SOCIETY FOR AMERICAN BASEBALL RESEARCH

The Pythagorean Won-Loss Formula in Baseball: An Introduction to Statistics and Modeling

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Ackowledgements

For many useful discussions:
Sal Baxamusa, Phil Birnbaum, Ray Ciccolella, Steve Johnston, Michelle Manes, Russ Mann.

Dedicated to my great uncle Newt Bromberg:
a lifetime Sox fan who promised me that I would live to see a World Series Championship in Boston.

Goals of the Talk

Derive James' Pythagorean Won-Loss formula from a reasonable model.

Introduce some of the techniques of modeling.

Discuss the mathematics behind the models and model testing.

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Discuss the 2004 World Champion Boston Red Sox!

Numerical Observation: Pythagorean Won-Loss Formula

Parameters:

- $\mathrm{RS}_{\mathrm{obs}}$: average number of runs scored per game;
- $\mathrm{RA}_{\mathrm{obs}}$: average number of runs allowed per game;
- γ: some parameter, constant for a sport.

Bill James' Won-Loss Formula (NUMERICAL Observation):

$$\mathrm{Won-Loss\ Percentage} = \frac{{\mathrm{RS}_{\mathrm{obs}}}^{\gamma}}{{\mathrm{RS}_{\mathrm{obs}}}^{\gamma} + {\mathrm{RA}_{\mathrm{obs}}}^{\gamma}}$$

γ originally taken as 2, numerical studies show best γ is about 1.82.

Applications of the Pythagorean Won-Loss Formula

Extrapolation: use half-way through season to predict a team's performance.

Evaluation: see if consistently over-perform or under-perform.

There are other statistics / formulas (example: run-differential per game); advantage is very easy to apply, depends only on two simple numbers for a team.

Probability Review

Probability density:

- $p(x) \geq 0$;
- $\int_{-\infty}^{\infty} p(x)\mathrm{d}x = 1$;
- X random variable with density $p(x)$: $\mathrm{Prob}\,(X \in [a, b]) = \int_a^b p(x)\mathrm{d}x$.

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Mean (average value) $\mu = \int_{-\infty}^{\infty} xp(x)\mathrm{d}x$.

Variance (how spread out) $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)\mathrm{d}x$.

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Independence: two random variables are independent if knowledge of one does not give knowledge of the other.

Modeling the Real World

Guidelines for Modeling:

- Model should capture key features of the system;
- Model should be mathematically tractable (solvable).

In general these are conflicting goals. How should we try and model baseball games?

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Possible Model:

- Runs Scored and Runs Allowed independent random variables;
- $f_{\mathrm{RS}}(x)$, $g_{\mathrm{RA}}(y)$: probability density functions for runs scored (allowed).
- Reduced to calculating

$$\int_x \left[\int_{y \leq x} f_{\mathrm{RS}}(x) g_{\mathrm{RA}}(y) \mathrm{d}y \right] \mathrm{d}x \quad \text{or} \quad \sum_i \left[\sum_{j<i} f_{\mathrm{RS}}(i) g_{\mathrm{RA}}(j) \right].$$

Problems with the Model

Reduced to calculating

$$\int_x \left[\int_{y \leq x} f_{\mathrm{RS}}(x) g_{\mathrm{RA}}(y) \mathrm{d}y \right] \mathrm{d}x \quad \text{or} \quad \sum_i \left[\sum_{j<i} f_{\mathrm{RS}}(i) g_{\mathrm{RA}}(j) \right].$$

Problems with the model:

- Can the integral (or sum) be completed in closed form?
- Are the runs scored and allowed independent random variables?
- What are f_{RS} and g_{RA}?

