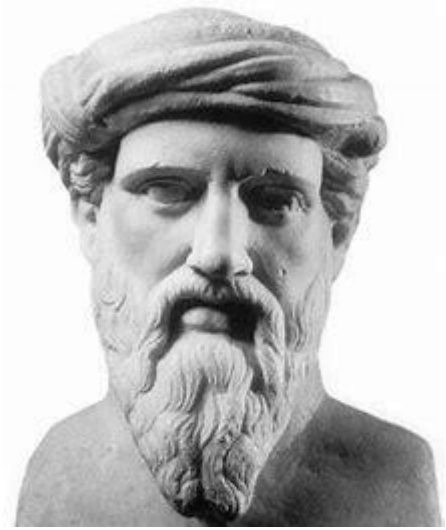
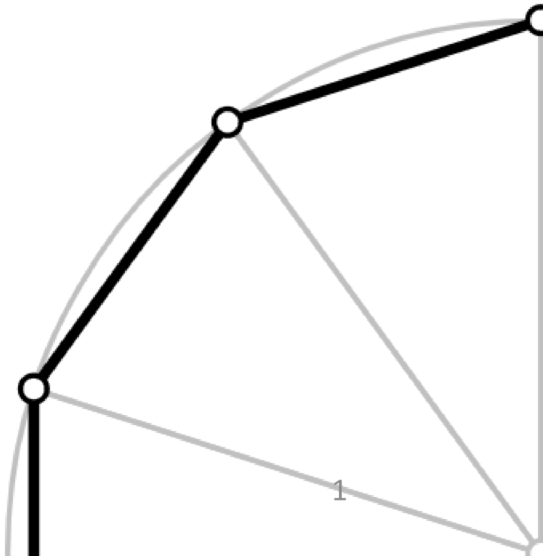


From Pythagoras to π

Steven J Miller, Williams College
sjm1@Williams.edu



There are many wonderful formulas for π.

Our goal is to derive enough **TRIGONOMETRY** to explain.

We assume two key formulas: the area of a circle of radius r is πr^2 and the perimeter of a circle of radius r is $2 \pi r$.

Gregory – Leibniz Formula for π

Claim: $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + 1/13 - 1/15 + 1/17 - 1/19 +$ you get the idea….

One of the best ways to prove this is to use the Geometric Series Formula, integration, and the arctangent function.

Instead of proving it let's test it.

Note we can make the algebra a little nicer:

$1 - 1/3 =$
$1/5 - 1/7 =$
$1/9 - 1/11 =$

Gregory – Leibniz Formula for π

Claim: $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + 1/13 - 1/15 + 1/17 - 1/19 +$ you get the idea….

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$1 - 1/3 = 2/3 =$
$1/5 - 1/7 = 2/35 =$
$1/9 - 1/11 = 2/99 =$

Gregory – Leibniz Formula for π

Claim: $\pi/4 = 1 – 1/3 + 1/5 – 1/7 + 1/9 – 1/11 + 1/13 – 1/15 + 1/17 – 1/19 +$ you get the idea….

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$1 – 1/3 = 2/3 = 2/(4-1)$
$1/5 – 1/7 = 2/35 = 2/(36-1)$
$1/9 – 1/11 = 2/99 = 2/(100-1)$

Gregory – Leibniz Formula for π

Claim: $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + 1/13 - 1/15 + 1/17 - 1/19 +$ you get the idea….

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$1 - 1/3 = 2/3 = 2/(4-1) = 2/(2^2 - 1)$
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$1/9 - 1/11 \;=\; 2/99 \;=\; 2/(100-1) = 2/(10^2 - 1)$

So we can write it a bit more concisely as $\pi/4 = 2(1/(2^2 - 1) + 1/(6^2 - 1) + 1/(10^2 - 1) + \ldots)$,
or $\pi =$

Gregory – Leibniz Formula for π

Claim: $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + 1/9 - 1/11 + 1/13 - 1/15 + 1/17 - 1/19 +$ you get the idea….

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So we can write it a bit more concisely as $\pi/4 = 2(1/(2^2 - 1) + 1/(6^2 - 1) + 1/(10^2 - 1) + \ldots)$,
or $\pi = 8(1/(2^2 - 1) + 1/(6^2 - 1) + 1/(10^2 - 1) + 1/(14^2 + 1) + (1/18^2 + 1) + \ldots.)$.

Gregory – Leibniz Formula for π

$\pi = 8(1/(2^2 - 1) + 1/(6^2 - 1) + 1/(10^2 - 1) + 1/(14^2 + 1) + (1/18^2 + 1) + \ldots).$

Let's get some values as we take more and more terms.

```
f[n_] := Sum[8 / ( (4k - 2)^2 - 1 ), {k, 1,n}]
Print["Pi is approximately ", N[Pi,20]]
For[n = 0, n <= 3, n++, Print["n = ",10n + 1, ", approximation is ", f[10n+1], " or approximately ", 1.0 f[10n+1]]]
```

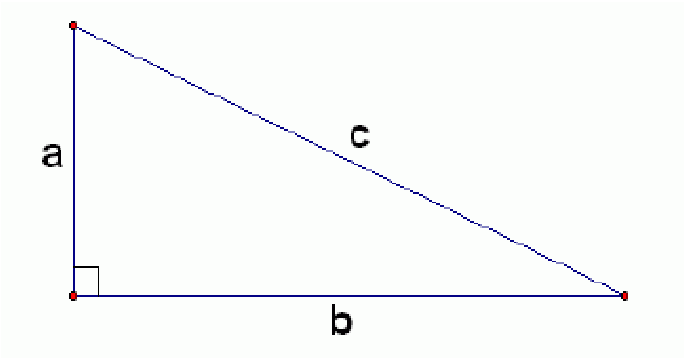
π is approximately 3.1415926535897932385
n = 1 , approximation is 8/3 or approximately 2.66667
n = 11 , approximation is 911392701638017048/294362129962575675 or approximately 3.09616
n = 21 , approximation is 30263630165326248720686195856232072/9706767974090987674271719115507175 or approximately 3.11779
n = 31 , approximation is 513492776497882117081648376432908105776082650140936 / 164293258993570516478884709700298185808795185630525 or approximately 3.12546

If we take n=1000 our approximation is about 3.14109, while for n=1,000,000 it is about 3.14159215. Notice it is only slowly converging…..

Remember π is approximately 3.1415926535897932385

Introduction to Trigonometry

Geometry Gem: Pythagorean Theorem



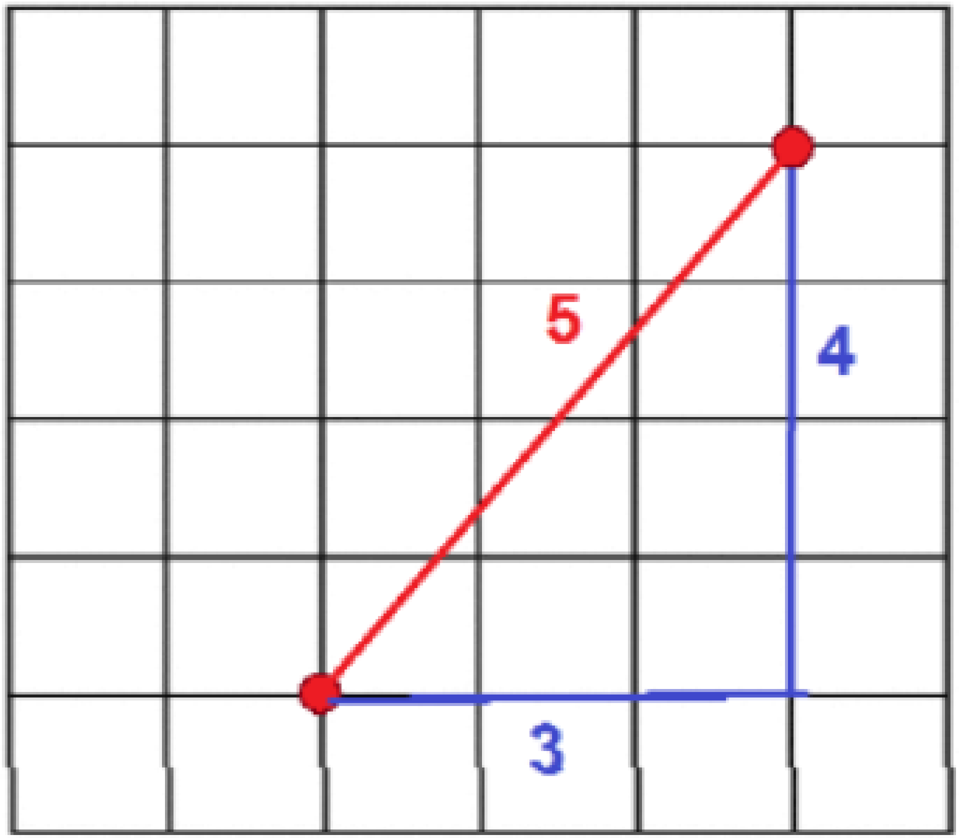
Theorem (Pythagorean Theorem)

Right triangle with sides a, b and hypotenuse c, then
$a^2 + b^2 = c^2$.

Why do we care about Pythagoras?

We use the Pythagorean formula to find the distance between two points. It works very well when we have grid coordinates for our start and our end.

Using $a^2 + b^2 = c^2$.



Geometric Proofs of Pythagoras

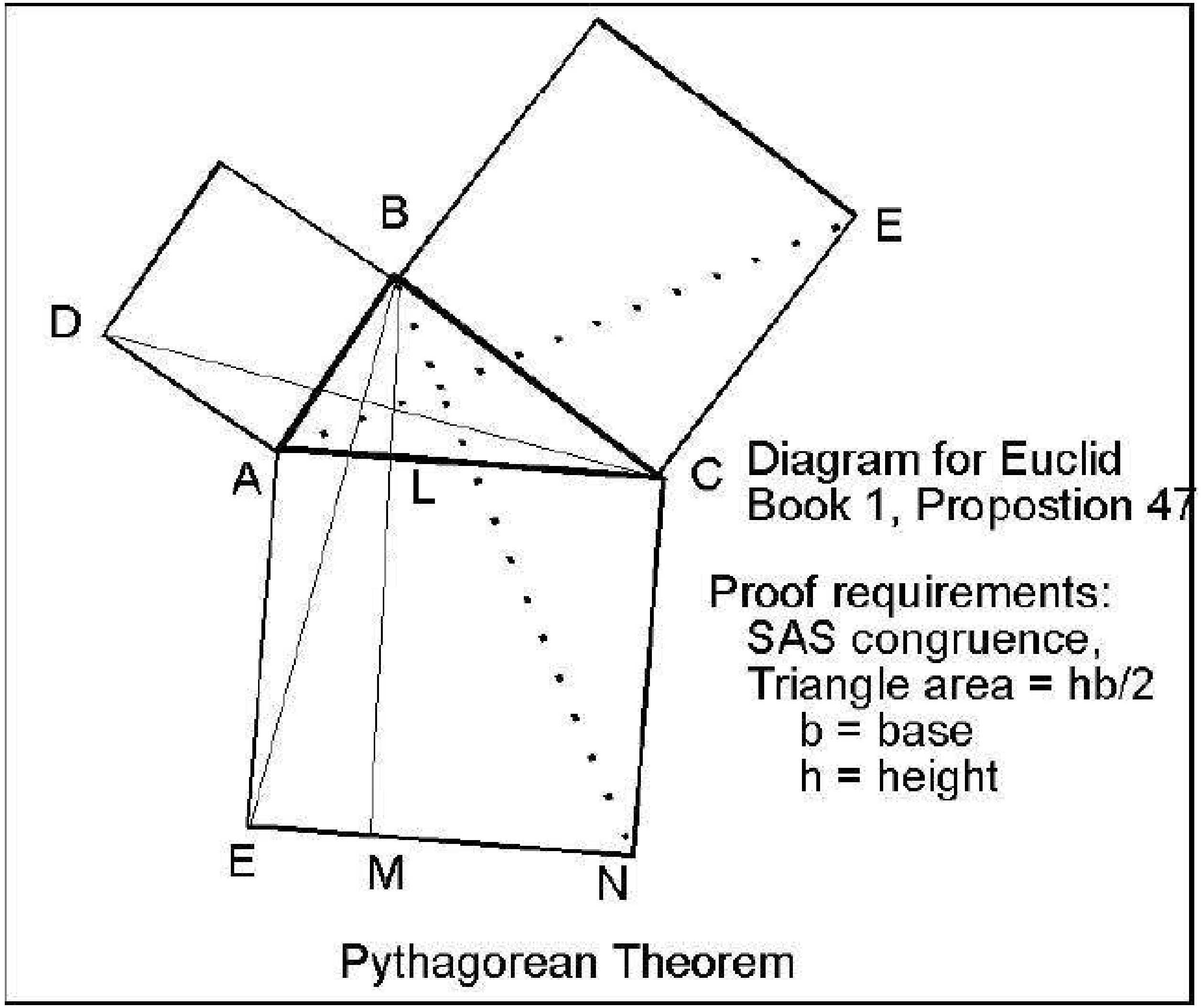


Figure: Euclid's Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?

