Analysis of a Four-Dimensional Intraguild Predation Model of Burmese Pythons and American Alligators in the Everglades

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SIAM Session on Applications in Computational and Mathematical Modeling in Biology

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

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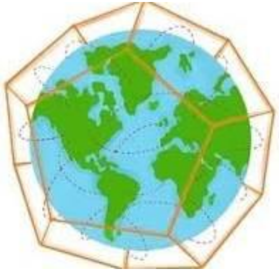
Outline

1) Project Context: Polymath Jr. 2025
2) Biological Background and Motivation
3) Model Formulation: ODE System

https://geometrynyc.wixsite.com/polymathreu

Polymath Junior Program



- Provide research opportunities to undergraduates.
- Online, runs in the spirit of the Polymath Project.
- Projects run by researcher with experience in undergraduate mentoring.
- Most 15-25 students, a main mentor, grad students / postdocs assisting.

INTRO to PP models and Holling types and Harvesting

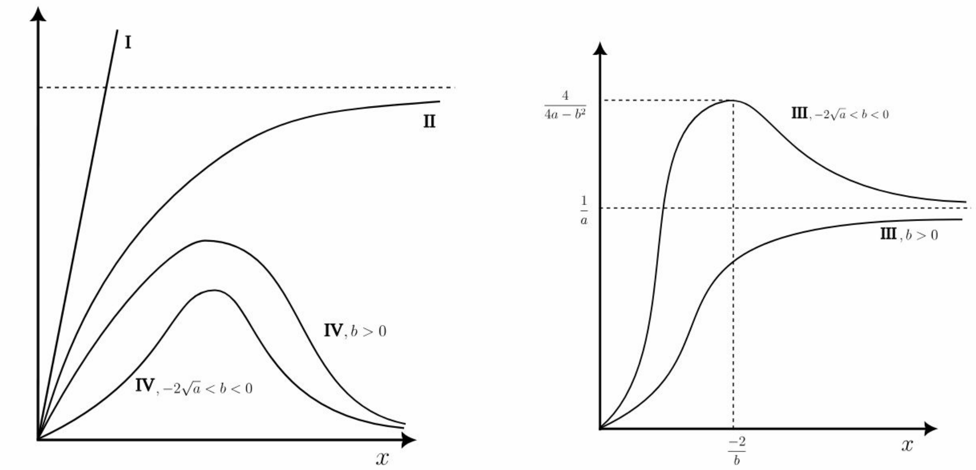
Lotka-Volterra Model: Let x(t) represent the prey population and y(t) represent the predator population. The model is given by:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta xy, \\ \frac{dy}{dt} = \delta xy - \gamma y, \end{cases}$$