Three Parameter Weibull

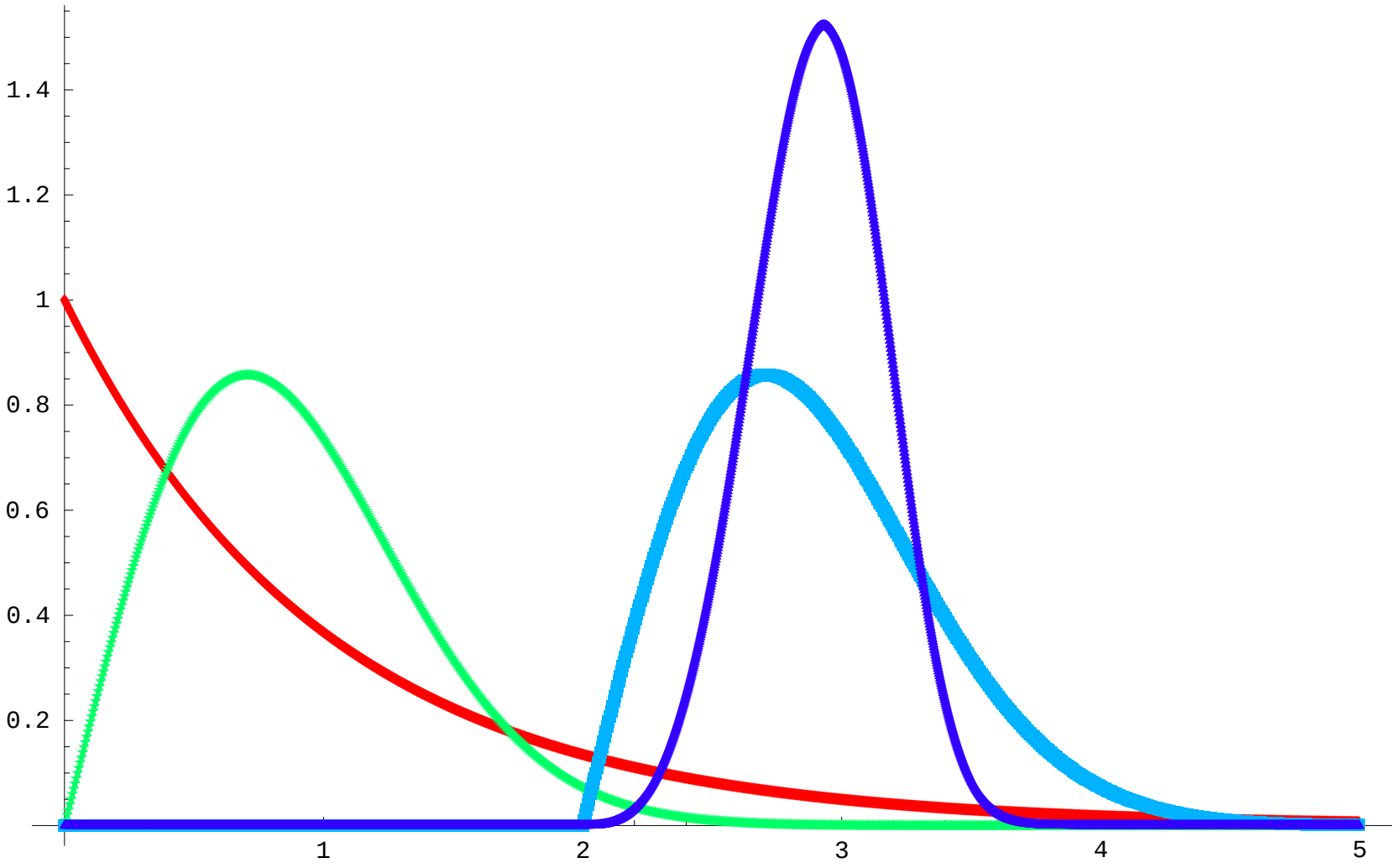
Weibull distribution:

$$f(x;\alpha,\beta,\gamma) \;=\; \begin{cases} \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^{\gamma}} & \text{if } x \geq \beta \\ 0 & \text{otherwise.} \end{cases}$$

- α: scale (variance: meters versus centimeters);
- β: origin (mean: translation, zero point);
- γ: shape (behavior near β and at infinity).

Various values give different shapes, but can we find α, β, γ such that it fits observed data? Is the Weibull theoretically tractable?

Weibull Plots: Parameters (α, β, γ)



Red:(1, 0, 1) (exponential); Green:(1, 0, 2); Cyan:(1, 2, 2); Blue:(1, 2, 4)

Weibull Integrations

Let $f(x;\alpha,\beta,\gamma)$ be the probability density of a Weibull(α,β,γ):

$$f(x;\alpha,\beta,\gamma) \;=\; \begin{cases} \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^{\gamma}} & \text{if } x \geq \beta \\ 0 & \text{otherwise.} \end{cases}$$

For $s \in \mathbb{C}$ with the real part of s greater than 0, recall the Γ-function:

$$\begin{aligned} \Gamma(s) &= \int_0^{\infty} e^{-u} u^{s-1} \mathrm{d}u \;=\; \int_0^{\infty} e^{-u} u^{s} \frac{\mathrm{d}u}{u}. \\ \Gamma(n) &= (n-1)! \quad (\text{i.e.}, \Gamma(4) = 3\cdot 2\cdot 1). \end{aligned}$$

Let $\mu_{\alpha,\beta,\gamma}$ denote the mean of $f(x;\alpha,\beta,\gamma)$.

Weibull Integrations (Continued)

$$\begin{aligned}
\mu_{\alpha,\beta,\gamma} &= \int_\beta^\infty x \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^\gamma}\mathrm{d}x \\
&= \int_\beta^\infty \alpha\frac{x-\beta}{\alpha} \cdot \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^\gamma}\mathrm{d}x \;+\; \beta.
\end{aligned}$$

Change variables: $u = \left(\frac{x-\beta}{\alpha}\right)^\gamma$. Then $\mathrm{d}u = \frac{\gamma}{\alpha}\left(\frac{x-\beta}{\alpha}\right)^{\gamma-1}\mathrm{d}x$ and

$$\begin{aligned}
\mu_{\alpha,\beta,\gamma} &= \int_0^\infty \alpha u^{\gamma^{-1}} \cdot e^{-u}\mathrm{d}u \;+\; \beta \\
&= \alpha\int_0^\infty e^{-u}\, u^{1+\gamma^{-1}}\,\frac{\mathrm{d}u}{u} \;+\; \beta \\
&= \alpha\Gamma(1+\gamma^{-1}) \;+\; \beta.
\end{aligned}$$

A similar calculation determines the variance.