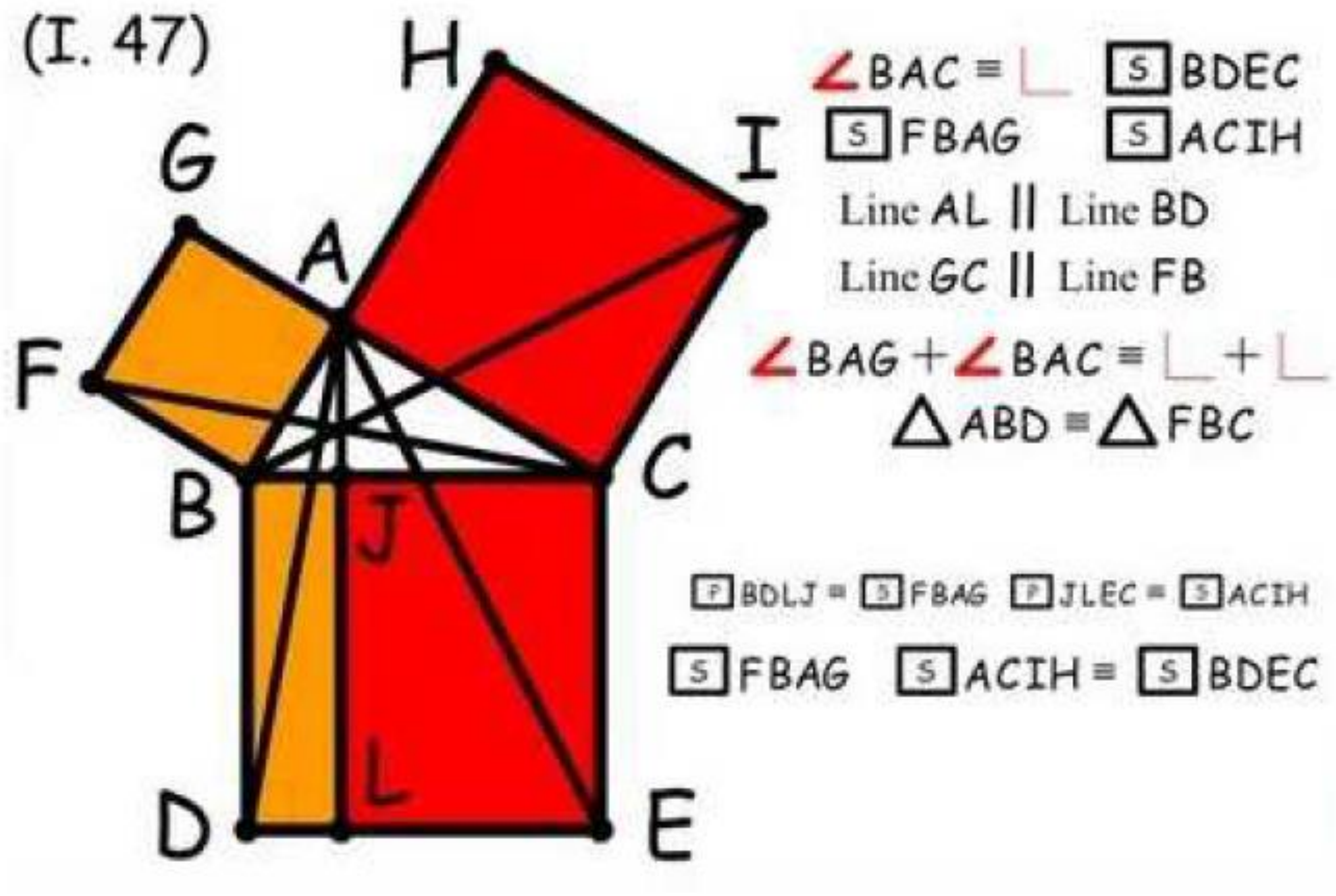


Figure: Euclid's Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?

Geometric Proofs of Pythagoras

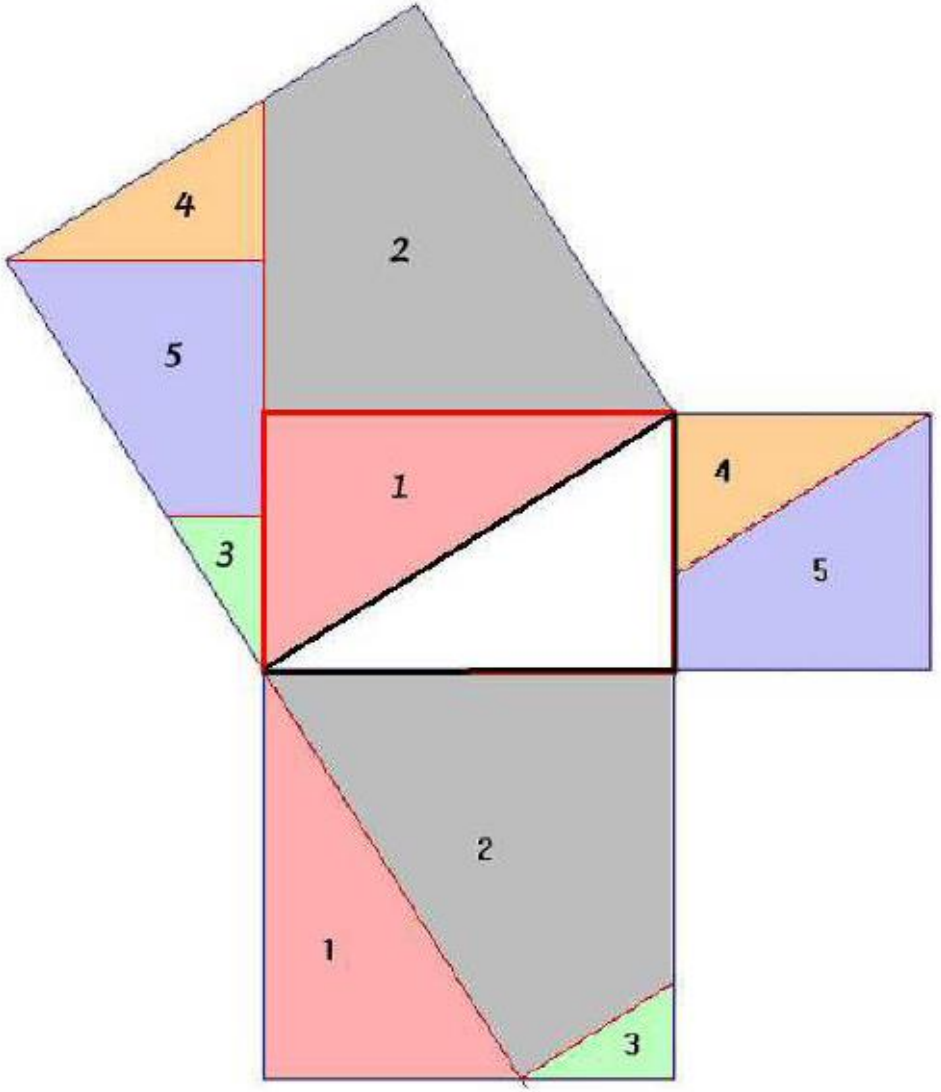
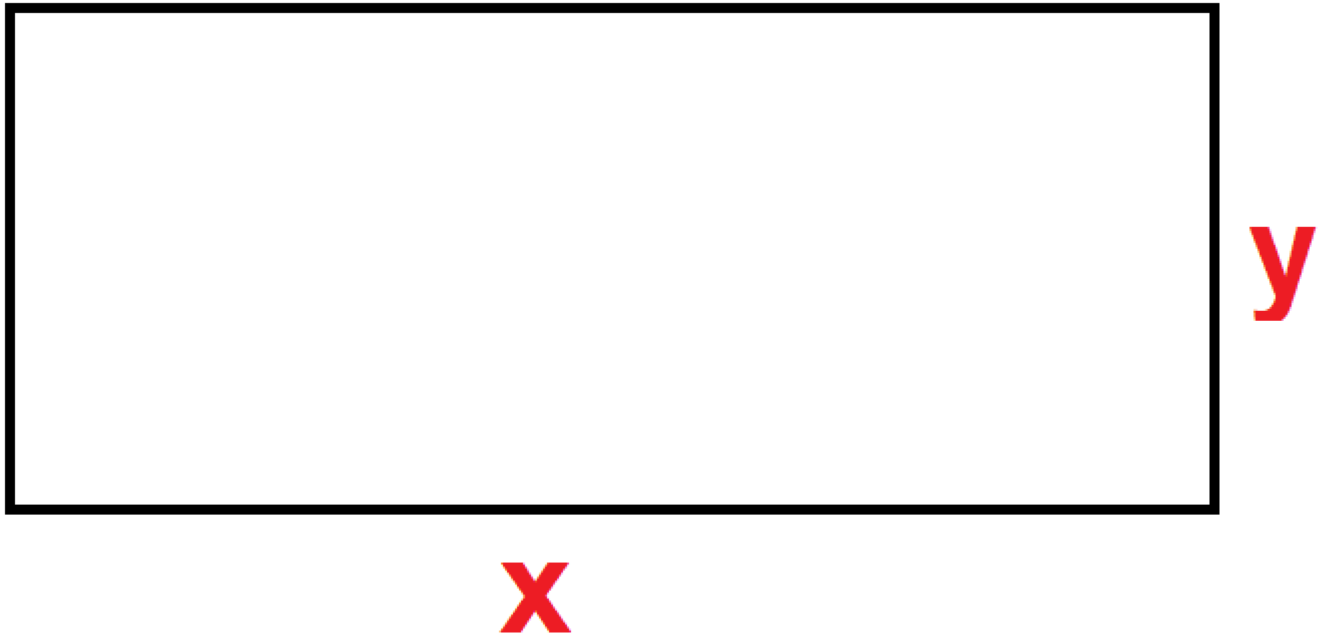


Figure: A nice matching proof, but how to find these slicings!

What are our assumptions?

We assume two items.

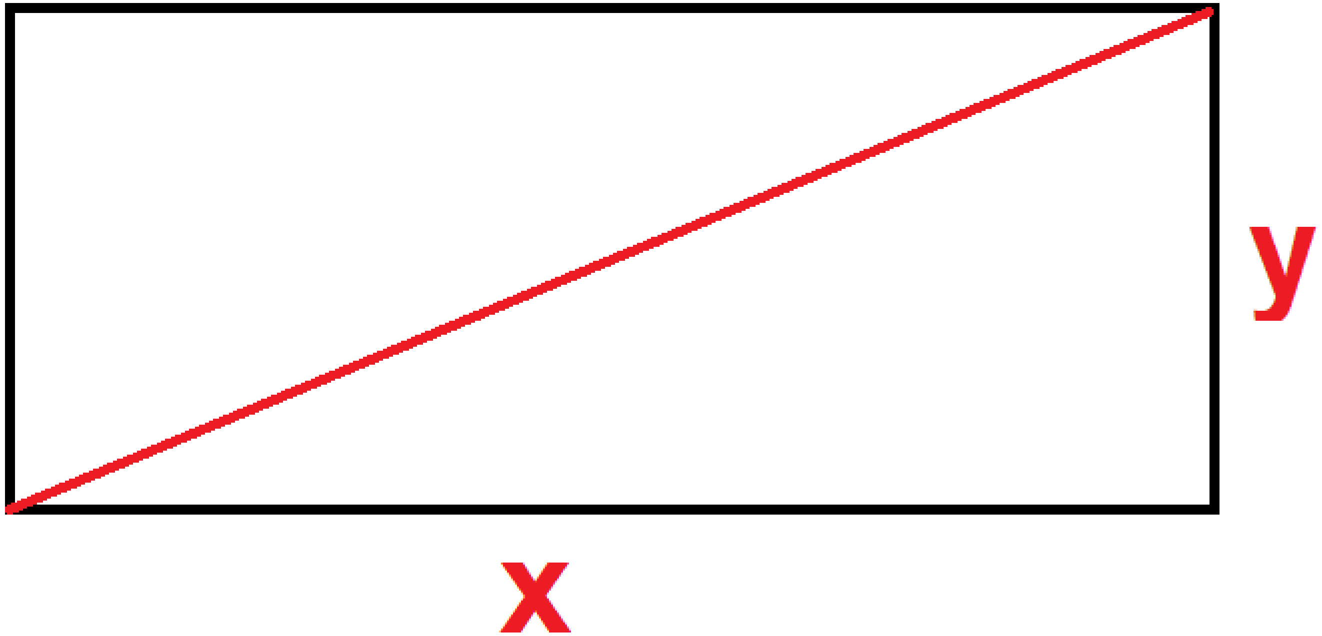
1. The area of a rectangle x by y is x y.



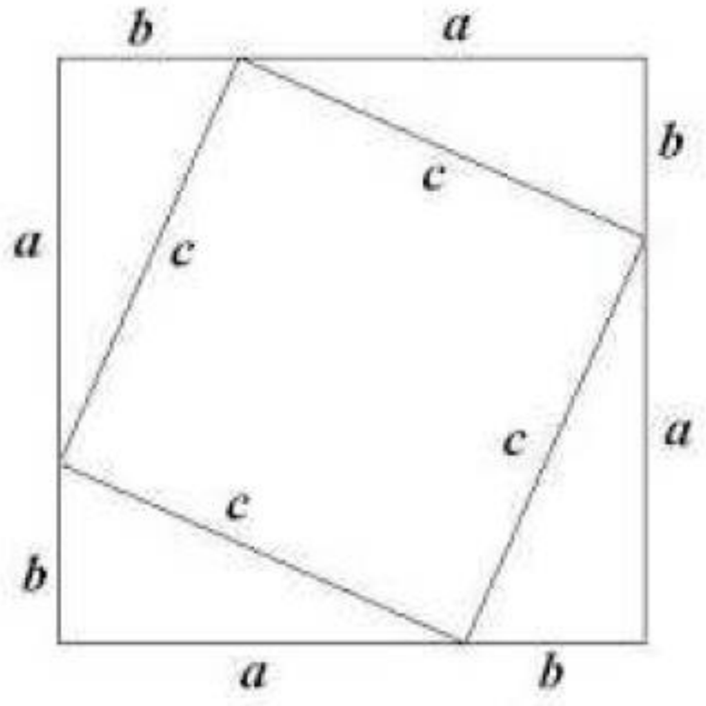
What are our assumptions?

We assume two items.

