, 5896, \ldots$$

$$b_{1,n}^{(2)} : 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, \ldots$$

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$$b_{1,1}^{(2)} = b_{1,2}^{(2)} = b_{1,3}^{(2)} = 1 \text{ and } b_{1,n}^{(2)} = b_{1,n-1}^{(2)} + b_{1,n-3}^{(2)}, n \geq 4$$

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$$b^{(2)}_{1,1} = b^{(2)}_{1,2} = b^{(2)}_{1,3} = 1 \text{ and } b^{(2)}_{1,n} = b^{(2)}_{1,n-1} + b^{(2)}_{1,n-3}, n \geq 4$$

$(b^{(2)}_{1,n})^{\infty}_{n=1}$ satisfies $p(x) = 1 - x - x^3$.

Sample case: $\mathcal{B}_{1,n}^{(2)}$

Step 2: To prove $(|\mathcal{B}_{1,n}^{2}|)_{n=1}^{\infty}$ satisfies a recurrence encoded by $q(x) = 1 - x - 2x^2 - x^3$, it suffices to verify $p(x)$ divides $q(x^2)$.

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$$q(x^2) \; = \; 1 - x^2 - 2x^4 - x^6 \; = \; \underbrace{(1 - x - x^3)}_{p(x)}(1 + x + x^3)$$

Future investigations

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Study the more general condition $q \min F \geq p|F|$

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Study nonlinear Schreier-type conditions for multisets:
$(\min F)^q \geq |F|^p$