Pythagorean Won-Loss Formula

Theorem: Pythagorean Won-Loss Formula: *Let the runs scored and allowed per game be two independent random variables drawn from Weibull distributions* $(\alpha_{\mathrm{RS}}, \beta, \gamma)$ *and* $(\alpha_{\mathrm{RA}}, \beta, \gamma)$*;* α_{RS} *and* α_{RA} *are chosen so that the means are* RS *and* RA*. If* $\gamma > 0$ *then*

$$\text{Won-Loss Percentage}(\mathrm{RS}, \mathrm{RA}, \beta, \gamma) \;=\; \frac{(\mathrm{RS} - \beta)^{\gamma}}{(\mathrm{RS} - \beta)^{\gamma} + (\mathrm{RA} - \beta)^{\gamma}}.$$

In baseball take $\beta = -.5$ (from runs must be integers).

$\mathrm{RS} - \beta$ estimates average runs scored, $\mathrm{RA} - \beta$ estimates average runs allowed.

Best Fit Weibulls to Data: Method of Least Squares

Minimized the sum of squares of the error from the runs scored data plus the sum of squares of the error from the runs allowed data.

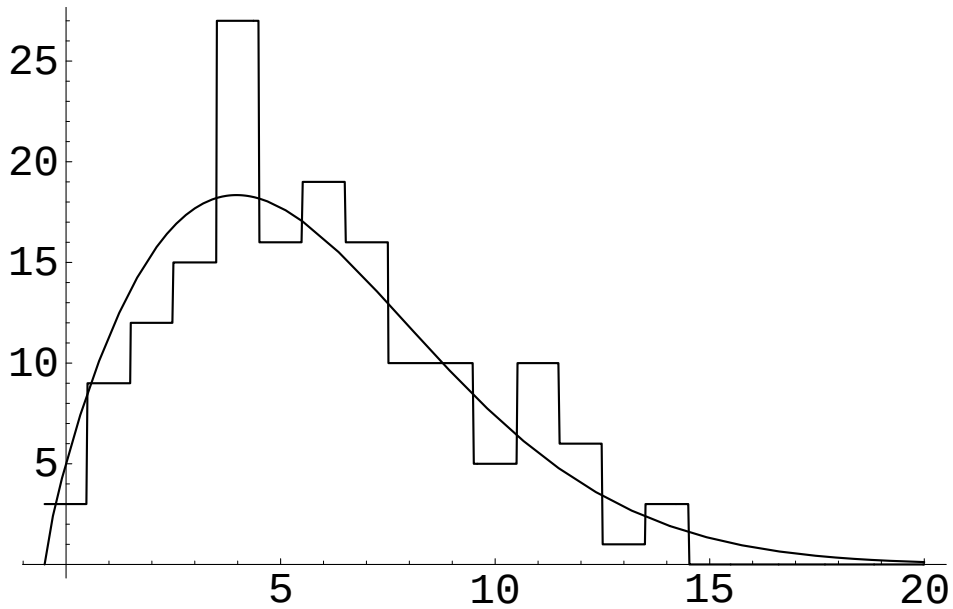
- $\mathrm{Bin}(k)$ is the kth bin;
- $\mathrm{RS}_{\mathrm{obs}}(k)$ (resp. $\mathrm{RA}_{\mathrm{obs}}(k)$) the observed number of games with the number of runs scored (allowed) in $\mathrm{Bin}(k)$;
- $A(\alpha, \beta, \gamma, k)$ the area under the Weibull with parameters (α, β, γ) in $\mathrm{Bin}(k)$.

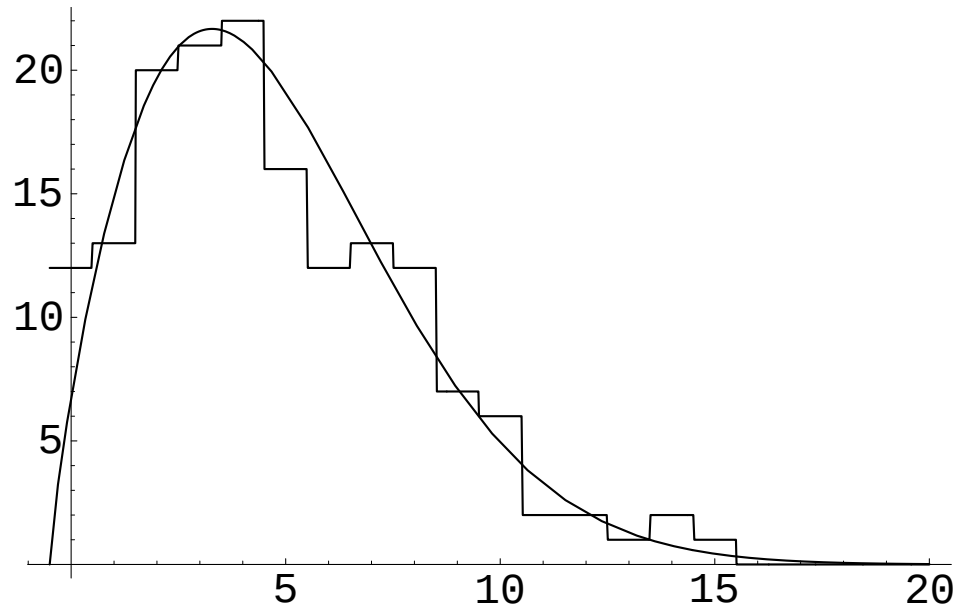
Find the values of $(\alpha_{\mathrm{RS}}, \alpha_{\mathrm{RA}}, \gamma)$ that minimize