1. The area of a rectangle x by y is x y.
2. The area of a right triangle is ½ x y.



Geometric Proofs of Pythagoras



Big square: $(a+b)^2$

$$= a^2 + 2ab + b^2$$

Four triangles $= 2ab$

Little square $= c^2$

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

$$a^2 + b^2 = c^2$$

Figure: Four triangles proof: I

Geometric Proofs of Pythagoras

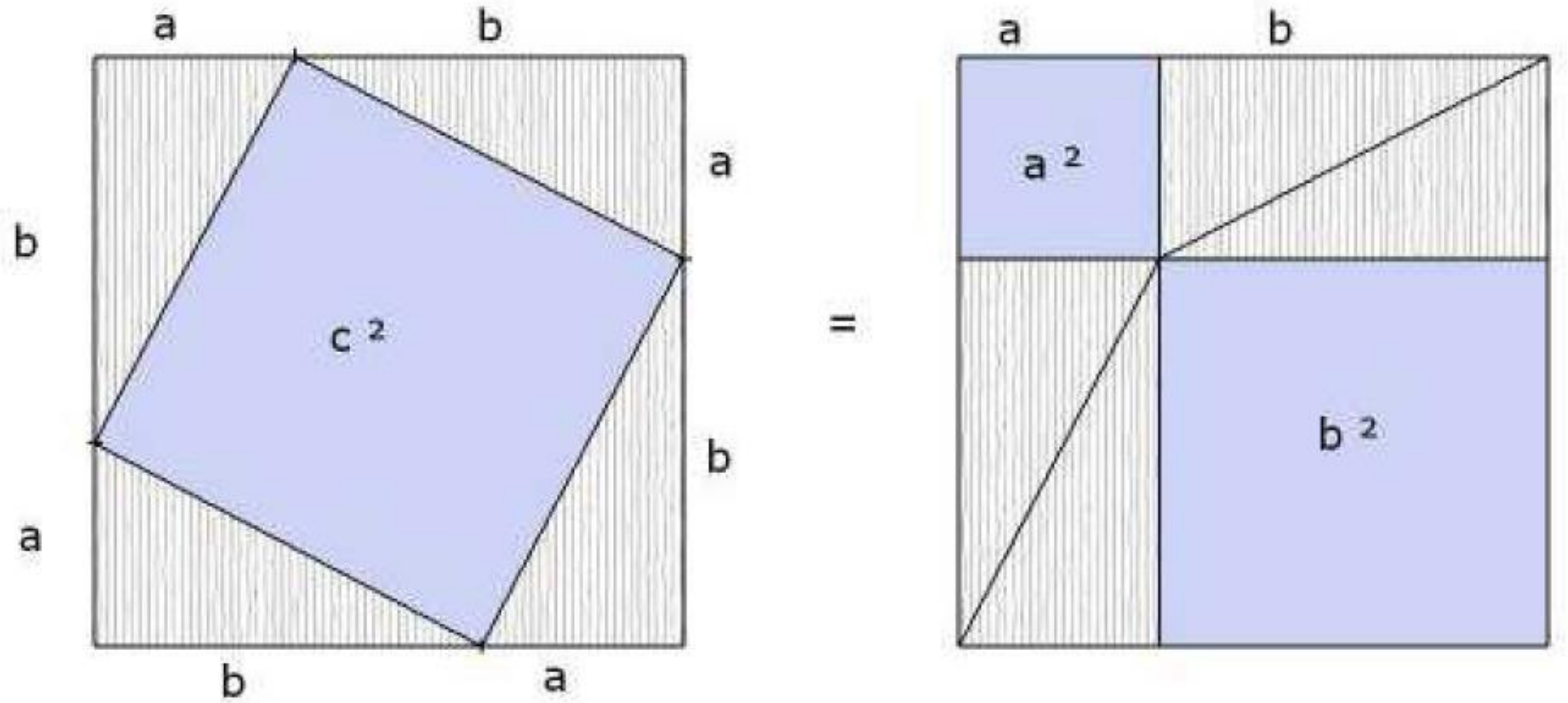


Figure: Four triangles proof: II

Geometric Proofs of Pythagoras

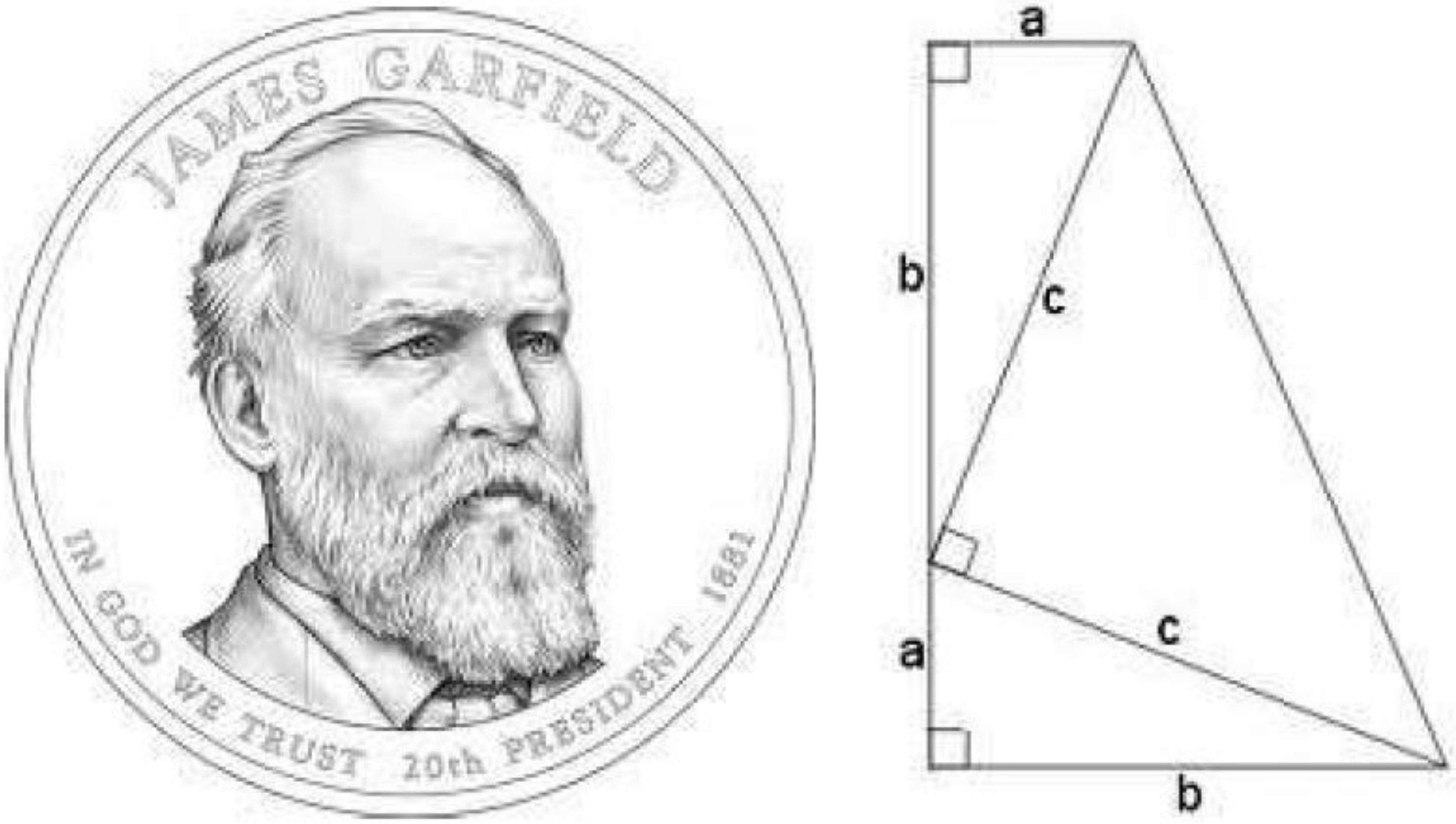
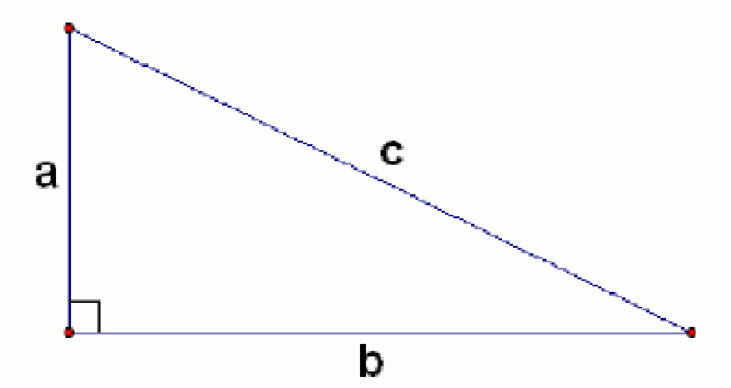


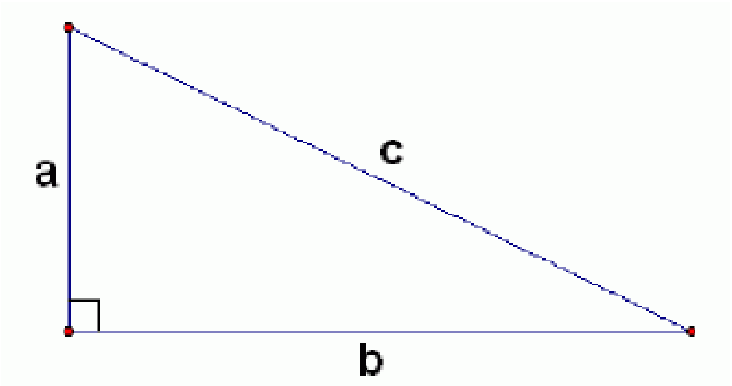
Figure: President James Garfield's (Williams 1856) Proof.

Possible Pythagorean Theorems....



- $c^2 = a^{17} + b^{17}$.
- $c^2 = a^2 + 17b^2$.
- $c^2 = a^2 - b^2$.
- $c^2 = a^2 + ab + b^2$.
- $c^2 = a^2 + 17ab + b^2$.

Possible Pythagorean Theorems....



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- $c^2 = a^2 + ab + b^2$.
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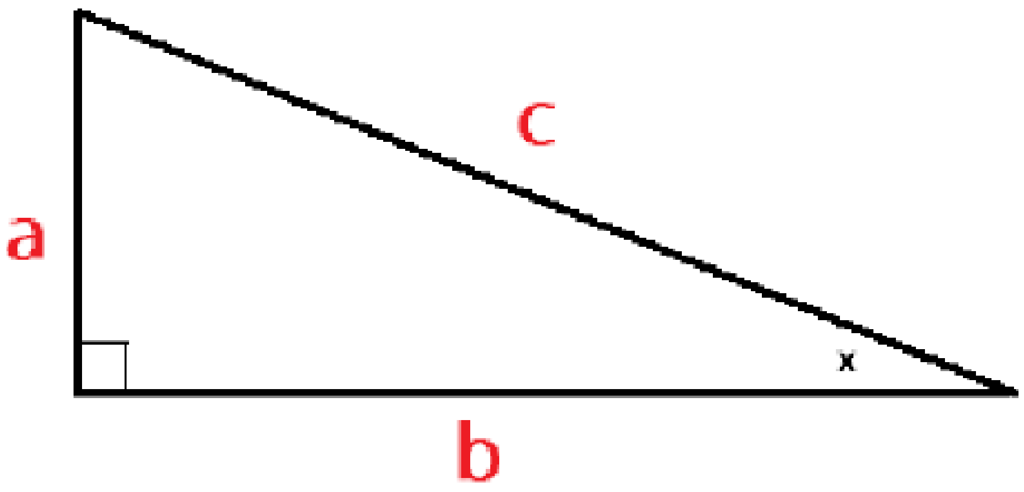
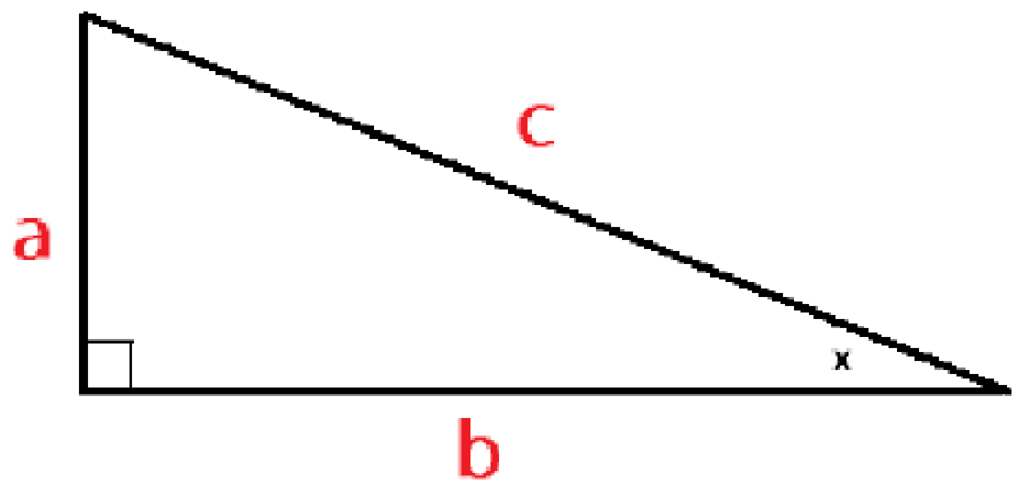
All of these formulas have problems, from symmetry issues to negatives to too large EXCEPT the fourth.

Dimensional Analysis Proof of the Pythagorean Theorem



◇ Area is a function of hypotenuse c and angle x.

Dimensional Analysis Proof of the Pythagorean Theorem



- ◇ Area is a function of hypotenuse c and angle x.
- ◇ $\text{Area}(c, x) = f(x)c^2$ for some function f (similar triangles).