INTRO to PP models and Holling types and Harvesting

$$\frac{dx}{dt} = G_1(x) - \alpha_1 F(x)y - H_1(x)$$

$$\frac{dy}{dt} = -G_2(y) + \alpha_2 F(x)y - H_2(y)$$



(a) Holling Type I,II, and IV

(b) Holling Type III

A point $E^* = (x^*, y^*)$ is called a **fixed point** of the dynamical system

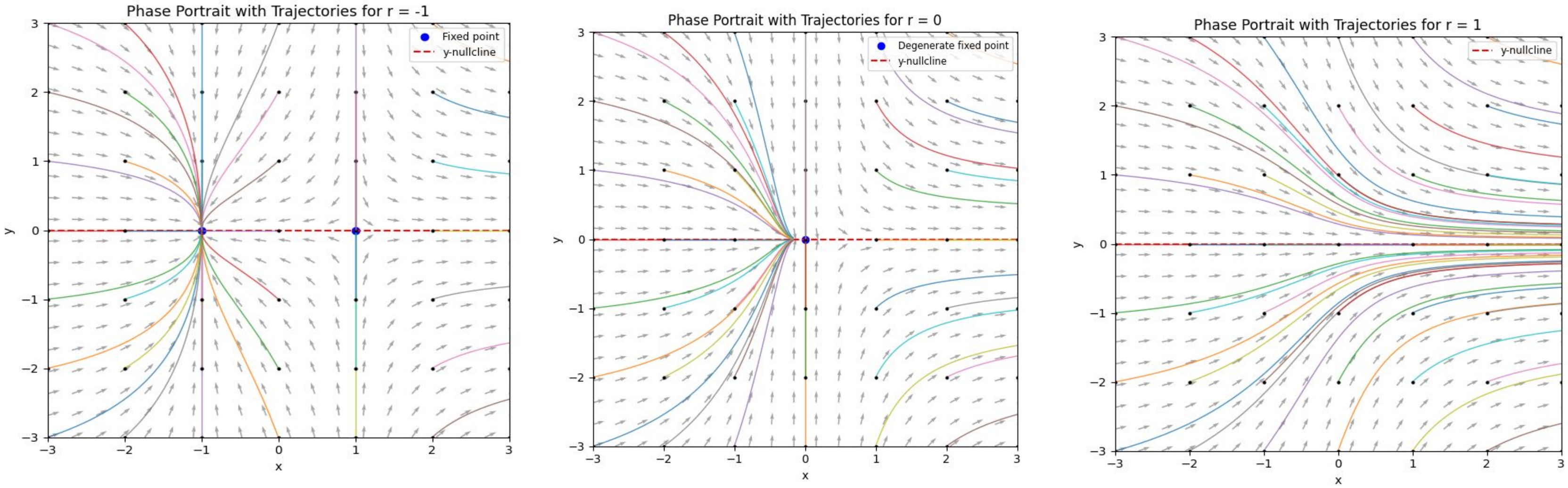
$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases}$$

if it satisfies f(E^*) = g(E^*) = 0. E^* Is said to be **hyperbolic** if both of the Jacobian eigenvalues are non-zero.

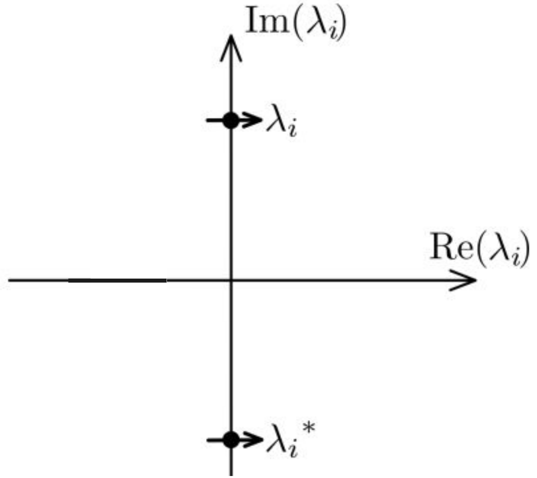
Hartman Grobman Theorem: Near a hyperbolic fixed point, the dynamics of a nonlinear system are topologically conjugate to its linearization.

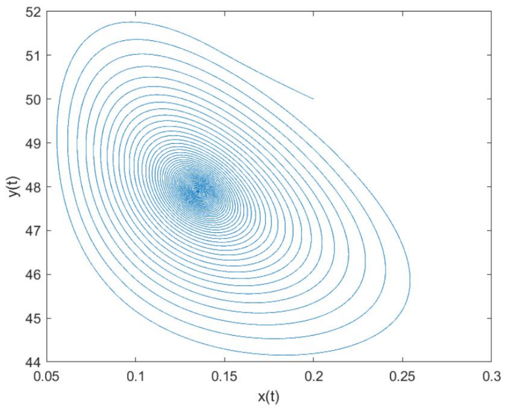
Bifurcations: qualitative change in the behavior of a system as a parameter is varied.