$$\sum_{k=1}^{\#\mathrm{Bins}} \left(\mathrm{RS}_{\mathrm{obs}}(k) - \#\mathrm{Games} \cdot A(\alpha_{\mathrm{RS}}, -.5, \gamma, k)\right)^2 + \sum_{k=1}^{\#\mathrm{Bins}} \left(\mathrm{RA}_{\mathrm{obs}}(k) - \#\mathrm{Games} \cdot A(\alpha_{\mathrm{RA}}, -.5, \gamma, k)\right)^2 .$$

Best Fit Weibulls to Data (Method of Maximum Likelihood)

Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Boston Red Sox

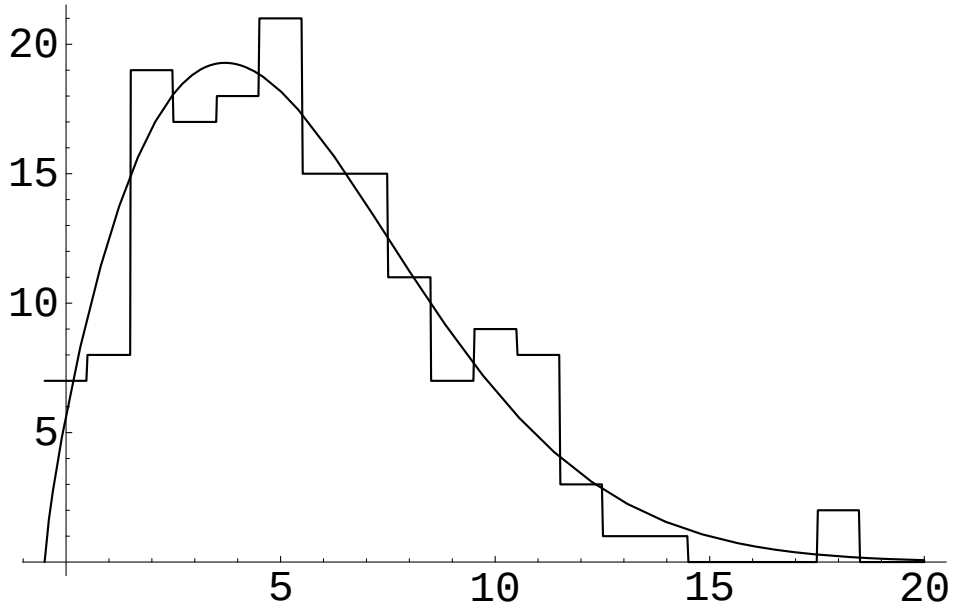


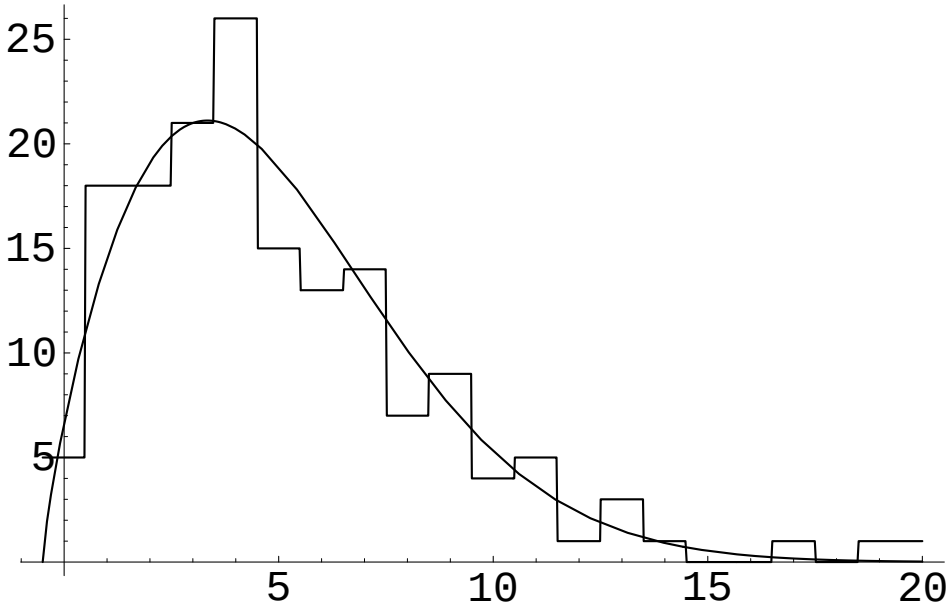


Using as bins

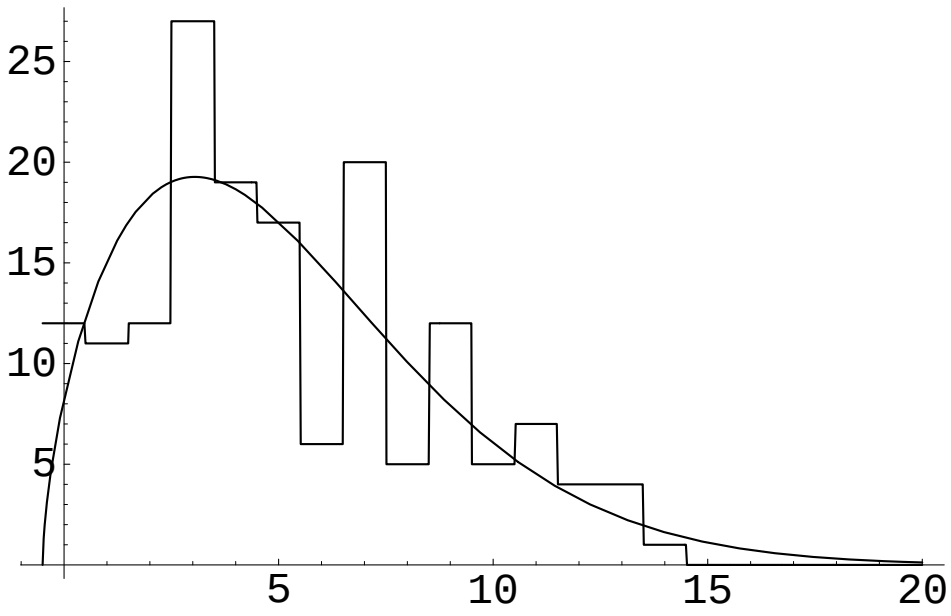
$$[-.5, .5] \cup [.5, 1.5] \cup \cdots \cup [7.5, 8.5] \cup [8.5, 9.5] \cup [9.5, 11.5] \cup [11.5, \infty).$$

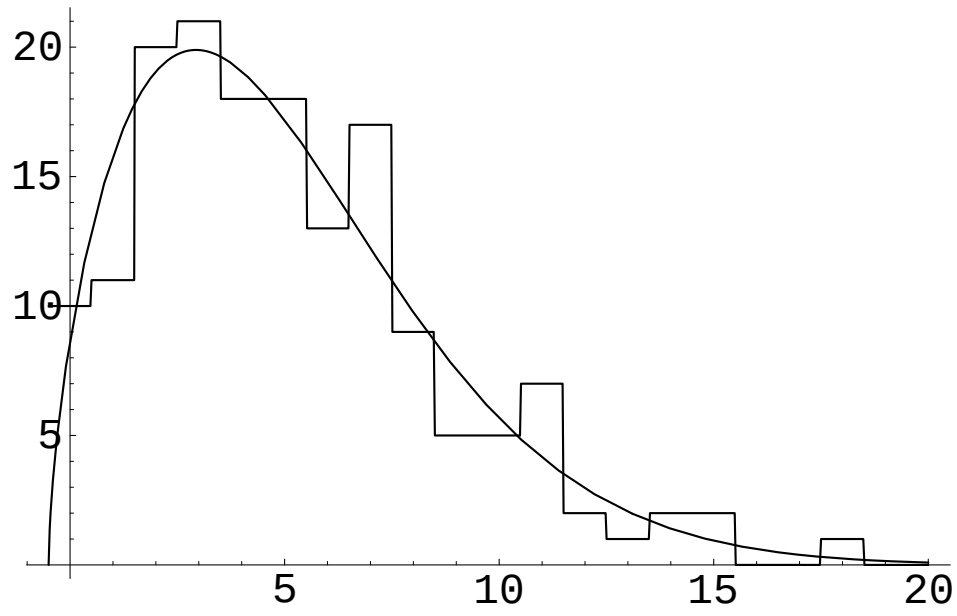
Plots of RS (predicted vs observed) and RA (predicted vs observed) for the New York Yankees