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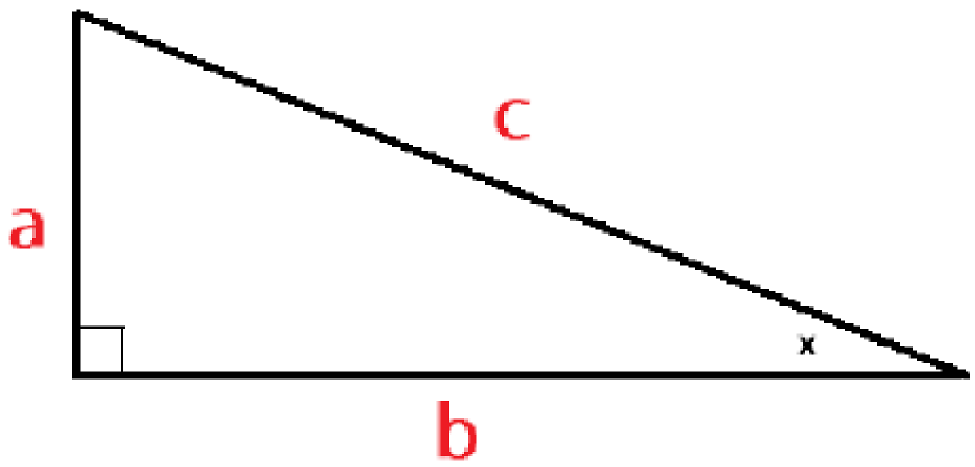

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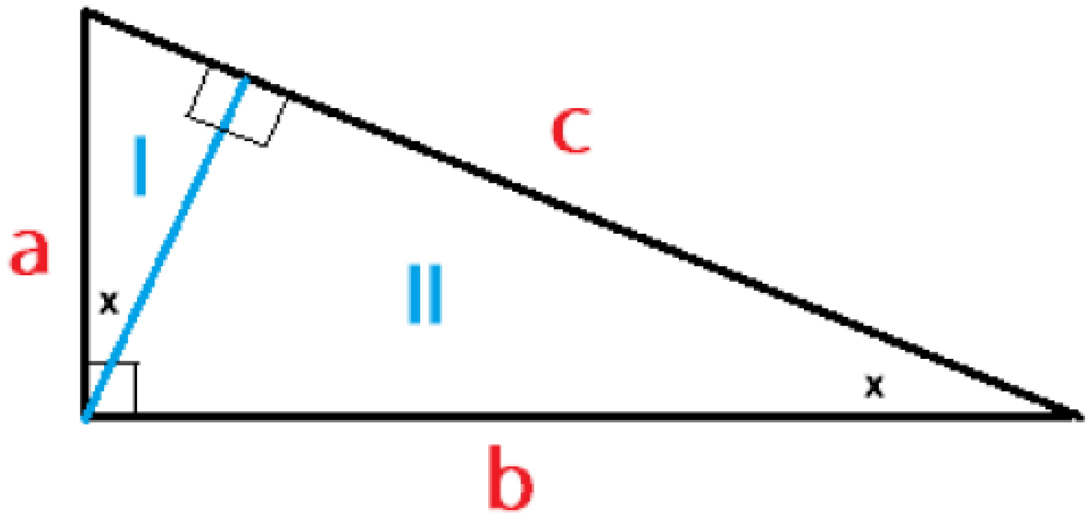

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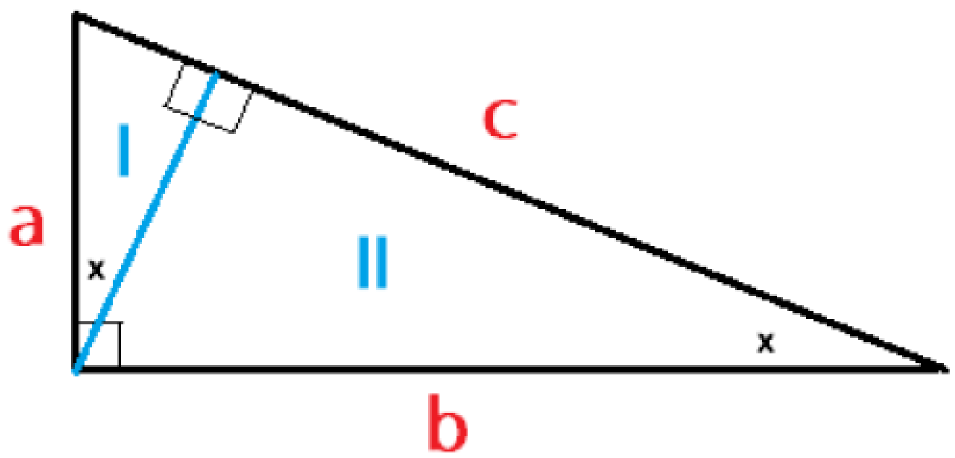

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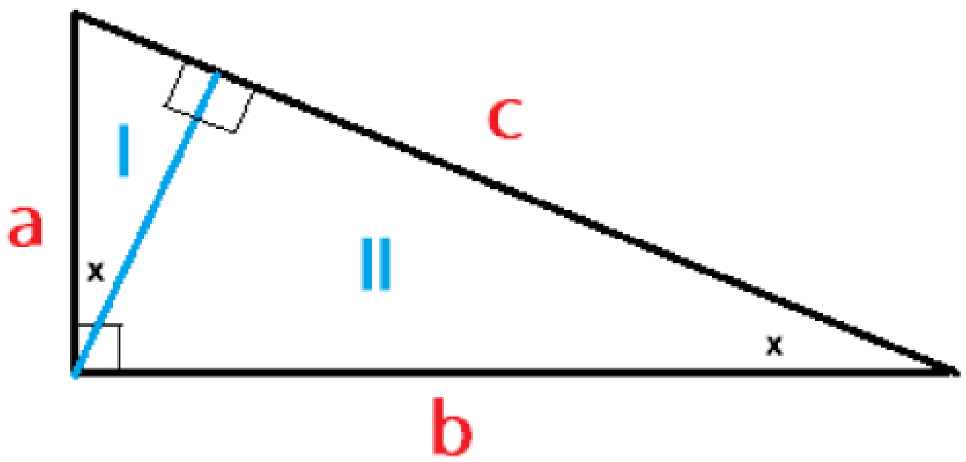
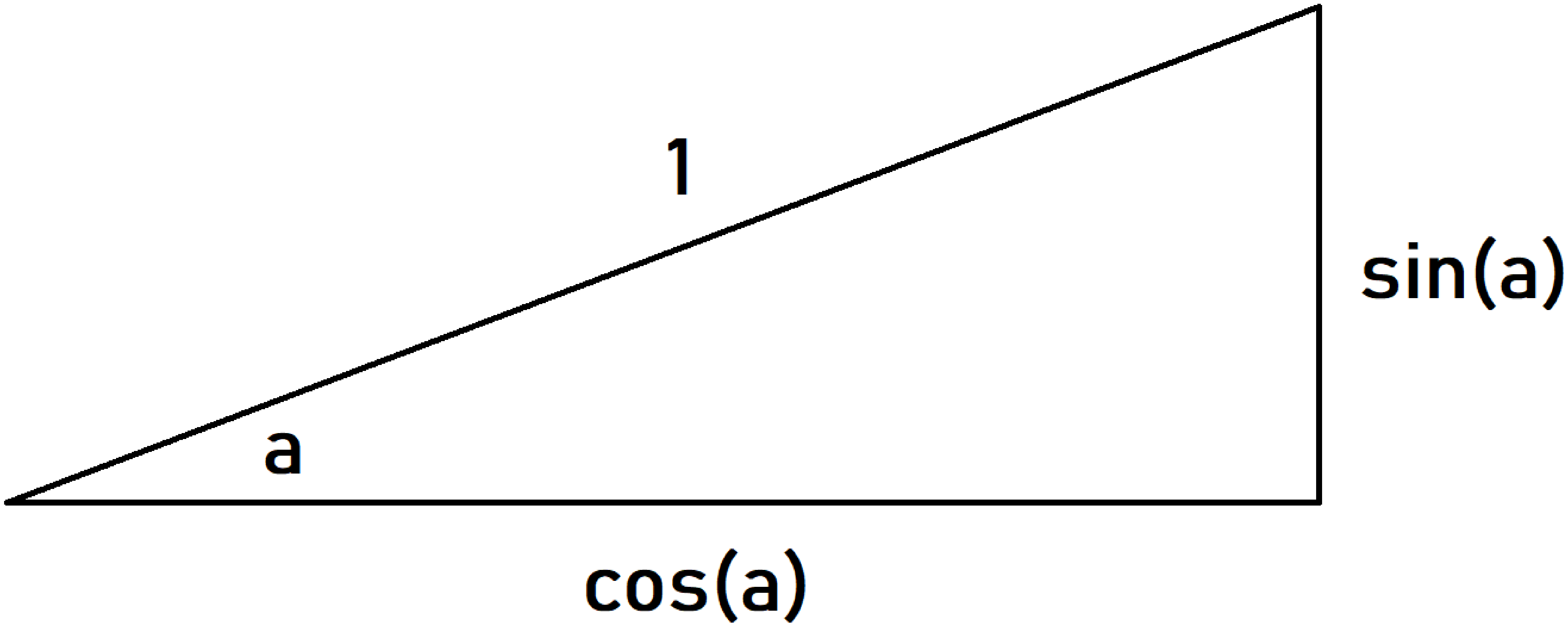
- $f(x)a^2 + f(x)b^2 = f(x)c^2$

Dimensional Analysis Proof of the Pythagorean Theorem



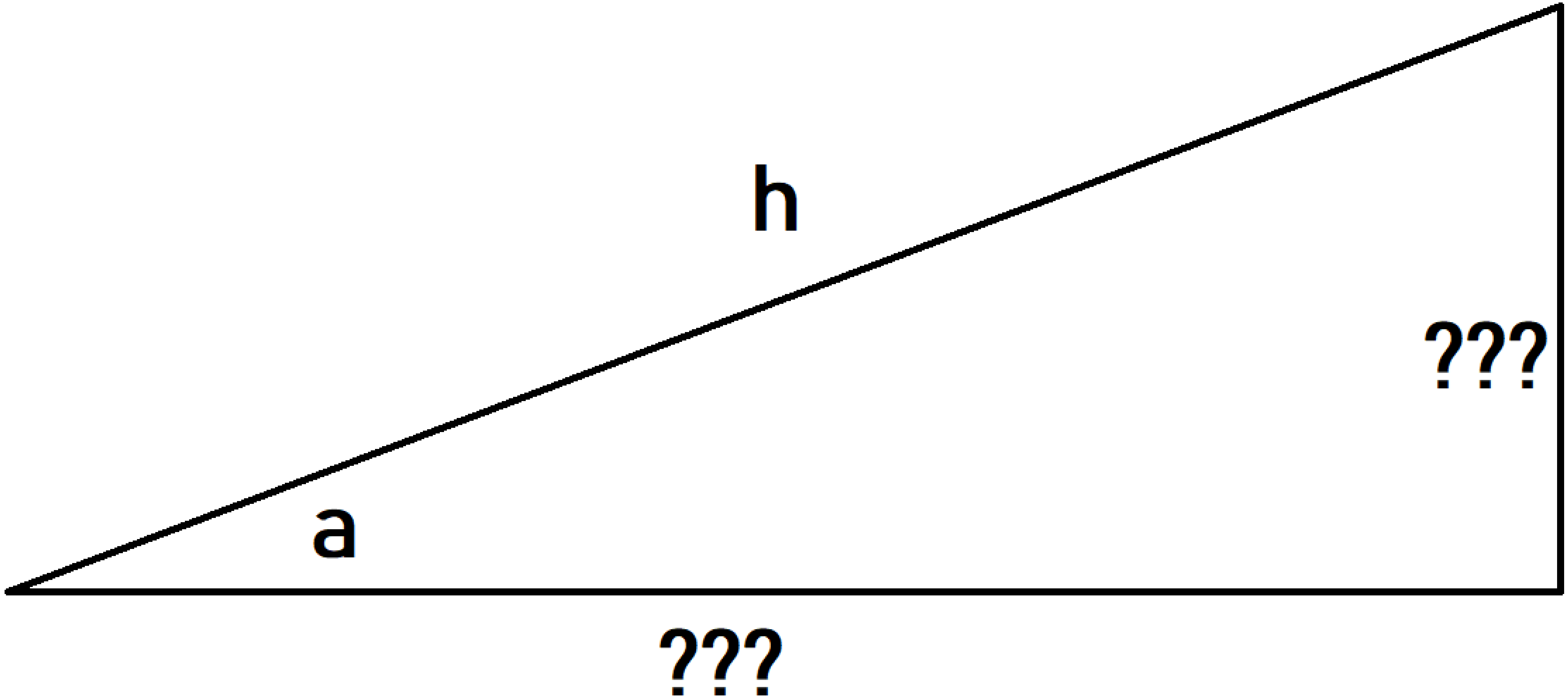
- ◇ Area is a function of hypotenuse c and angle x.
- ◇ $\mathrm{Area}(c, x) = f(x)c^2$ for some function f (CPCTC).
- ◇ Must draw an auxiliary line, but where? Need right angles!
- ◇ $f(x)a^2 + f(x)b^2 = f(x)c^2 \Rightarrow a^2 + b^2 = c^2$.

Introduction to Trigonometry Functions: sine and cosine.