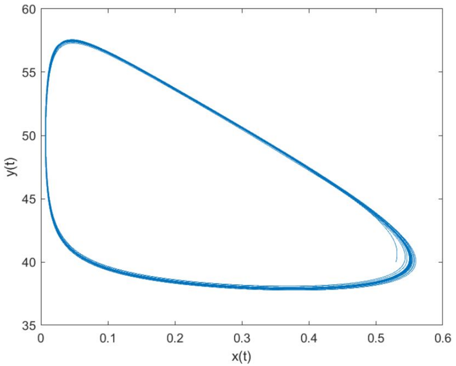
$$\begin{cases} \dfrac{dx}{dt} = r + x^2, \\ \dfrac{dy}{dt} = -y. \end{cases}$$



A **Hopf bifurcation** occurs when a fixed point of a dynamical system loses stability as a pair of complex conjugate eigenvalues crosses the imaginary axis, resulting in the emergence or disappearance of a periodic orbit of small amplitude.







Bogdanov Takens Bifurcation:

$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^2, \; \mu = (\mu_1, \mu_2)$$

A Bogdanov Takens Bifurcation occurs when the system has an equilibrium point x* such that the Jacobian $Df(x^*, \mu^*)$ has a double zero eigenvalue:

$$\mathrm{tr}(Df) = 0, \quad \det(Df) = 0, \quad Df \neq 0$$

And $Df(x^*, \mu^*)$ has a single Jordan block (geometric multiplicity = 1). The system can then be transformed to the normal form:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = \beta_1 + \beta_2 u + u^2 + s\,uv. \end{cases}$$

Background (Biology)

1800s to mid-1900s: Intense hunting for leather and meat.

1950s to 1960s: Populations crash; poaching persists; weak enforcement.

1967: Listed as endangered in the United States.

1973: Endangered Species Act; stronger enforcement; interstate hide trade curbed.

1970s: States close hunting, protect nests, begin science-based management.

Background (Biology)

Late 1970s to 1980s: Rebound via protection, anti-poaching, wetland conservation.

1987: Delisted federally; managed as "threatened due to similarity of appearance."

Today: Millions across the Southeast, especially Florida and Louisiana.

FWC lottery permits; two alligators per permit.

Burmese Pythons

Invasive to the everglades, they were released by means of

- **Intentional releases by owners who could no longer keep large snakes.**
- **Accidental escapes from pet stores, breeders, and private collections.**
- **Facility damage during Hurricane Andrew (1992) likely released additional snakes into the Everglades.**

Ecological implications

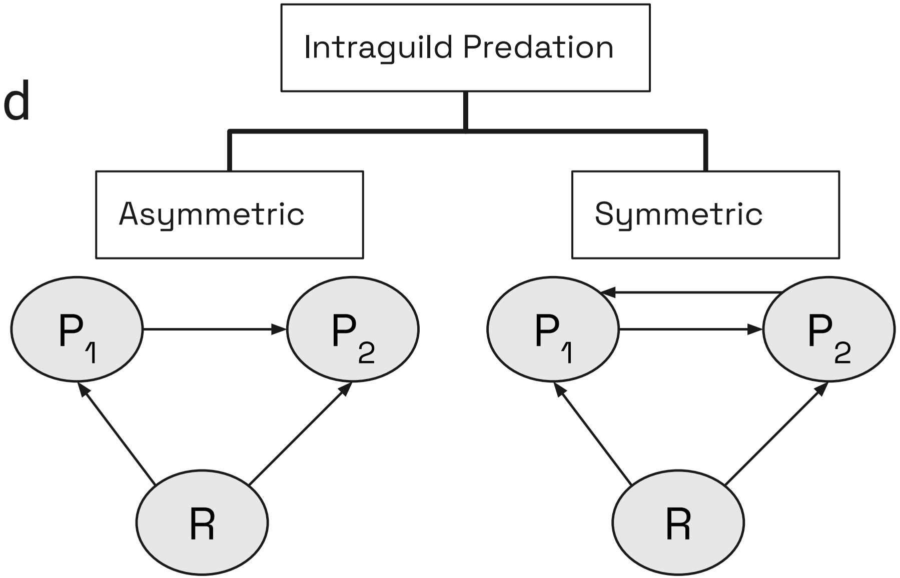
- Sharp declines of midsized mammals in parts of the Everglades (raccoons, opossums, marsh rabbits)
- Predation on native birds, reptiles, and occasional juvenile alligators
- Pressure on threatened species that nest or forage on the ground
- Competition with native apex predators for prey, altering food webs