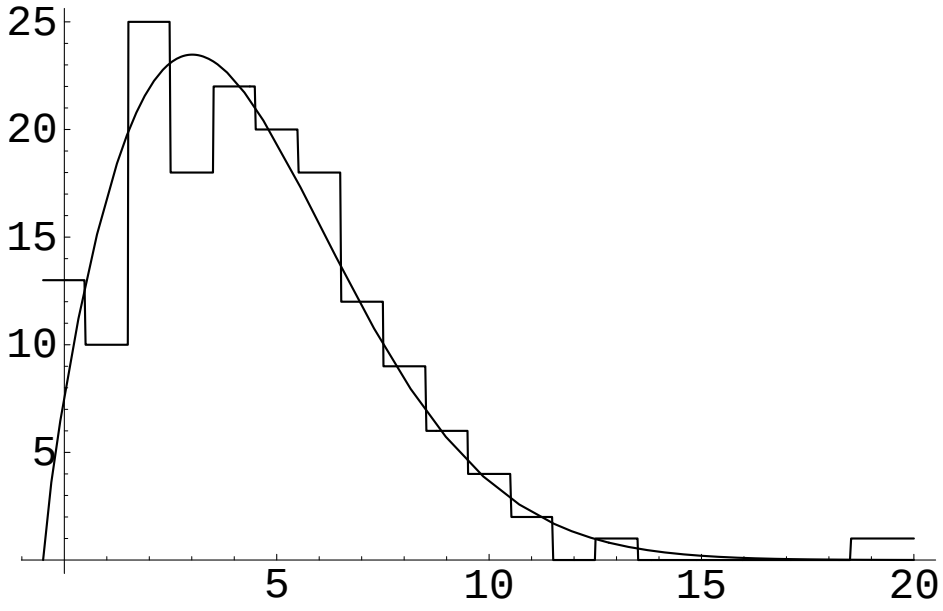


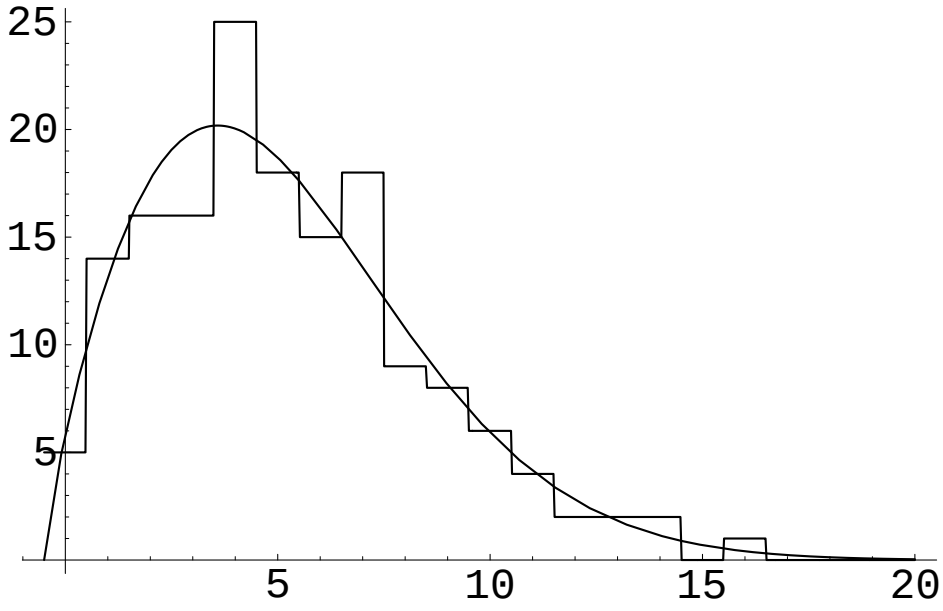
Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Baltimore Orioles