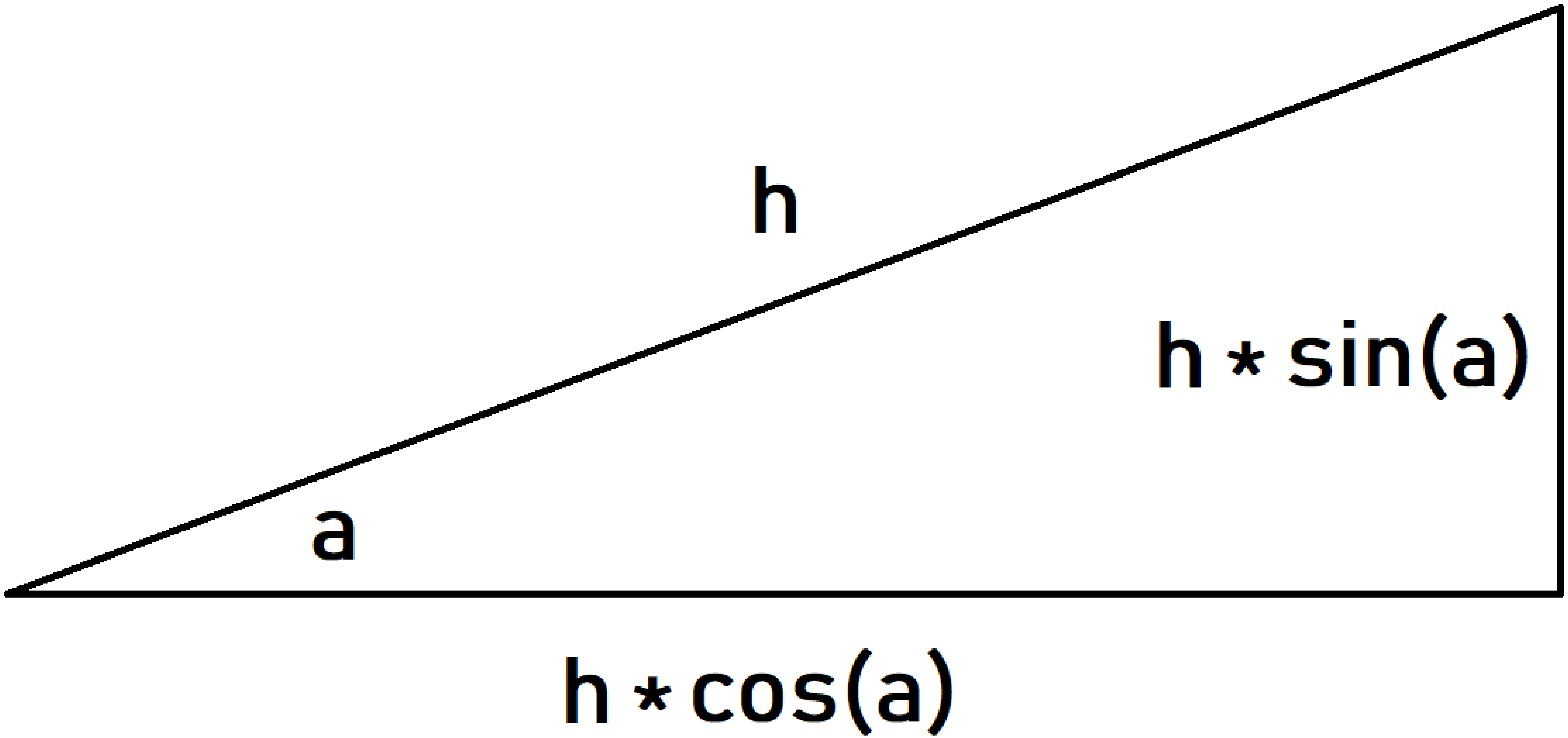
In a right triangle with angle a and hypotenuse 1, we define two functions: cosine and sine, abbreviated cos(a) and sin(a), as above. Note cosine is the side adjacent to the angle a.

Introduction to Trigonometry Functions: sine and cosine.



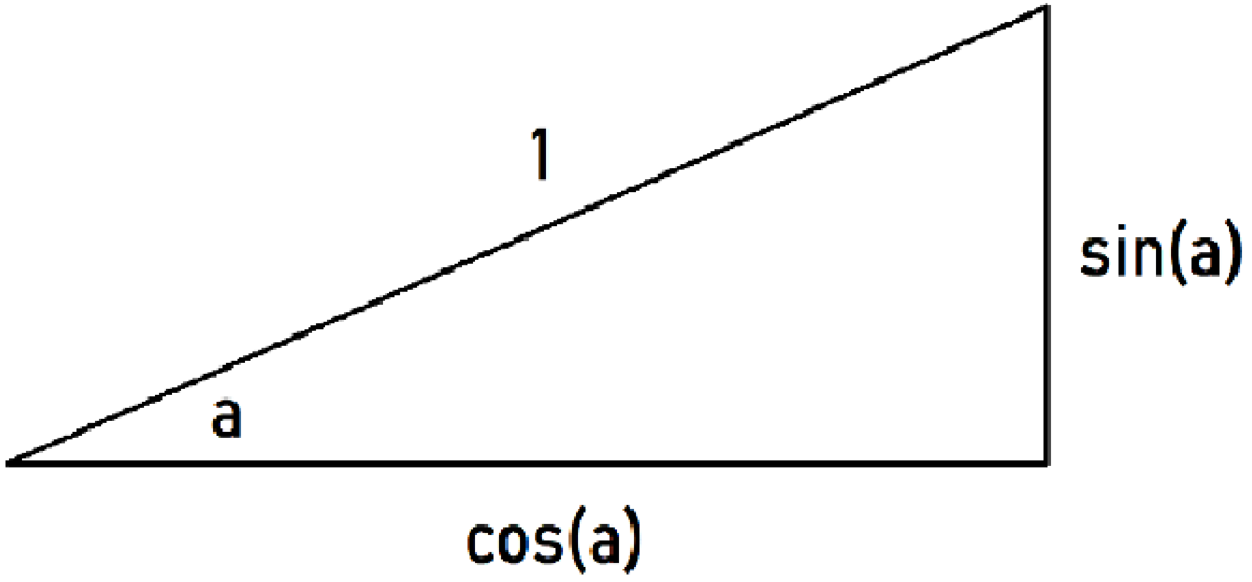
What happens to the lengths of the two sides if the hypotenuse is now h units, not 1 unit?

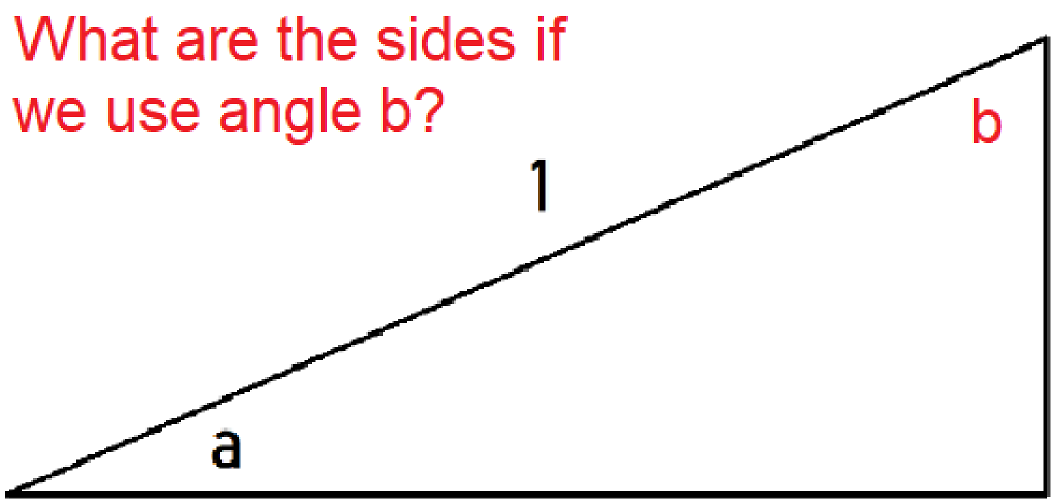
Introduction to Trigonometry Functions: sine and cosine.



We just rescale: think about going from 1 yard to 3 feet; the triangle has the same relative proportions, but the numbers will all be increased by a factor of 3.

Introduction to Trigonometry Functions: sine and cosine.





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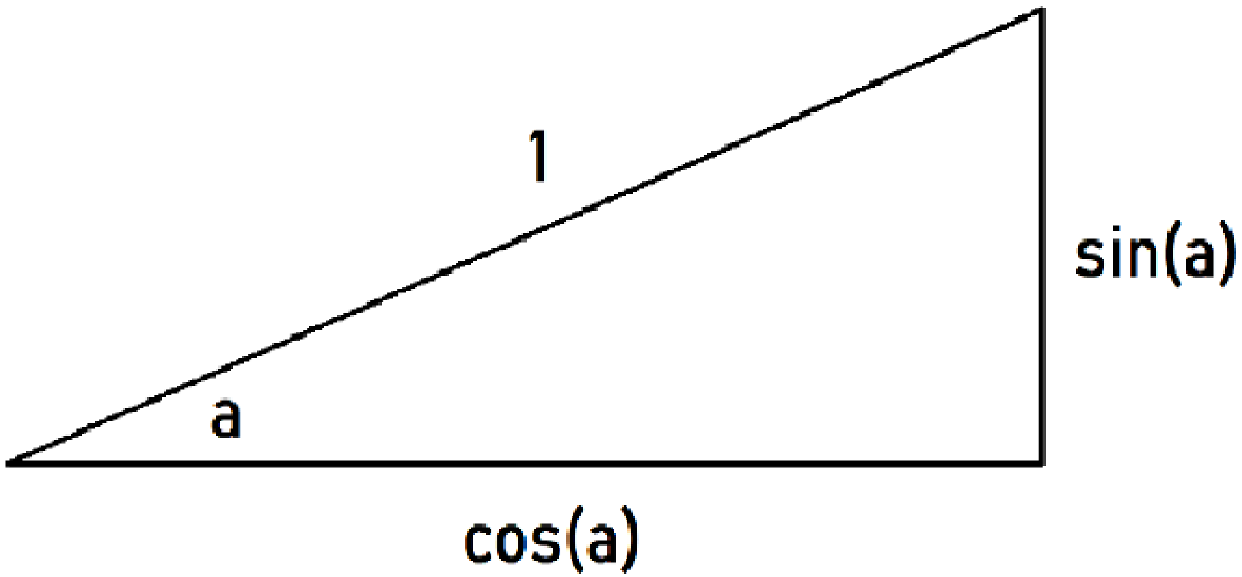
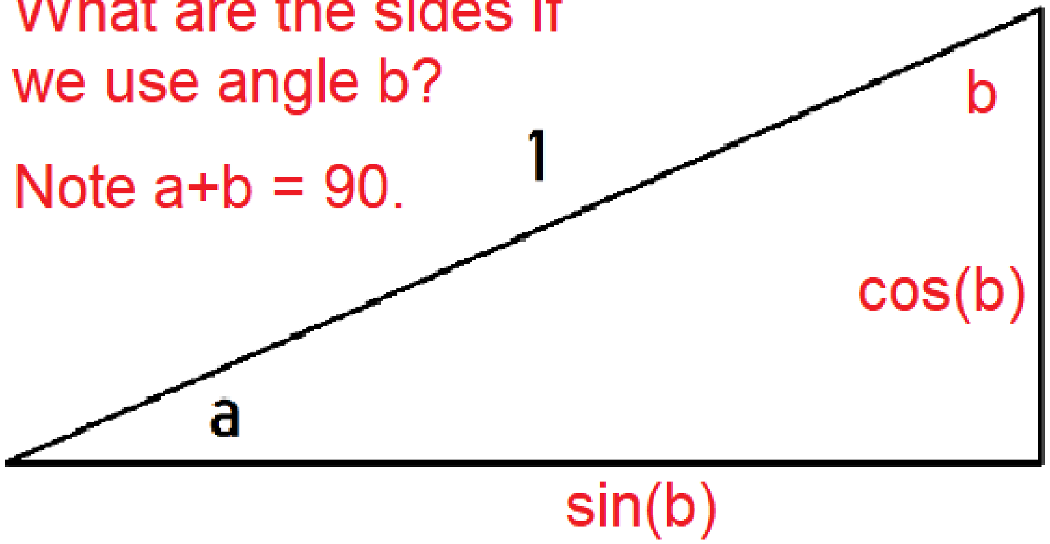
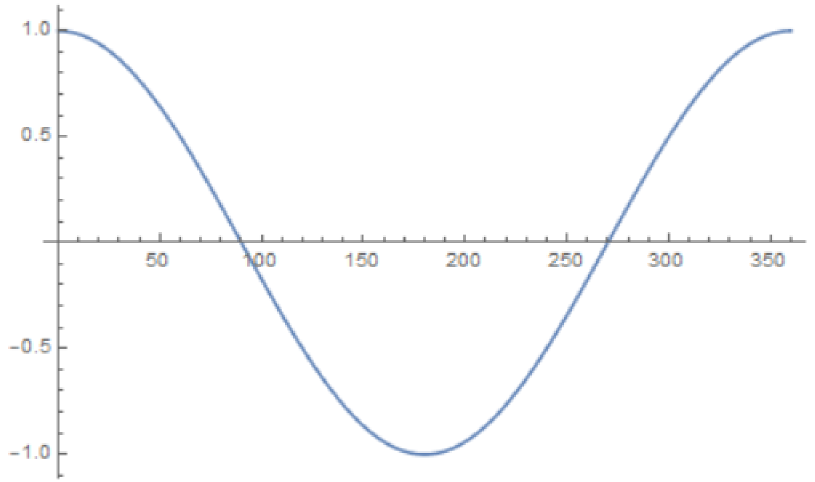


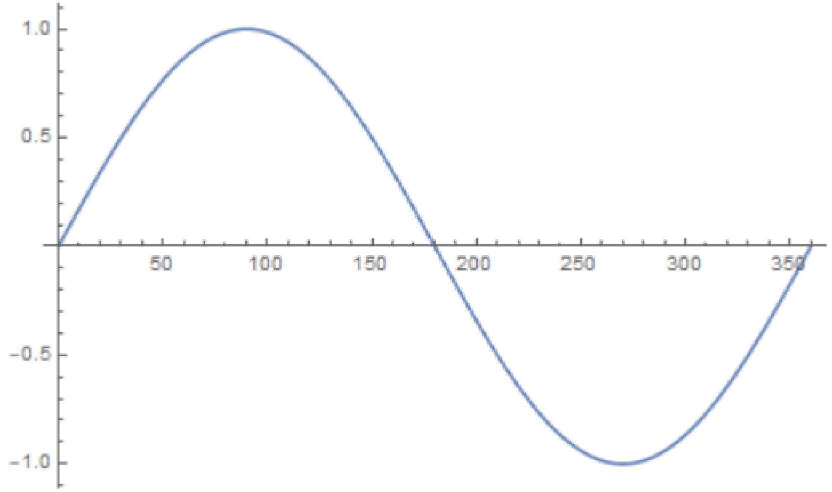


In a right triangle with angle a and hypotenuse 1, we define two functions: cosine and sine, abbreviated cos(a) and sin(a), as above. Note cosine is the side adjacent to the angle a.

We proved sin(90-a) = cos(a) and cos(90-a) = sin(a) (at least if a and b are at most 90)

Trigonometry Plots: Cos(90-a) = Sin(a)   and Sin(90-a) = Cos(a)





Plot of Cos(a) top, plot of Sin(a) bottom.

Notice that they do appear to be shifted by 90 degrees.

Note also they appear to be periodic, returning to where they started.

Finally, note they are very different than the linear and quadratic functions you may have seend.

