A Pair of Diophantine Equations Involving Fibonacci Numbers

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A Pair of Equations

For relatively prime $a, b \in \mathbb{N}$, consider

$$ax + by = \frac{(a-1)(b-1)}{2}$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2}$$

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Theorem (Beiter (1964), extended by Chu (2020))

Exactly one of the equations has a **nonnegative integral** solution (x, y). The solution is unique.

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$$ax + by = \frac{(a-1)(b-1)}{2} \quad (1)$$

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Define $\Gamma : \{(a, b) \in \mathbb{N}^2 : \gcd(a, b) = 1\} \to \{1, 2\}$:

$$\Gamma(a, b) = \begin{cases} 1, & \text{if (1) has a solution;} \\ 2, & \text{if (2) has a solution.} \end{cases}$$

Fibonacci Numbers

$$F_1 = 1, \; F_2 = 1, \; F_3 = 2, \; F_4 = 3, \; F_5 = 5, \; F_6 = 8, \ldots$$

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$$a \boxed{x} + b \boxed{y} = \frac{(a-1)(b-1)}{2} \tag{1}$$

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$$F_n \boxed{x} + F_{n+1} \boxed{y} = \frac{(F_n - 1)(F_{n+1} - 1)}{2} \quad (1)$$

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$$\Gamma(F_n, F_{n+1}) = ?$$

Previous work

Chu 2020:

n	3	4	5	6	7	8
$\Gamma(F_n, F_{n+1})$	2	2	2	1	1	1

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$$F_{6k} \cdot \boxed{\frac{F_{6k-1}-1}{2}} + F_{6k+1} \cdot \boxed{\frac{F_{6k-1}-1}{2}} = \frac{(F_{6k}-1)(F_{6k+1}-1)}{2}$$

$$1 + F_{6k+3} \cdot \boxed{\frac{F_{6k+2}-1}{2}} + F_{6k+4} \cdot \boxed{\frac{F_{6k+2}-1}{2}} = \frac{(F_{6k+3}-1)(F_{6k+4}-1)}{2}$$

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R. K. Davala 2023: Identities for other famous sequences

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R. K. Davala 2023: Identities for other famous sequences

Polymath Jr. 2023: Characterize (a, b) for $\Gamma(a, b) = 1$ or 2

Our goals

Chu 2020:

$$F_{6k} \cdot \boxed{\frac{F_{6k-1}-1}{2}} + F_{6k+1} \cdot \boxed{\frac{F_{6k-1}-1}{2}} = \frac{(F_{6k}-1)(F_{6k+1}-1)}{2}$$

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$$\Gamma(F_n^2, F_{n+1}^2) = ?$$

Are there similar identities for Fibonacci squared?

Fibonacci Squared - The Data

$$ax + by = \frac{(a-1)(b-1)}{2} \quad (1)$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2} \quad (2)$$

n	F_n^2	F_{n+1}^2	x	y	$\Gamma(F_n^2, F_{n+1}^2)$
2	1	4	0	0	(1)
3	4	9	3	0	(1)
4	9	25	5	2	(2)
5	25	64	20	4	(1)
6	64	169	51	12	(1)
7	169	441	83	52	(2)

Table: Fibonacci squared

Fibonacci Squared - The Data

n	F_n^2	F_{n+1}^2	x	y	$\Gamma(F_n^2, F_{n+1}^2)$	$F_n^2 - x - y$
2	1	4	0	0	(1)	**1**
3	4	9	3	0	(1)	**1**
4	9	25	5	2	(2)	2
5	25	64	20	4	(1)	**1**
6	64	169	51	12	(1)	**1**
7	169	441	83	52	(2)	34
8	441	1156	356	84	(1)	**1**
9	1156	3025	935	220	(1)	**1**
10	3025	7921	1513	934	(2)	578
⋮	⋮	⋮	⋮	⋮	⋮	⋮
13	54289	142129	27143	16776	(2)	10370

Table: Fibonacci Squared Pattern

Fibonacci Squared - The Data

n	F_n^2	F_{n+1}^2	x	y	$\Gamma(F_n^2, F_{n+1}^2)$	$F_n^2 - x - y$
2	1	4	0	0	(1)	1
3	4	9	3	0	(1)	1
4	**9**	25	**5**	2	(2)	2
5	25	64	20	4	(1)	1
6	64	169	51	12	(1)	1
7	**169**	441	**83**	52	(2)	34
8	441	1156	356	84	(1)	1
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Identities for Fibonacci Squared

Theorem

For $n \geq 7$ with $n \equiv 1 \mod 6$, the following holds

$$F_n^2 \cdot \boxed{\frac{F_n^2 - 3}{2}} + F_{n+1}^2 \cdot \boxed{\frac{F_n^2 - F_{n-1}^2 - 1}{2}} + 1 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