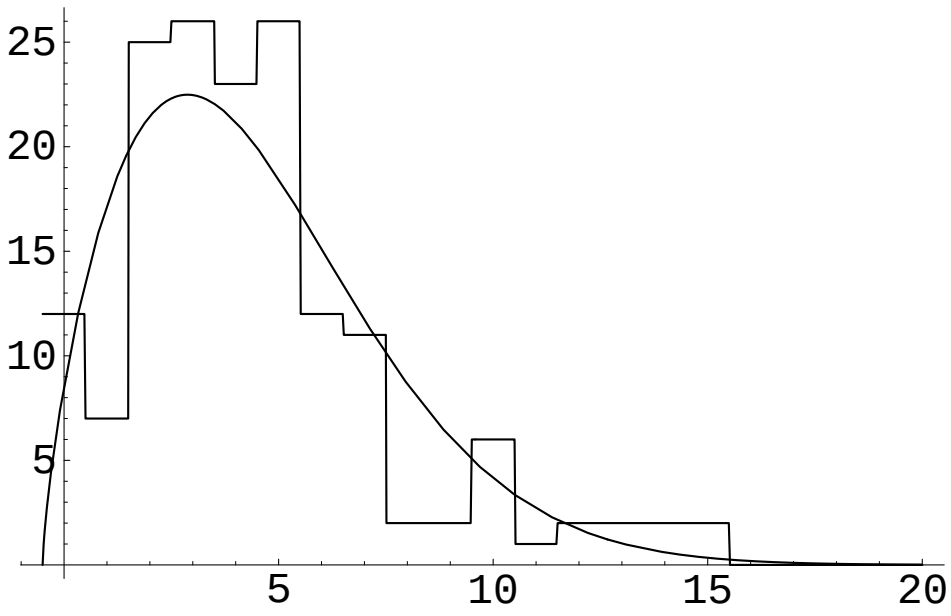


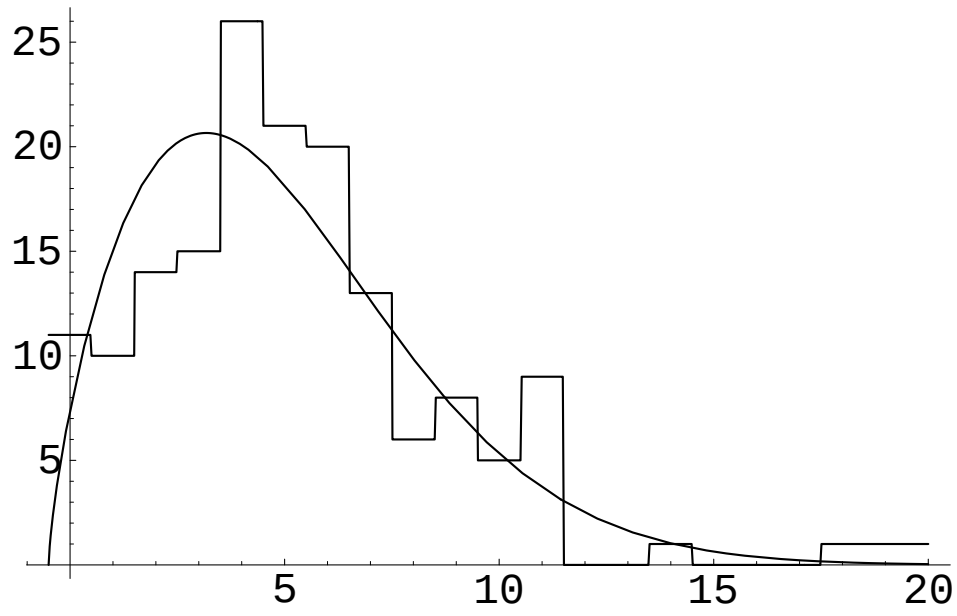
Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Tampa Bay Devil Rays





Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Toronto Blue Jays





Bonferroni Adjustments

Imagine a fair coin. If flip 1,000,000 times expect 500,000 heads.

Approximately 95% of the time $499,000 \leq \#\text{Heads} \leq 501,000$.

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N	Prob
5	22.62
10	40.13
14	51.23
20	64.15
50	92.31

Team	RS+RA χ2: 20 d.f.	Indep χ2: 109 d.f
Boston Red Sox	15.63	83.19
New York Yankees	12.60	129.13
Baltimore Orioles	29.11	116.88
Tampa Bay Devil Rays	13.67	111.08
Toronto Blue Jays	41.18	100.11
Minnesota Twins	17.46	97.93
Chicago White Sox	22.51	153.07
Cleveland Indians	17.88	107.14
Detroit Tigers	12.50	131.27
Kansas City Royals	28.18	111.45
Los Angeles Angels	23.19	125.13
Oakland Athletics	30.22	133.72
Texas Rangers	16.57	111.96
Seattle Mariners	21.57	141.00

20 d.f.: 31.41 (at the 95% level) and 37.57 (at the 99% level).
109 d.f.: 134.4 (at the 95% level) and 146.3 (at the 99% level).

Bonferroni Adjustment:
20 d.f.: 41.14 (at the 95% level) and 46.38 (at the 99% level).
109 d.f.: 152.9 (at the 95% level) and 162.2 (at the 99% level).

Testing the Model: Data from Method of Maximum Likelihood

Team	Obs Wins	Pred Wins	ObsPerc	PredPerc	GamesDiff	γ
Boston Red Sox	98	93.0	0.605	0.574	5.03	1.82
New York Yankees	101	87.5	0.623	0.540	13.49	1.78
Baltimore Orioles	78	83.1	0.481	0.513	-5.08	1.66
Tampa Bay Devil Rays	70	69.6	0.435	0.432	0.38	1.83
Toronto Blue Jays	67	74.6	0.416	0.464	-7.65	1.97
Minnesota Twins	92	84.7	0.568	0.523	7.31	1.79
Chicago White Sox	83	85.3	0.512	0.527	-2.33	1.73
Cleveland Indians	80	80.0	0.494	0.494	0.	1.79
Detroit Tigers	72	80.0	0.444	0.494	-8.02	1.78
Kansas City Royals	58	68.7	0.358	0.424	-10.65	1.76
Los Angeles Angels	92	87.5	0.568	0.540	4.53	1.71
Oakland Athletics	91	84.0	0.562	0.519	6.99	1.76
Texas Rangers	89	87.3	0.549	0.539	1.71	1.90
Seattle Mariners	63	70.7	0.389	0.436	-7.66	1.78

γ: mean $= 1.74$, standard deviation $= .06$, median $= 1.76$;
close to numerically observed value of 1.82.

The mean number of the difference between observed and predicted wins was $-.13$ with a standard deviation of 7.11 (and a median of 0.19).
If we consider just the absolute value of the difference then we have a mean of 5.77 with a standard deviation of 3.85 (and a median of 6.04).

Conclusions

- Can find parameters such that the Weibulls are good fits to the data;

- The runs scored and allowed per game are statistically independent;

- The Pythagorean Won-Loss Formula is a consequence of our model;

- Method of Least Squares: best value of γ of about 1.79;
  Method of Maximum Likelihood: best value of γ of about 1.74;
  both are close to the observed best 1.82.

Future Work

- Micro-analysis: runs scored and allowed are not entirely independent (big lead, close game), run production smaller for inter-league games in NL parks, et cetera.

- What about other sports? Does the same model work? How does γ depend on the sport?

- Are there other probability distributions that give integrals which can be determined in closed form?

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   ◇ Weibull worksheet:

   `http://www.beyondtheboxscore.com/story/2006/4/30/114737/251`

   ◇ Run distribution plots for various teams:

   `http://www.beyondtheboxscore.com/story/2006/2/23/164417/484`

2. Miller, Steven J.:

   ◇ *A Derivation of James' Pythagorean projection*, By The Numbers – The Newsletter of the SABR Statistical Analysis Committee, vol. 16 (February 2006), no. 1, 17–22.