Intraguild Predation

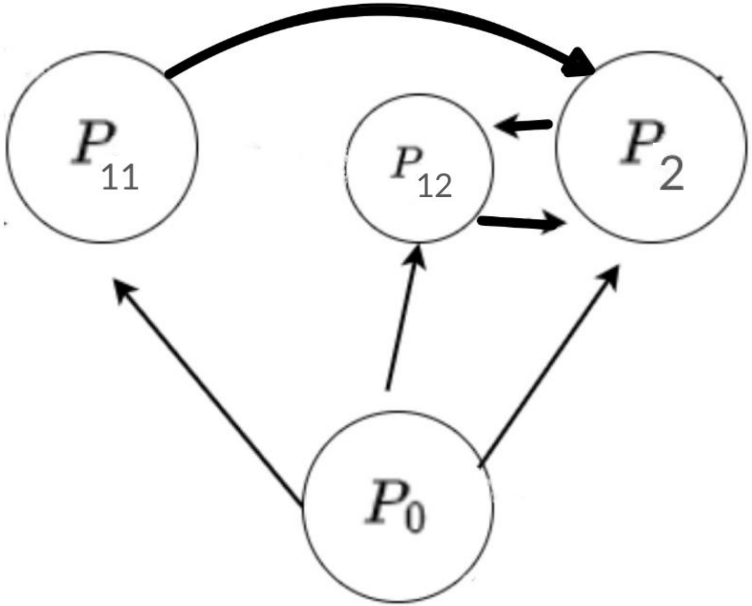
Concept of intraguild predation:

> Intraguild predation is the killing and sometimes eating of a potential competitor of a different species. This interaction represents a combination of predation and competition.

Typically modeled with 3 equations.



Model Illustrated:



P_0 - Racoon/prey population

P_{11} - Juvenile Alligator population

P_{12} - Adult Alligator population

P_2 - Burmese Python population

Initial Model

$$\begin{cases} \frac{dP_0}{dt} &= f_1(P_0) - m_{20}f_2(P_0)P_{12} - m_{30}f_3(P_0)P_2 \\ \frac{dP_{11}}{dt} &= c_{20}f_2(P_0)P_{12} - m_{31}f_4(P_{11})P_3 - d_1P_{11} \\ \frac{dP_{12}}{dt} &= d_1P_{11} - d_2P_{12} - H_2(P_{12}) \\ \frac{dP_2}{dt} &= c_{30}f_3(P_0)P_2 + c_{31}f_4(P_{11})P_2 - d_3P_2 - H_3(P_2) \end{cases}$$

$f_m(P_n) = P_n$

$H_2(P_2) = h_2$

$H_3(P_3) = h_3P_3$

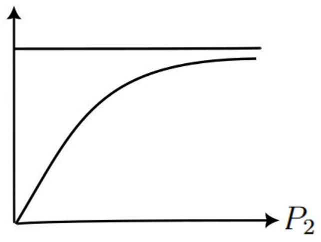
$$\begin{cases} \frac{dP_0}{dt} &= rP_0 - m_{20}P_0P_{12} - m_{30}P_0P_2 \\ \frac{dP_{11}}{dt} &= c_{20}P_0P_{12} - m_{31}P_{11}P_3 - d_{11}P_{11} \\ \frac{dP_{12}}{dt} &= d_{12}P_{11} - d_2P_{12} - h_2 \\ \frac{dP_2}{dt} &= c_{30}P_0P_2 + c_{31}P_2P_{11} - d_3P_2 - h_3P_2 \end{cases}$$