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Theorem

For $n \geq 1$ with $n \equiv 4 \mod 6$, the following holds

$$F_n^2 \cdot \boxed{\frac{F_n^2 + 1}{2}} + F_{n+1}^2 \cdot \boxed{\frac{F_n^2 - F_{n-1}^2 - 1}{2}} + 1 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

Identities for Fibonacci Squared

Theorem

For $n \geq 1$ with $n \equiv 0, 2, 3, 5 \mod 6$, the following holds

$$F_n^2 \cdot \boxed{\left(F_n^2 - \frac{F_{n-1}^2 + 1}{2}\right)} + F_{n+1}^2 \cdot \boxed{\frac{F_{n-1}^2 - 1}{2}} = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

Proof

Theorem

For $n \geq 1$ with $n \equiv 0, 2, 3, 5 \mod 6$, the following identity holds

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Proof:

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Proof:

$$F_n = F_{n+1} - F_{n-1} \Longrightarrow F_n^2 - F_{n-1}^2 - F_{n+1}^2 = -2F_{n-1}F_{n+1}$$

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Theorem

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$$\left(F_{n-1}F_{n+1} - F_n^2\right)^2 = ((-1)^n)^2 \quad \text{(Cassini Identity)}$$

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$$\Longrightarrow F_n^4 - 2F_n^2 F_{n-1}F_{n+1} + F_{n-1}^2 F_{n+1}^2 = 1$$

Proof

$$F_n^2 - F_{n-1}^2 - F_{n+1}^2 = -2F_{n-1}F_{n+1}$$
$$F_n^4 - 2F_n^2 F_{n-1} F_{n+1} + F_{n-1}^2 F_{n+1}^2 = 1$$

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$$(2F_n^2 - F_{n-1}^2 - 1) \cdot F_n^2 + (F_{n-1}^2 - 1) \cdot F_{n+1}^2 = (F_n^2 - 1)(F_{n+1}^2 - 1)$$

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$$(2F_n^2 - F_{n-1}^2 - 1) \cdot F_n^2 + (F_{n-1}^2 - 1) \cdot F_{n+1}^2 = (F_n^2 - 1)(F_{n+1}^2 - 1)$$

$$\left(F_n^2 - \frac{F_{n-1}^2 + 1}{2}\right) \cdot F_n^2 + \left(\frac{F_{n-1}^2 - 1}{2}\right) \cdot F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}$$

Fibonacci Cubed - The Data

What about Fibonacci cubed?

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What about Fibonacci cubed?

n	F_n^3	F_{n+1}^3	x_n	y_n	$\Gamma(F_n^3, F_{n+1}^3)$
2	1	8	0	0	(1)
3	8	27	8	1	(1)
4	27	125	18	9	(2)
5	125	512	106	36	(1)
6	512	2197	405	161	(2)
7	2197	9261	1791	673	(1)

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Table: Fibonacci cubed

$$x_4 = 3^3 - x_3 - 1; \quad y_4 = y_3 + 2^3$$
$$x_5 = 5^3 - x_4 - 1; \quad y_5 = y_4 + 3^3$$
$$x_6 = 8^3 - x_5 - 1; \quad y_6 = y_5 + 5^3$$

Fibonacci Cubed - The Data

$$\begin{cases} x_n &= F_n^3 - x_{n-1} - 1 \\ y_n &= y_{n-1} + F_{n-1}^3. \end{cases}$$

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Conjecture

Let $n \in \mathbb{N}$ with $n \geq 2$. The following hold

$$\left[\sum_{k=1}^{2n-1} (-1)^{k-1} F_k^3\right] F_{2n-1}^3 + \left(\sum_{k=2}^{2n-2} F_k^3\right) F_{2n}^3 = \frac{(F_{2n-1}^3 - 1)(F_{2n}^3 - 1)}{2},$$

$$1 + \left[\sum_{k=1}^{2n} (-1)^k F_k^3 - 1\right] F_{2n}^3 + \left(\sum_{k=2}^{2n-1} F_k^3\right) F_{2n+1}^3 = \frac{(F_{2n}^3 - 1)(F_{2n+1}^3 - 1)}{2}.$$

Ongoing proof

Frontczak's formulas (2018) to replace sums of Fibonacci cubes:

$$\left(\frac{F_{6n}}{4} - \frac{F_{6n-3}}{4} - F_{2n}^3 + F_{2n-1}^3 - \frac{1}{2}\right) F_{2n-1}^3$$
$$+ \left(\frac{F_{6n-3}}{4} + \frac{F_{6n-6}}{4} - F_{2n-1}^3 - F_{2n-2}^3 - \frac{1}{2}\right) F_{2n}^3$$
$$= \frac{(F_{2n-1}^3 - 1)(F_{2n}^3 - 1)}{2}$$