Equations of Lines

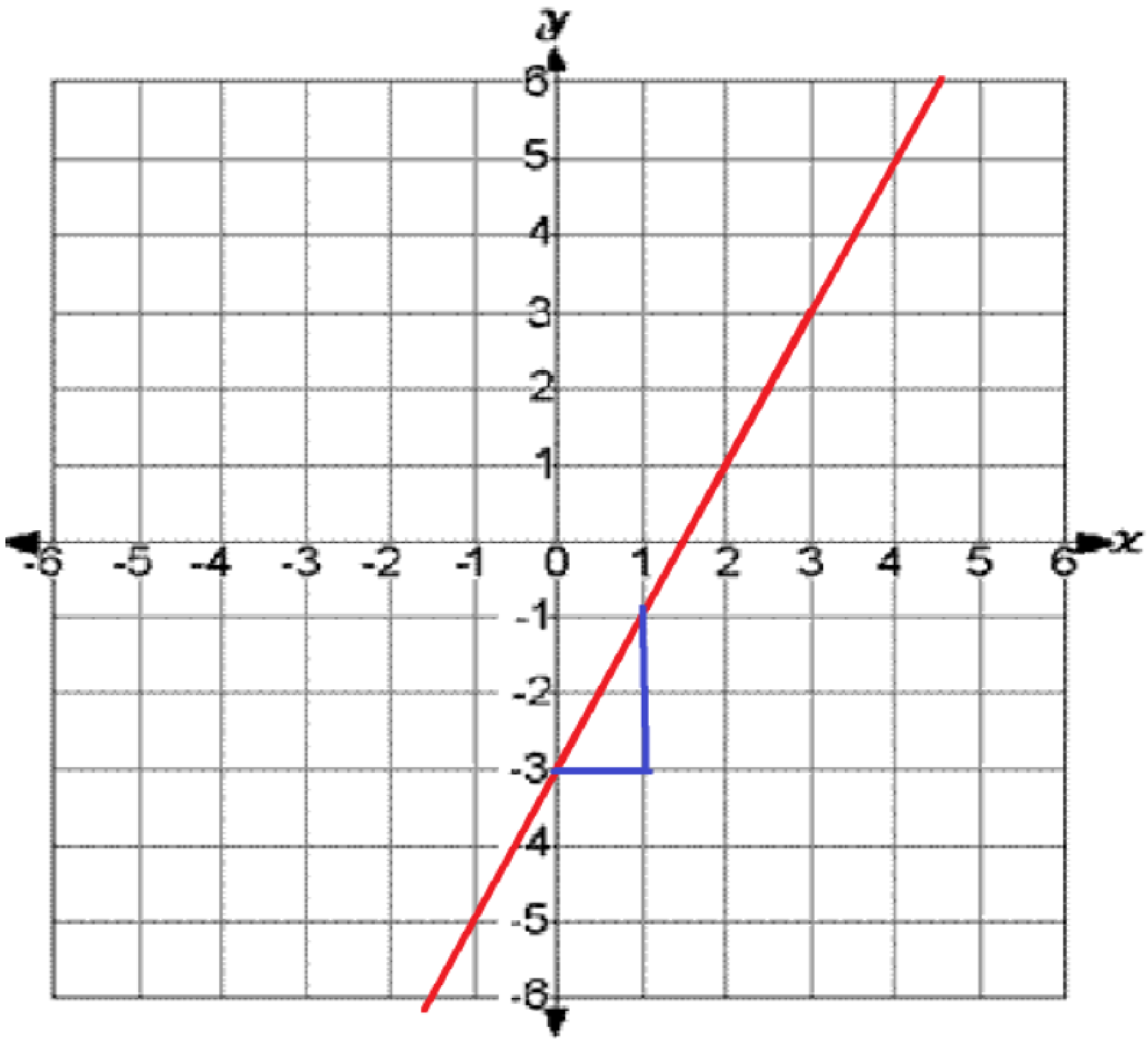
There are many ways to write down the equation of a line. One of the most commons is the slope-intercept, where we write y = m x + b; m is the slope and b is the intercept.

The slope m is the rise over the run; if we increase x by 1 it tells us how much y increases.

The intercept b is where the line crosses the y-axis; in other words, it is the value of y when x = 0.

What is the slope here: m =

What is the intercept: b =



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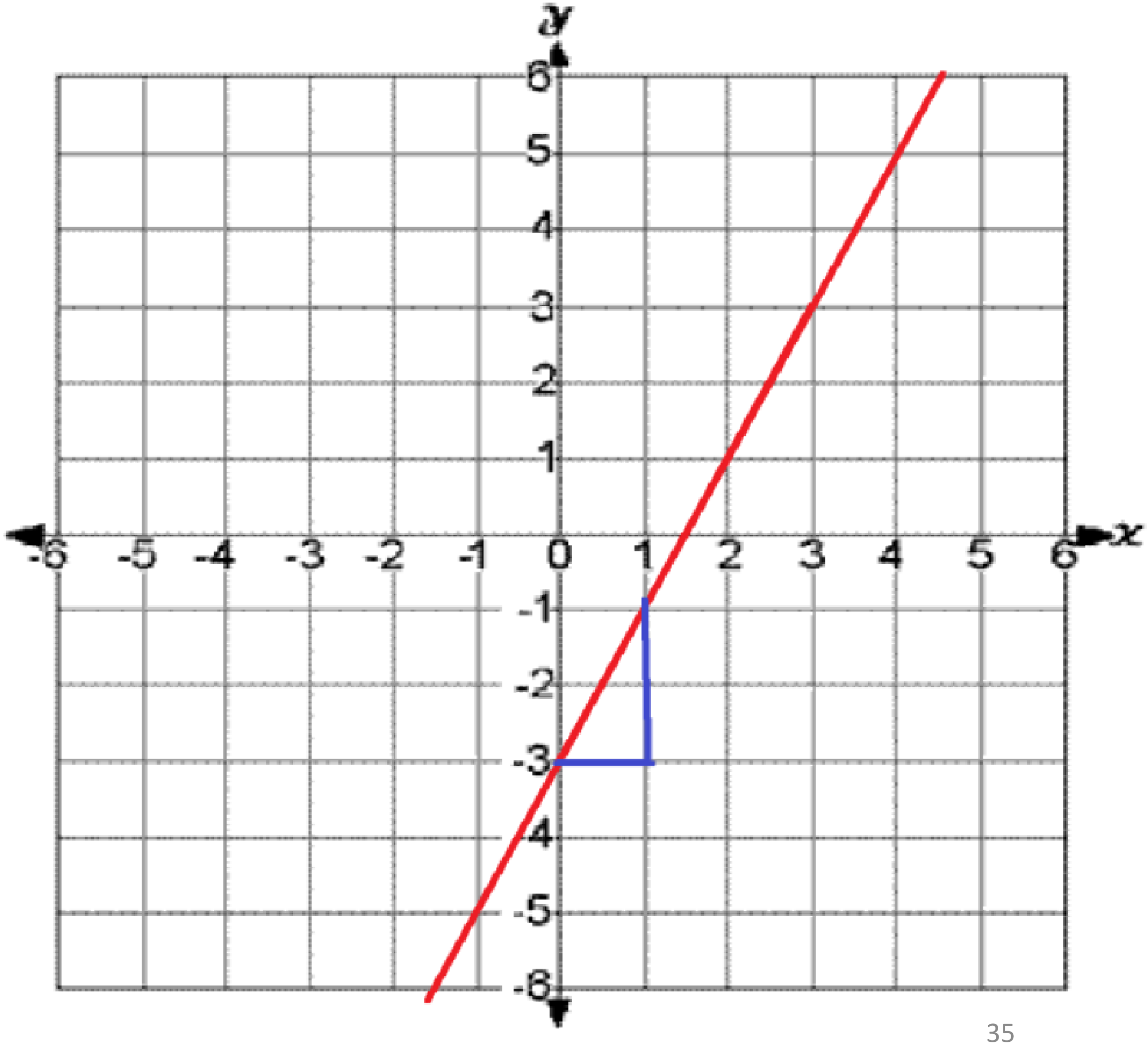
What is the slope here: m = 2

What is the intercept: b = -3

When x increases by 1, y increases by 2, so slope is 2.



Algebra: Expanding Expressions

We need just a bit of algebra: we need FOIL.

It stands for First, Outside, Inside, Last, and is a way to multiply two expressions.

If we have (a+b) (c+d) this is ac + ad + bc + bd.

Thus $(x+y)(x-y) = xx - xy + yx - yy = x^2 - y^2$.

We need $(mx - 1)^2 = (mx - 1)(mx - 1) =$

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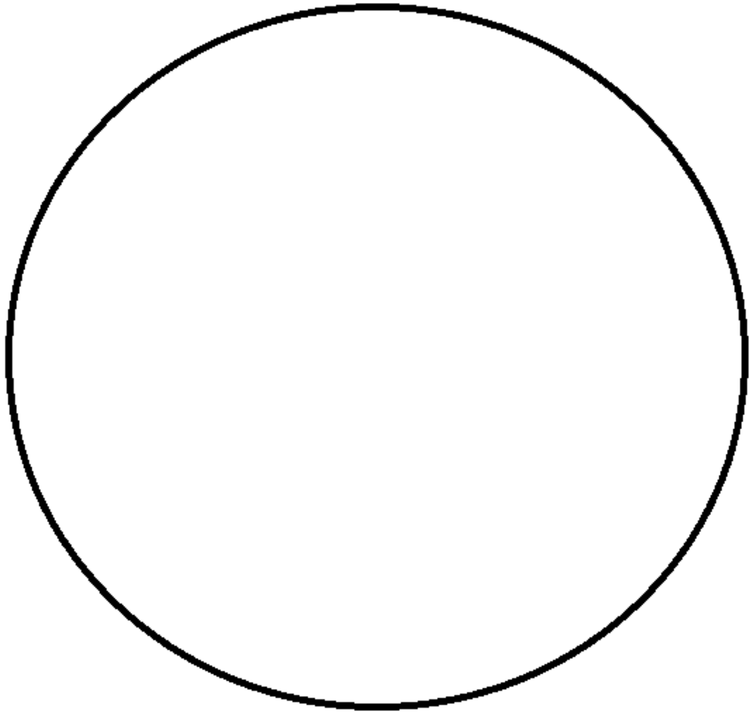
We need $(mx - 1)^2 = (mx - 1)(mx - 1) = mx\, mx - mx - mx + 1 = m^2 x^2 - 2m x + 1$.

Finding Pythagorean Triples: $a^2 + b^2 = c^2$.

A circle of radius r and centered at the origin (0,0) is all points that are r units from (0,0); thus it is $x^2 + y^2 = r^2$.

Without loss of generality, we can divide both sides by c^2 and consider $(a/c)^2 + (b/c)^2 = 1$. Thus we are looking for pairs of rational numbers on the circle of radius 1 (rationals are ratios of two integers, like 1, 11/17 and 3/2). We let $x = a/c$ and $y = b/c$. Can you find a point on the circle with radius 1 centered at the origin? Thus can you find a point (x,y) such that $x^2 + y^2 = 1$?



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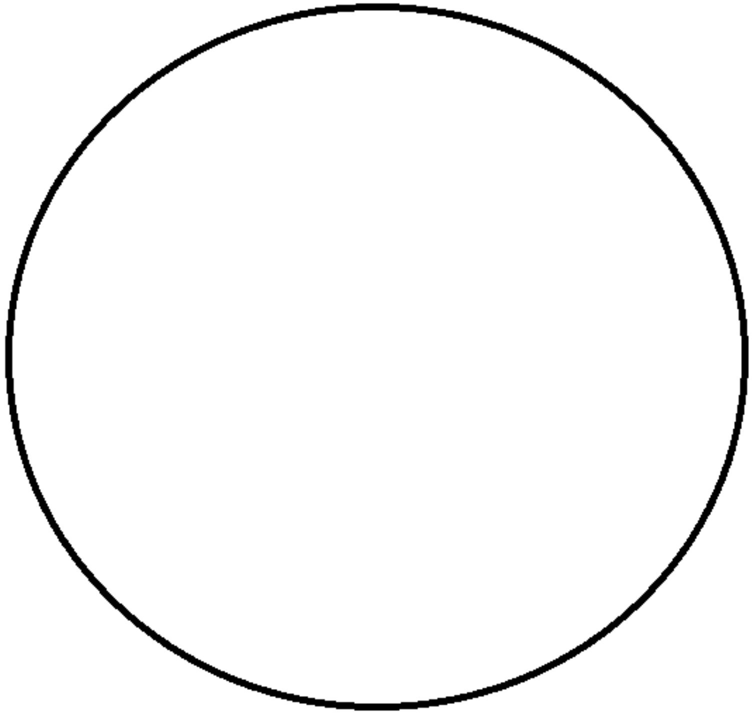
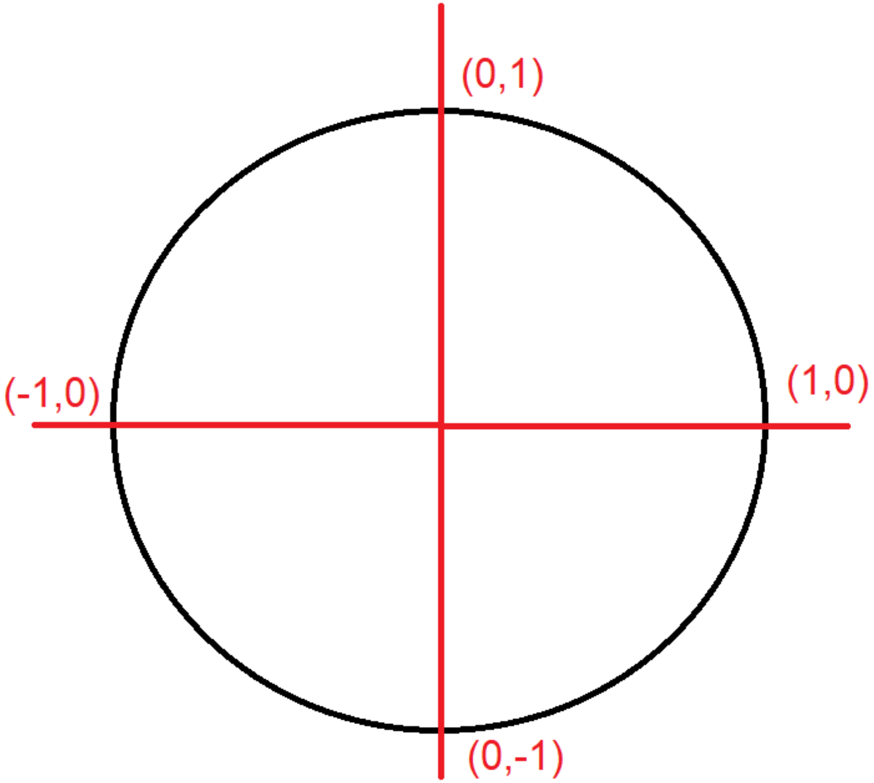
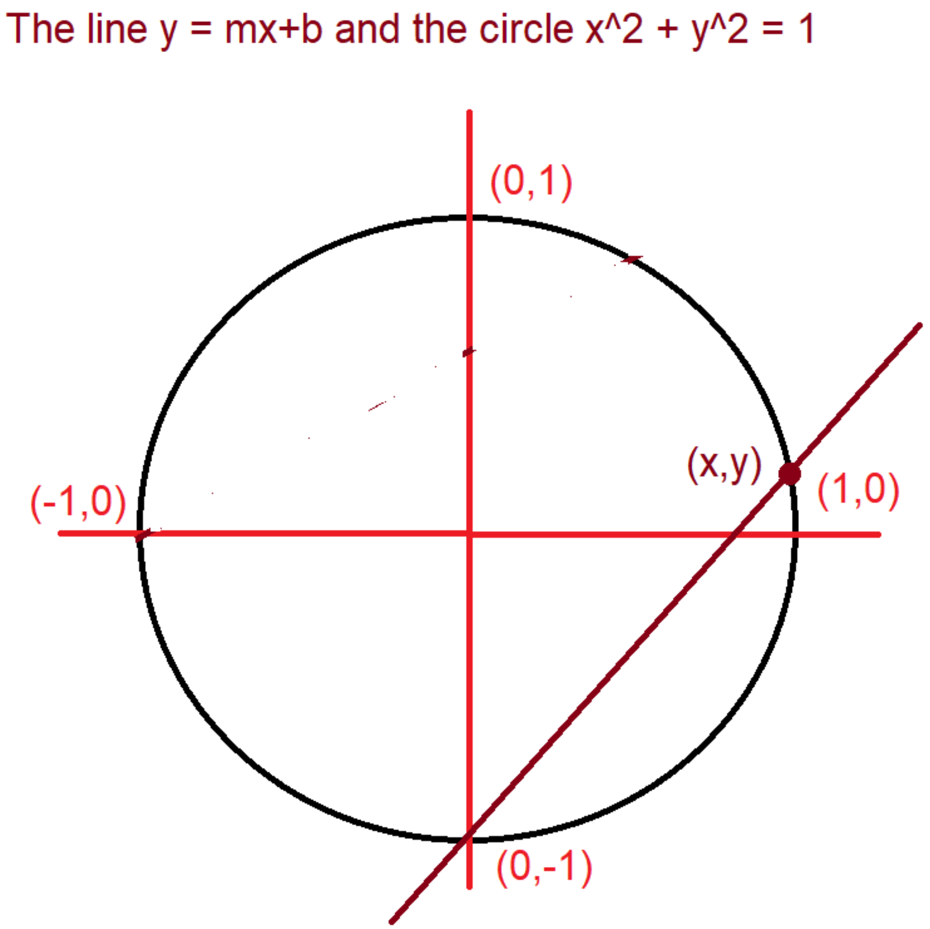