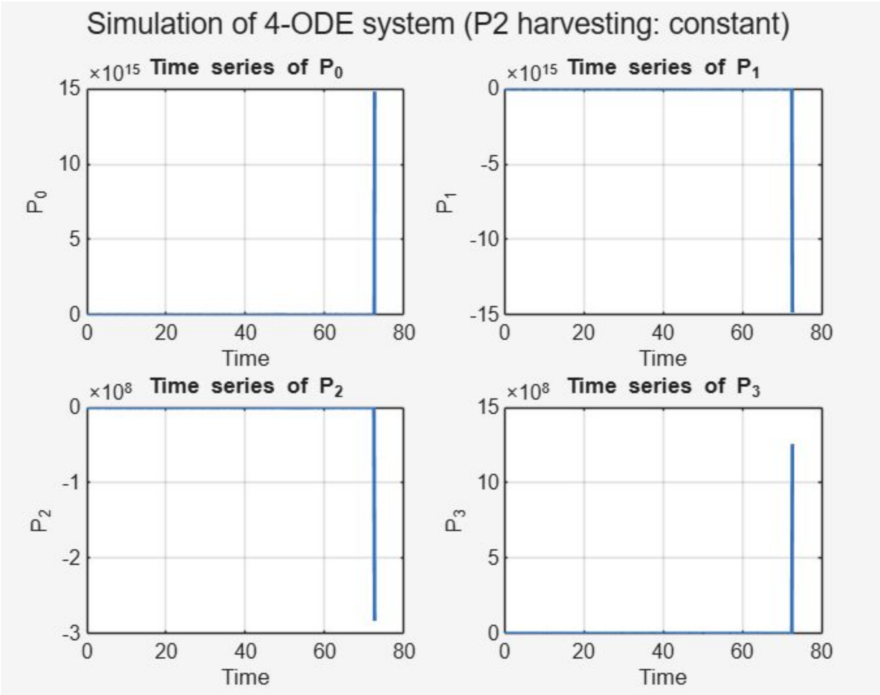
$h_2 \Rightarrow$ *Saturated*

$$\begin{cases} \frac{dP_0}{dt} &= rP_0 - m_{20}P_0P_{12} - m_{30}P_0P_2 \\ \frac{dP_{11}}{dt} &= c_{20}P_0P_{12} - m_{31}P_{11}P_3 - d_{11}P_{11} \\ \frac{dP_{12}}{dt} &= d_{12}P_{11} - d_2P_{12} - h_2\frac{P_{12}}{a+P_{12}} \\ \frac{dP_2}{dt} &= c_{30}P_0P_2 + c_{31}P_2P_{11} - d_3P_2 - h_3P_2 \end{cases}$$

Why constant harvesting doesn’t work

Applying constant harvesting in MATLAB shows that P_1 and P_2 eventually become **negative**, which is biologically unrealistic and highlights why **constant harvesting does not work.**





Model

Prey, Juvenile Alligator, Adult Alligator, Python

$$\begin{cases} \frac{dP_0}{dt} = rP_0 - m_{20}P_0P_{12} - m_{30}P_0P_2 \\ \frac{dP_{11}}{dt} = c_{20}P_0P_{12} - m_{31}P_{11}P_3 - d_{11}P_{11} \\ \frac{dP_{12}}{dt} = d_{12}P_{11} - P_{12}P_2 - h_2\frac{P_{12}}{a+P_{12}} \\ \frac{dP_2}{dt} = c_{30}P_0P_2 + c_{31}P_2P_{11} - d_3P_2 - h_3P_2 \end{cases}$$

↓

Prey

$$\frac{dP_0}{dt} = rP_0\left(1 - \frac{P_0}{K}\right) - m_{10}P_0P_1 - m_{20}P_0P_2,$$

Alligator

$$\frac{dP_1}{dt} = c_{10}P_0P_1 - d_1P_1 - \frac{h_1P_1}{a+P_1} - m_{31}P_1P_2,$$

Python

$$\frac{dP_2}{dt} = c_{20}P_0P_2 + c_{21}P_1P_2 - d_2P_2 - h_2P_2.$$

Equilibria: 7 Equilibrium points

- 2 Equilibria on the P_0 axis
- 2 equilibria on the $P_0 - P_1$ plane suggesting that the prey can coexist with the alligators
- 2 non zero equilibria where all 3 species coexist.
- 1 equilibria in the $P_0 - P_2$ plane.