   `http://www.philbirnbaum.com/btn2006-02.pdf`

   ◇ *A derivation of the Pythagorean Won-Loss Formula in baseball* (preprint, 13 pgs).

   `http://www.math.brown.edu/~sjmiller/math/papers/PythagWonLoss_Paper.pdf`

Appendix: Proof of the Pythagorean Won-Loss Formula

Proof. Let X and Y be independent random variables with Weibull distributions $(\alpha_{\mathrm{RS}}, \beta, \gamma)$ and $(\alpha_{\mathrm{RA}}, \beta, \gamma)$ respectively.

$$\alpha_{\mathrm{RS}} = \frac{\mathrm{RS} - \beta}{\Gamma(1+\gamma^{-1})}, \qquad \alpha_{\mathrm{RA}} = \frac{\mathrm{RA} - \beta}{\Gamma(1+\gamma^{-1})}.$$

We need only calculate the probability that X exceeds Y. We use the integral of a probability density is 1.

$$
\begin{aligned}
\text{Prob}(X > Y) &= \int_{x=\beta}^{\infty}\int_{y=\beta}^{x} f(x;\alpha_{\mathrm{RS}},\beta,\gamma) f(y;\alpha_{\mathrm{RA}},\beta,\gamma)\mathrm{d}y\ \mathrm{d}x \\
&= \int_{x=\beta}^{\infty}\int_{y=\beta}^{x} \frac{\gamma}{\alpha_{\mathrm{RS}}}\left(\frac{x-\beta}{\alpha_{RS}}\right)^{\gamma-1} e^{-\left(\frac{x-\beta}{\alpha_{\mathrm{RS}}}\right)^{\gamma}} \frac{\gamma}{\alpha_{\mathrm{RA}}}\left(\frac{y-\beta}{\alpha_{\mathrm{RA}}}\right)^{\gamma-1} e^{-\left(\frac{y-\beta}{\alpha_{\mathrm{RA}}}\right)^{\gamma}}\mathrm{d}y \\
&= \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{\mathrm{RS}}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-\left(\frac{x}{\alpha_{\mathrm{RS}}}\right)^{\gamma}} \left[\int_{y=0}^{x} \frac{\gamma}{\alpha_{\mathrm{RA}}}\left(\frac{y}{\alpha_{\mathrm{RA}}}\right)^{\gamma-1} e^{-\left(\frac{y}{\alpha_{\mathrm{RA}}}\right)^{\gamma}}\mathrm{d}y\right] \\
&= \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{\mathrm{RS}}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-(x/\alpha_{\mathrm{RS}})^{\gamma}} \left[1 - e^{-(x/\alpha_{\mathrm{RA}})^{\gamma}}\right]\mathrm{d}x \\
&= 1 - \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{\mathrm{RS}}}\left(\frac{x}{\alpha_{RS}}\right)^{\gamma-1} e^{-(x/\alpha)^{\gamma}}\mathrm{d}x,
\end{aligned}
$$

where we have set

$$
\frac{1}{\alpha^{\gamma}} = \frac{1}{\alpha_{\mathrm{RS}}^{\gamma}} + \frac{1}{\alpha_{\mathrm{RA}}^{\gamma}} = \frac{\alpha_{\mathrm{RS}}^{\gamma} + \alpha_{\mathrm{RA}}^{\gamma}}{\alpha_{\mathrm{RS}}^{\gamma}\alpha_{\mathrm{RA}}^{\gamma}}.
$$

$$
\begin{aligned}
\text{Prob}(X > Y) &= 1 - \frac{\alpha^\gamma}{\alpha_{\mathrm{RS}}^\gamma} \int_0^\infty \frac{\gamma}{\alpha} \left(\frac{x}{\alpha}\right)^{\gamma-1} e^{(x/\alpha)^\gamma} \mathrm{d}x \\
&= 1 - \frac{\alpha^\gamma}{\alpha_{\mathrm{RS}}^\gamma} \\
&= 1 - \frac{1}{\alpha_{\mathrm{RS}}^\gamma} \frac{\alpha_{\mathrm{RS}}^\gamma \alpha_{\mathrm{RA}}^\gamma}{\alpha_{\mathrm{RS}}^\gamma + \alpha_{\mathrm{RA}}^\gamma} \\
&= \frac{\alpha_{\mathrm{RS}}^\gamma}{\alpha_{\mathrm{RS}}^\gamma + \alpha_{\mathrm{RA}}^\gamma}.
\end{aligned}
$$

We substitute the relations for α_{RS} and α_{RA} and find that

$$
\text{Prob}(X > Y) = \frac{(\mathrm{RS} - \beta)^\gamma}{(\mathrm{RS} - \beta)^\gamma + (\mathrm{RA} - \beta)^\gamma}.
$$

Note $\mathrm{RS} - \beta$ estimates $\mathrm{RS}_{\mathrm{obs}}$, $\mathrm{RA} - \beta$ estimates $\mathrm{RA}_{\mathrm{obs}}$. $\square$