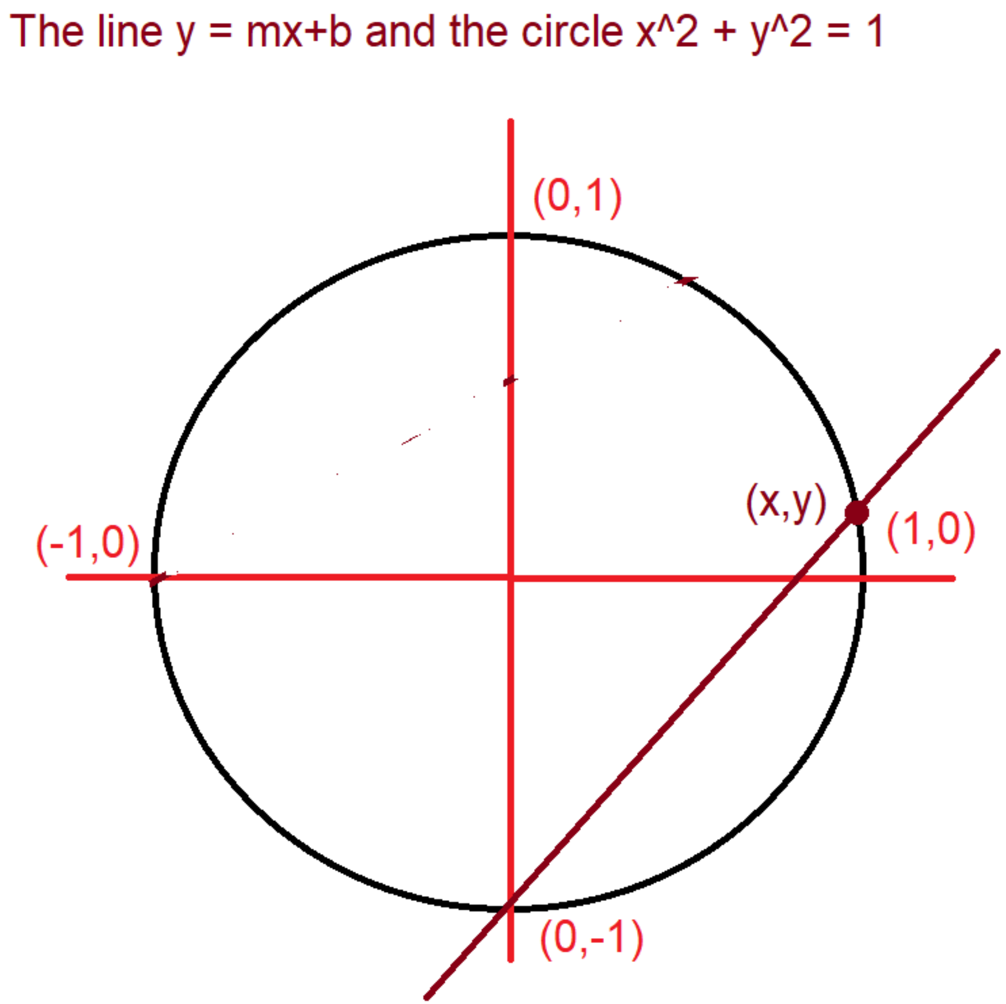
Finding Pythagorean Triples: $a^2 + b^2 = c^2$.

Let's find all points (x,y) such that $x^2 + y^2 = 1$. Consider a line through the point (0,-1) with rational slope. So there are rational numbers m and b such that y = mx + b. As y = 0 when x = -1, b must equal



Finding Pythagorean Triples: $a^2 + b^2 = c^2$.

Let's find all points (x,y) such that $x^2 + y^2 = 1$. Consider a line through the point (0,-1) with rational slope. So there are rational numbers m and b such that y = mx + b. As y = -1 when x = 0, b must equal -1, so equation is y = mx – 1.



We must simultaneously solve:
$x^2 + y^2 = 1$ and $y = mx – 1$.

We can replace y with mx - 1 and we now have
$x^2 + (mx – 1)^2 = 1$.

Expanding gives $x^2 + m^2 x^2 – 2 m x + 1 = 1$.

Thus $(1 + m^2) x^2 = 2 m x$, so either x = 0 or
$(1 + m^2) x = 2m$, thus $x = 2m / (1 + m^2)$.

Substituting for y gives $\frac{2m^2}{1+m^2} - 1 = \frac{m^2-1}{1+m^2}$.

So point is $\left(\frac{2m}{1+m^2}, \frac{m^2-1}{1+m^2}\right)$, note clear denominators
have $(2m)^2 + (m^2 – 1)^2 = (1 + m^2)^2$, which works!
Write m = p/q to get $(2pq)^2 + (p^2 – q^2)^2 = (p^2+ q^2)^2$.

Finding Pythagorean Triples: $a^2 + b^2 = c^2$.

So we have found that $(2pq)^2 + (p^2 - q^2)^2 = (p^2 + q^2)^2$ are Pythagorean triples, with a bit more work we see these are the only ones. Thus if we take different values of p and q we can get some interesting triples! What should we try?

p	q	2 p q	p^2 - q^2	p^2 + q^2

Finding Pythagorean Triples: $a^2 + b^2 = c^2$.

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p	q	2 p q	p^2 - q^2	p^2 + q^2
2	1	4	3	5
3	1	6	8	10
4	1	8	15	17
5	1	10	24	26
6	1	12	35	37
3	2	12	5	13
4	2	16	12	20
5	2	20	21	29
6	2	24	32	40
4	3	24	7	25
5	3	30	16	34
6	3	36	27	45
5	4	40	9	41
6	5	60	11	61

Trigonometry Identities:

$1 = \cos(a)^2 + \sin(a)^2$

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\sin(a+b) = \sin(a)\cos(b) + \cos(b)\sin(a)$.

Have we seen the first identity before?

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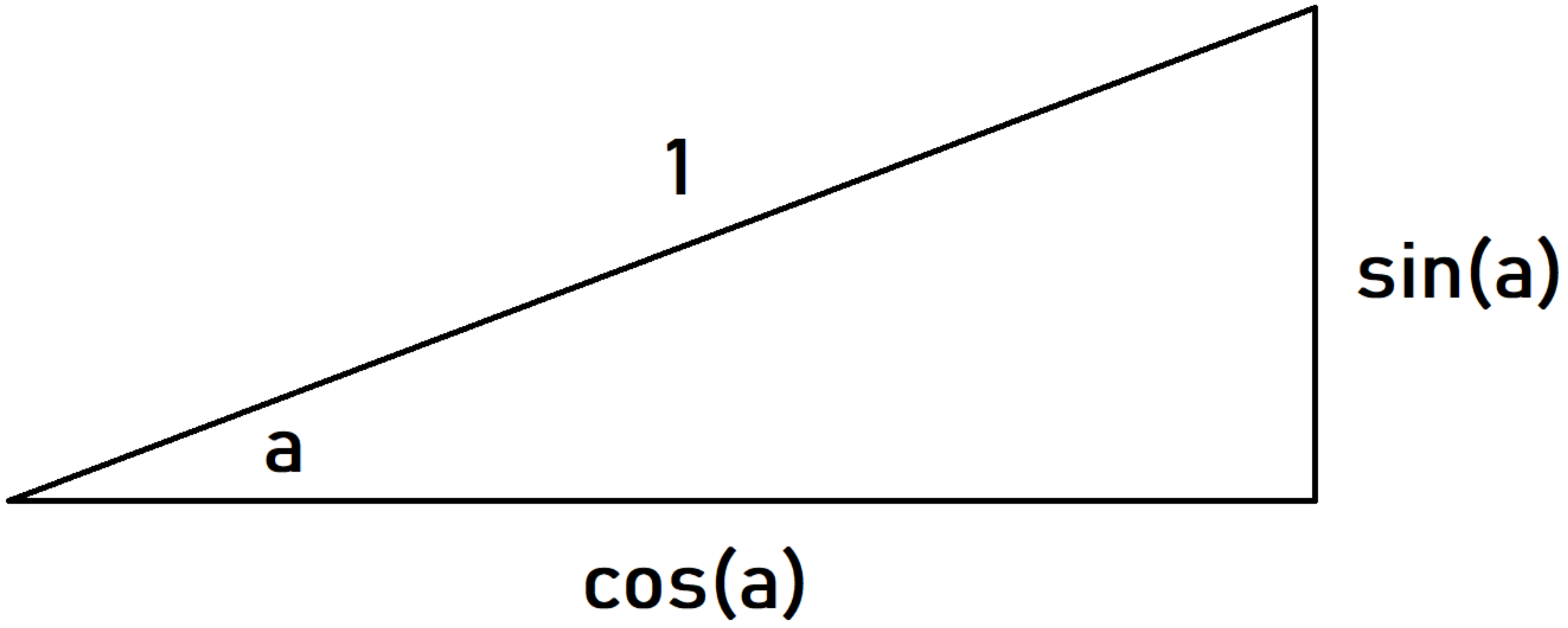
The first is just the Pythagorean Theorem when the hypotenuse is 1 and the angle is a.



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$\cos(x) + 1 = 2\cos(x/2)^2$. Now what? Divide by 2: $\cos(x/2)^2 = (\cos(x) + 1) / 2$. Now what?

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We can replace $\sin(x/2)^2$ with $1 - \cos(x/2)^2$, and thus the first identity becomes

$\cos(x/2)^2 = (\cos(x) + 1) / 2$ so $\cos(x/2) = \sqrt{\frac{\cos(x)+1}{2}}$.

The second is $\sin(x/2) = \sin(x) / (2\cos(x/2))$ so $\sin(x/2) = \sin(x) / \sqrt{2\,(\cos(x) + 1)}$.