Equilibria: Analysis of Straightforward Points

Jacobian

$$J(P_0, P_1, P_2) = \begin{bmatrix} 1 - \frac{2P_0}{K} - P_1 m_{10} - P_2 & -m_{10} P_0 & -P_0 \\ c_{10} P_1 & c_{10} P_0 - d_1 - \frac{h_1}{a+P_1} + \frac{h_1 P_1}{(a+P_1)^2} - m_{31} P_2 & -m_{31} P_1 \\ P_2 & P_2 & P_0 + P_1 - d_2 - h_2 \end{bmatrix}$$

Equilibria 1

$$J(P_0 = 0, P_1 = 0, P_2 = 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -d_1 - \frac{h_1}{a} & 0 \\ 0 & 0 & -d_2 - h_2 \end{bmatrix}$$

$$\text{Eigenvalues: } \begin{bmatrix} 1 \\ -d_2 - h_2 \\ -\frac{d_1 a + h_1}{a} \end{bmatrix}$$

1 positive, 2 negative
→ Saddle Point

Equilibria: Analysis of Straightforward Points

Equilibria 2

$$J(P_0 = K, P_1 = 0, P_2 = 0) = \begin{bmatrix} -1 & -m_{10}K & -K \\ 0 & Kc_{10} - d_1 - \frac{h_1}{a} & 0 \\ 0 & 0 & K - d_2 - h_2 \end{bmatrix}$$

$$\text{Eigenvalues: } \begin{bmatrix} -1 \\ K - d_2 - h_2 \\ Kc_{10} - d_1 - \frac{h_1}{a} \end{bmatrix}$$

Two Negative, One Positive

$$\frac{d_1}{c_{10}} + \frac{h_1}{ac_{10}} < K < d_2 + h_2$$

OR

$$d_2 + h_2 < K < \frac{d_1}{c_{10}} + \frac{h_1}{ac_{10}}$$

One negative, Two positive

$$K > d_2 + h_2$$

$$K > \frac{d_1}{c_{10}} + \frac{h_1}{ac_{10}}$$

Three Negative

$$K < \frac{d_1}{c_{10}} + \frac{h_1}{ac_{10}}$$

$$K < d_2 + h_2$$

Equilibria: Analysis of Straightforward Points

Equilibria 3

$$J\left(P_0 = d_2 + h_2, P_1 = 0, P_2 = \frac{K-d_2-h_2}{K}\right) =$$

$$\begin{bmatrix} 1 - \frac{2(d_2+h_2)}{K} - \frac{K-d_2-h_2}{K} & -m_{10}\left(d_2+h_2\right) & -d_2 - h_2 \\ 0 & c_{10}\left(d_2+h_2\right) - d_1 - \frac{h_1}{a} - \frac{m_{31}(K-d_2-h_2)}{K} & 0 \\ \frac{K-d_2-h_2}{K} & \frac{K-d_2-h_2}{K} & 0 \end{bmatrix}$$

Eigenvalues:
$$\begin{bmatrix} \frac{-d_2-h_2+\sqrt{-4(d_2+h_2)\left(K^2+(-d_2-h_2)K-\frac{d_2}{4}-\frac{h_2}{4}\right)}}{2K} \\ \frac{-d_2-h_2-\sqrt{-4(d_2+h_2)\left(K^2+(-d_2-h_2)K-\frac{d_2}{4}-\frac{h_2}{4}\right)}}{2K} \\ \frac{((-m_{31}+c_{10}(d_2+h_2)-d_1)K+m_{31}(d_2+h_2))a-h_1K}{Ka} \end{bmatrix}$$

Equilibria 3 first eigenvalue

$$Z = -4\left(d_2+h_2\right)\left(K^2 + \left(-d_2-h_2\right)K - \frac{d_2}{4} - \frac{h_2}{4}\right)$$

$$\frac{-d_2-h_2+\sqrt{Z}}{2K} \rightarrow \begin{cases} Z < 0 \rightarrow \text{imaginary} \\ Z = 0 \rightarrow \text{negative} \\ Z > 0 \end{cases}$$