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$$\left(\frac{F_{6n}}{4} - \frac{F_{6n-3}}{4} - F_{2n}^3 + F_{2n-1}^3 - \frac{1}{2}\right) F_{2n-1}^3 + \left(\frac{F_{6n-3}}{4} + \frac{F_{6n-6}}{4} - F_{2n-1}^3 - F_{2n-2}^3 - \frac{1}{2}\right) F_{2n}^3 = \frac{(F_{2n-1}^3 - 1)(F_{2n}^3 - 1)}{2}$$

$$\left(F_{6n-2} + 2F_{2n-1}^3\right) F_{2n-1}^3 + \left(F_{6n-4} - 2F_{2n-2}^3\right) F_{2n}^3 = 5F_{2n-1}^3 F_{2n}^3 + 1$$

Ongoing proof

$$\left(F_{6n-2} + 2F_{2n-1}^3\right) F_{2n-1}^3 + \left(F_{6n-4} - 2F_{2n-2}^3\right) F_{2n}^3 \;=\; 5F_{2n-1}^3 F_{2n}^3 + 1$$

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Binet's formula to obtain an identity involving $\phi = (1 + \sqrt{5})/2$.

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Linearize ϕ^n by $\phi F_n + F_{n-1}$

Ongoing proof

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Linearize ϕ^n by $\phi F_n + F_{n-1}$

An identity linear in both ϕ and F_n's

Another Pair of Diophantine Equations

Theorem

Given $a, b \in \mathbb{N}$ with $(a, b) = 1$, consider the two following equations.

$$\begin{aligned} xa + yb &= \frac{(a-1)(b-1)}{2}, \\ 1 + xa + yb &= \frac{(a-1)(b-1)}{2}. \end{aligned}$$

Exactly one of the equations above has a **nonnegative integral** solution (x, y), and the solution is unique.

Another Pair of Diophantine Equations

$(a, b) = (7, 8)$ gives $(x, y) = (3, 0)$:

$$3 \cdot 7 + 0 \cdot 8 = \frac{(7-1)(8-1)}{2}.$$

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Can we find a pair of equations that give only **positive** solutions?

Another Pair of Diophantine Equations

Theorem

Given $a, b \in \mathbb{N}$ with $(a, b) = 1$, consider the two following equations

$$\begin{aligned} xa + yb &= \frac{(a-1)(b-1)}{2} + (a+b), \\ 1 + xa + yb &= \frac{(a-1)(b-1)}{2} + (a+b). \end{aligned}$$

Exactly one of the equations above has a **positive integral** solution (x, y), and the solution is unique.

Another Pair of Diophantine Equations

$(a, b) = (7, 8)$:

$$\text{Old:} \quad 3 \cdot 7 + 0 \cdot 8 = \frac{(7-1)(8-1)}{2}$$

$$\text{New:} \quad 4 \cdot 7 + 1 \cdot 8 = \frac{(7-1)(8-1)}{2} + (7+8)$$

$$(3, 0) \rightarrow (3+1, 0+1)$$

Another Pair of Diophantine Equations

Theorem

Given $a, b \in \mathbb{N}$ with $(a, b) = 1$, $b \geq 2$, and $(a+1)b \equiv 0 \mod 2$, consider the two following equations

$$xa + yb = \frac{(a+1)b}{2} + 1, \tag{3}$$

$$xa + yb = \frac{(a+1)b}{2} - 1. \tag{4}$$

Exactly one of the two equations has a **positive integral** solution (x, y), and the solution is unique.

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Given $a, b \in \mathbb{N}$ with $(a, b) = 1$, $b \geq 2$, and $(a+1)b \equiv 0 \mod 2$, consider the two following equations

$$xa + yb = \frac{(a+1)b}{2} + 1, \qquad (3)$$

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Exactly one of the two equations has a **positive integral** solution (x, y), and the solution is unique.

Define $\Gamma' : \{(a, b) : (a, b) = 1, b \geq 2, 2 \text{ divides } (a+1)b\} \to \{1, 2\}$:

$$\Gamma'(a, b) = \begin{cases} 1, & \text{if (3) has a solution;} \\ 2, & \text{if (4) has a solution.} \end{cases}$$

Another Pair of Diophantine Equations

Asymmetric: possible that $\Gamma'(a, b) \neq \Gamma'(b, a)$.

$(a, b) = (3, 5)$ and $(a, b) = (5, 3)$

$$2 \cdot 3 + 1 \cdot 5 = \frac{(3+1)5}{2} + 1$$
$$1 \cdot 5 + 1 \cdot 3 = \frac{(5+1)3}{2} - 1$$

Proof Sketch

Let $k = (a+1)b/2$. Choose $1 \leq r_1, r_2 \leq b-1$ s.t.

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$s_1 + s_2 = 1 \implies$ exactly one is positive

What's next

Finish the proof for Fibonacci cubed

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Characterize when $\Gamma'(a, b) \neq \Gamma'(b, a)$