- $Z > 0$:
  - $\sqrt{Z} > d_2 + h_2 \rightarrow$ positive
  - $\sqrt{Z} < d_2 + h_2 \rightarrow$ negative

Equilibria: Analysis of Straightforward Points

Equilibria 3

Eigenvalues: $$\begin{bmatrix} \frac{-d_2-h_2+\sqrt{-4(d_2+h_2)\left(K^2+(-d_2-h_2)K-\frac{d_2}{4}-\frac{h_2}{4}\right)}}{2K} \\ \frac{-d_2-h_2-\sqrt{-4(d_2+h_2)\left(K^2+(-d_2-h_2)K-\frac{d_2}{4}-\frac{h_2}{4}\right)}}{2K} \\ \frac{((-m_{31}+c_{10}(d_2+h_2)-d_1)K+m_{31}(d_2+h_2))a-h_1K}{Ka} \end{bmatrix}$$

Equilibria 3 second eigenvalue

$$Z = -4\left(d_2+h_2\right)\left(K^2+\left(-d_2-h_2\right)K-\frac{d_2}{4}-\frac{h_2}{4}\right)$$

$$\frac{-d_2-h_2-\sqrt{Z}}{2K} \longrightarrow \begin{cases} Z<0 & \longrightarrow \text{imaginary} \\ Z=0 & \longrightarrow \text{negative} \\ Z>0 & \longrightarrow \text{negative} \end{cases}$$

2 equilibria on the $P_0 - P_1$

When the two equilibria collide and merge, a **saddle-node bifurcation** occurs.

$$K^2a^2{c_{10}}^2{m_{10}}^2+2K^2a\,{c_{10}}^2m_{10}-2Kac_{10}d_1m_{10}+K^2{c_{10}}^2-4Kc_{10}h_1m_{10}-2Kc_{10}d_1+{d_1}^2=0$$

By introducing additional parameters, this bifurcation can give rise to a **Bogdanov–Takens bifurcation**.

In other words, a saddle-node bifurcation takes place initially, and under further parameter constraints, the new equilibrium point exhibits **two zero eigenvalues**.

Steps followed:

1. **Shift the equilibrium:** Find the equilibrium $P^* = (P_0^*, P_1^*, P_2^*)$ and shift variables:

$$u = P_0 - P_0^*, \quad v = P_1 - P_1^*, \quad w = P_2 - P_2^*$$

so that the system becomes $\dot{z} = F(z)$ with $F(0) = 0$.

2. **Linearize and identify eigen-directions:** Compute the Jacobian $J = DF(0)$. At the BT point: $\operatorname{spec}(J) = \{0, 0, \lambda_s\}$ with $\lambda_s < 0$. The eigenvectors corresponding to zero eigenvalues define the center subspace.

3. **Center manifold reduction (3D → 2D):** There exists a smooth invariant manifold $w = h(u, v)$ with $h(0) = 0$, $Dh(0) = 0$. Substituting $w = h(u, v)$ reduces the system to:

$$\dot{u} = \tilde{F}_1(u, v), \quad \dot{v} = \tilde{F}_2(u, v)$$

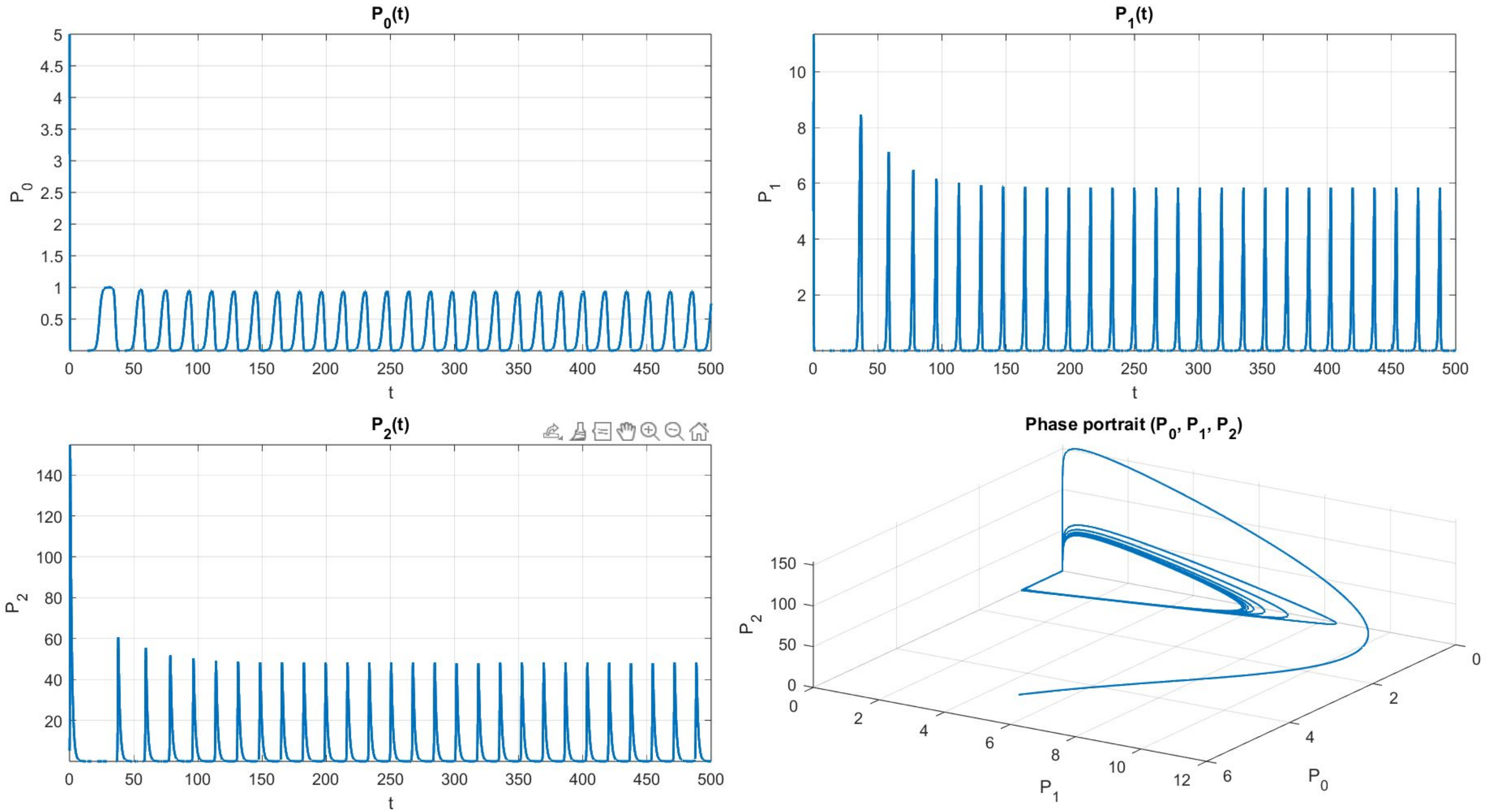
4. **Reduced dynamics:** Expand $\tilde{F}_1, \tilde{F}_2$ in a Taylor series near the origin and retain quadratic terms:

$$\dot{u} = v + \mathcal{O}(2), \quad \dot{v} = au^2 + buv + \cdots$$

5. **Transform to BT normal form:** Apply smooth coordinate and parameter changes to obtain:

$$\boxed{\dot{x} = y, \quad \dot{y} = \beta_1 + \beta_2 x + ax^2 + bxy + \mathcal{O}(3)}$$

What happens at the remaining equilibria?



Future Steps and Ultimate Goal

- The system exhibits even higher-codimension bifurcations, including **codimension-3** phenomena.

- The existence of an equilibrium indicating that prey and pythons can coexist **without the presence of alligators** is concerning, as it may lead to the **extinction of the alligator population**.

- As a next step, we aim to **fit real data from Florida wildlife** to the model to validate and refine our theoretical predictions.

Thank you!