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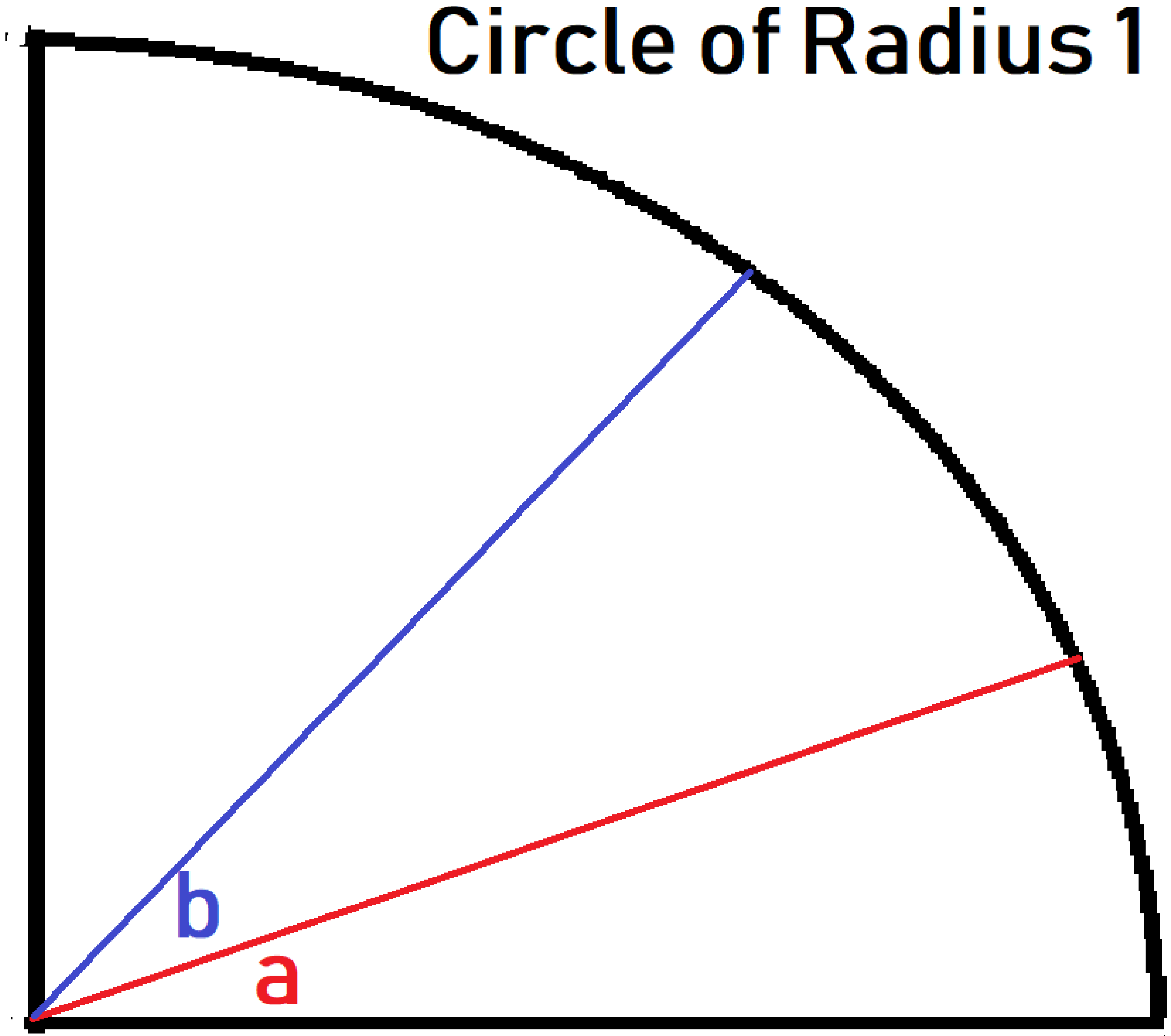

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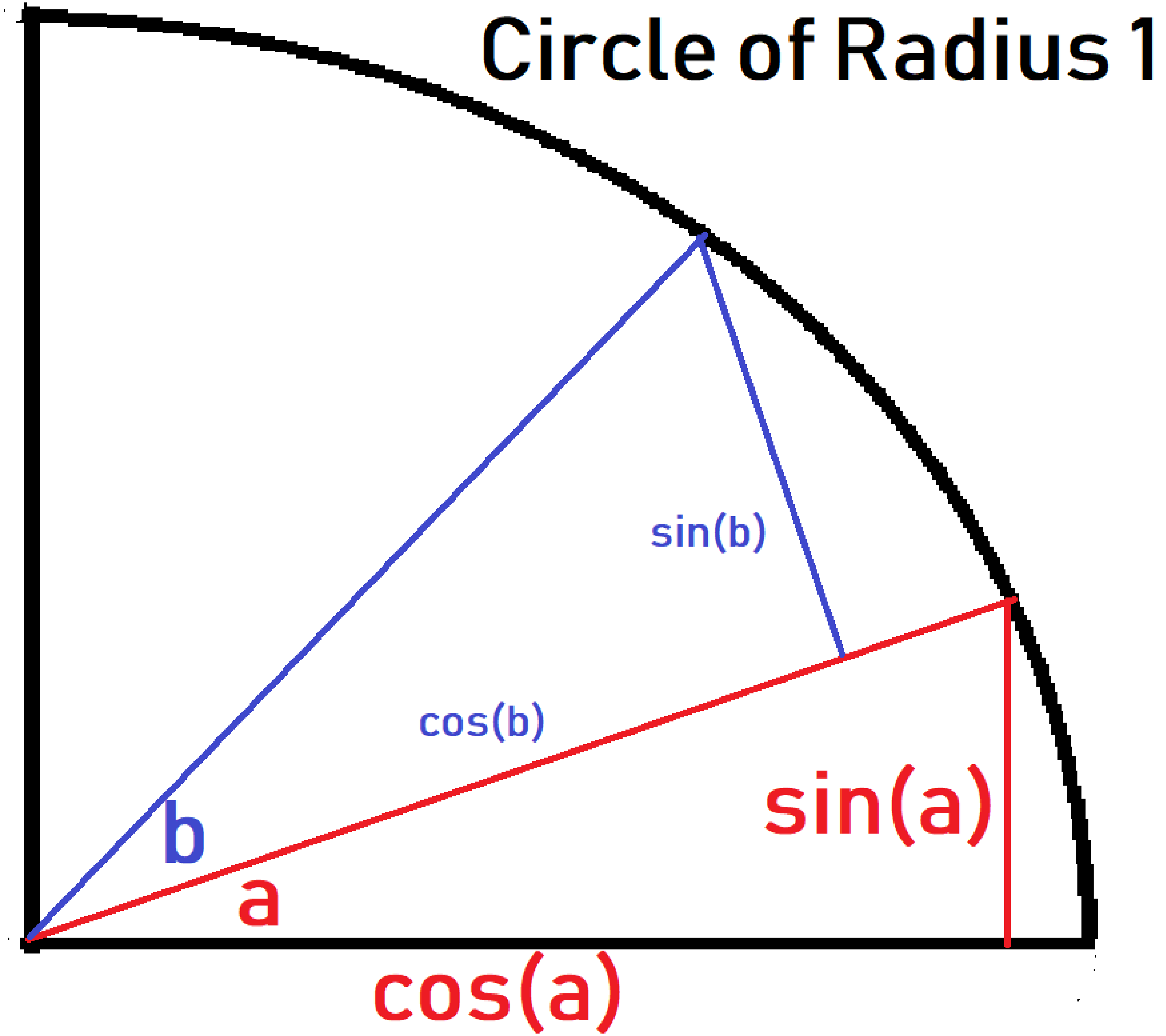

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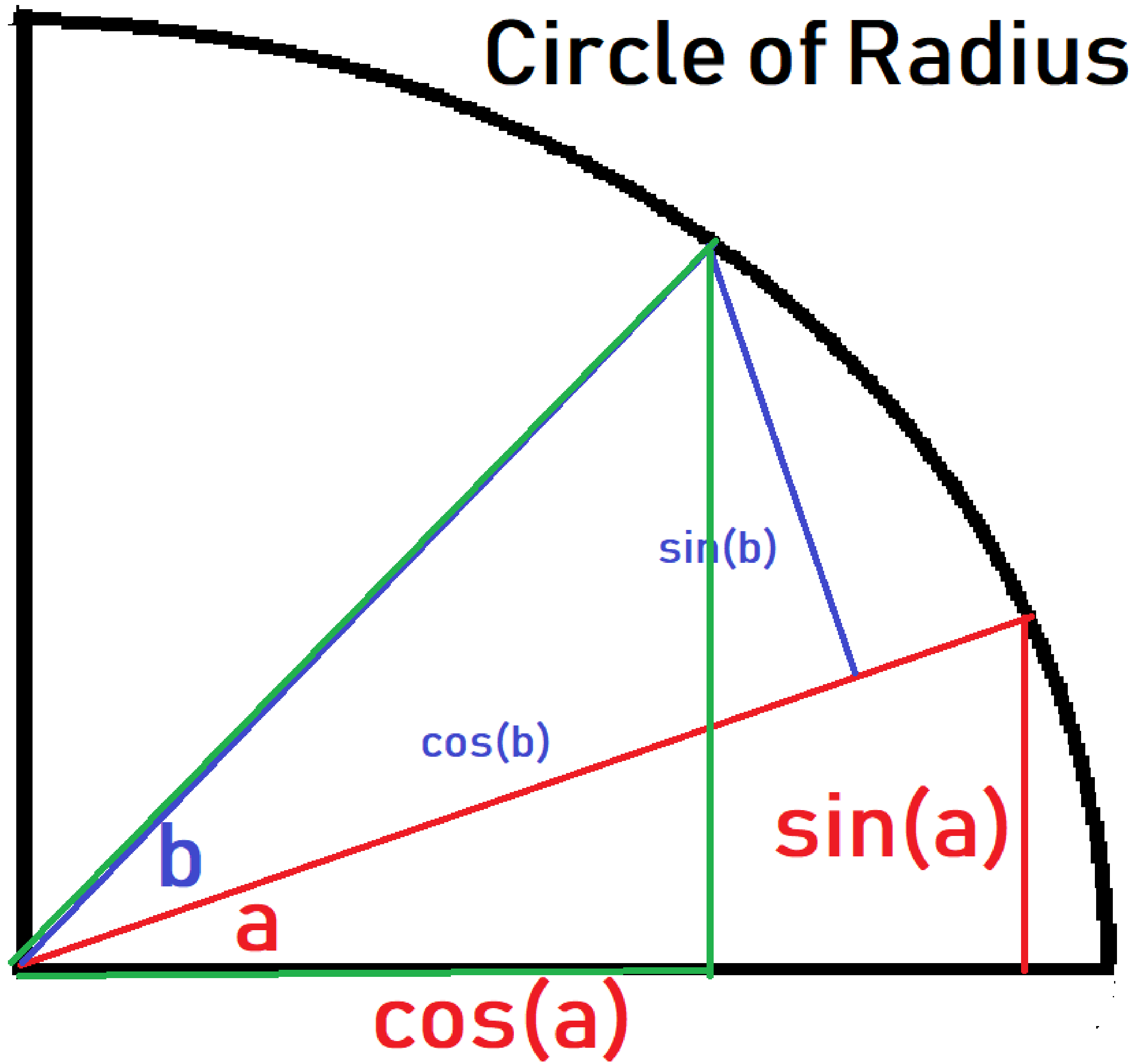

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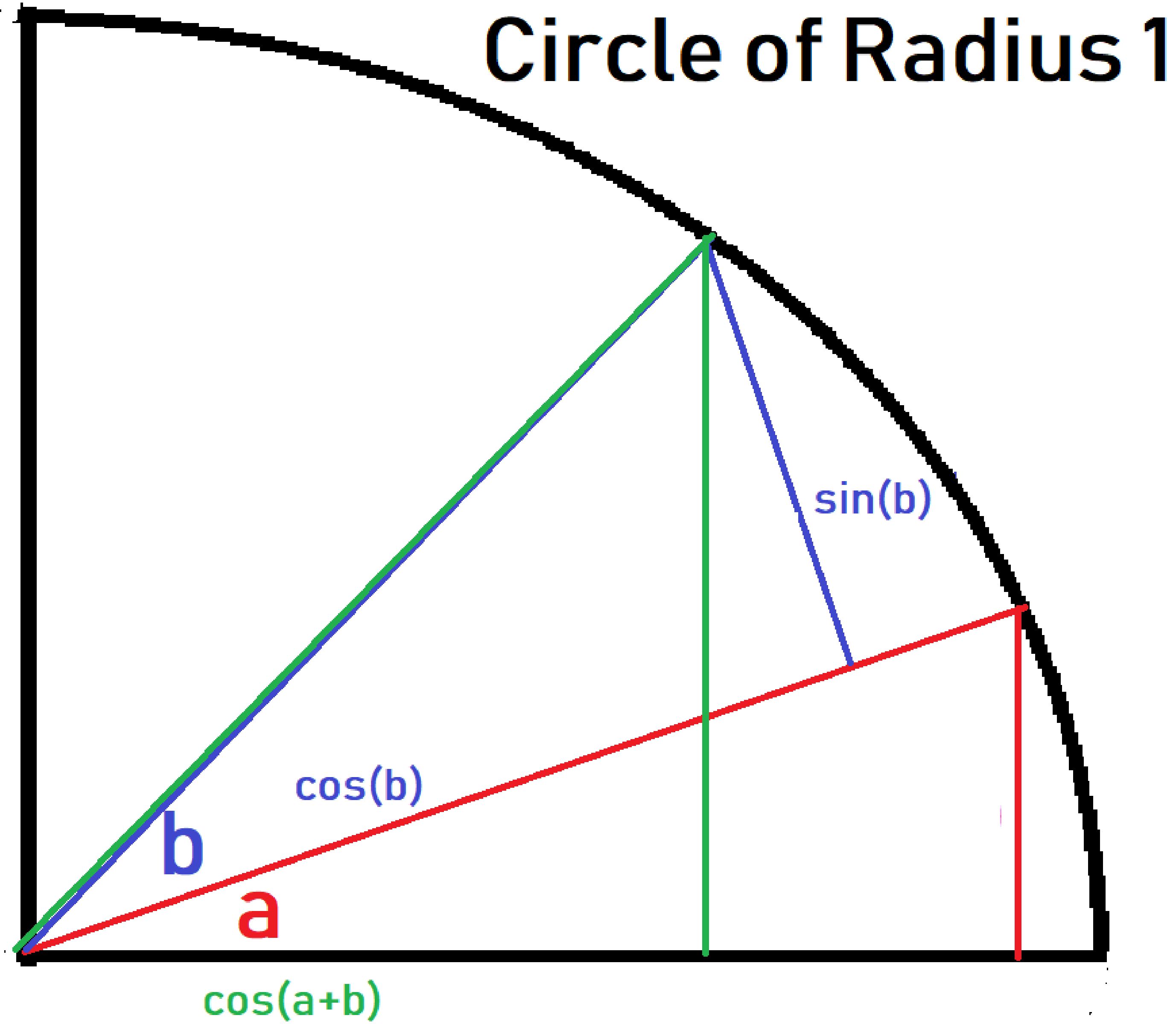
NEED MORE RIGHT TRIANGLES!



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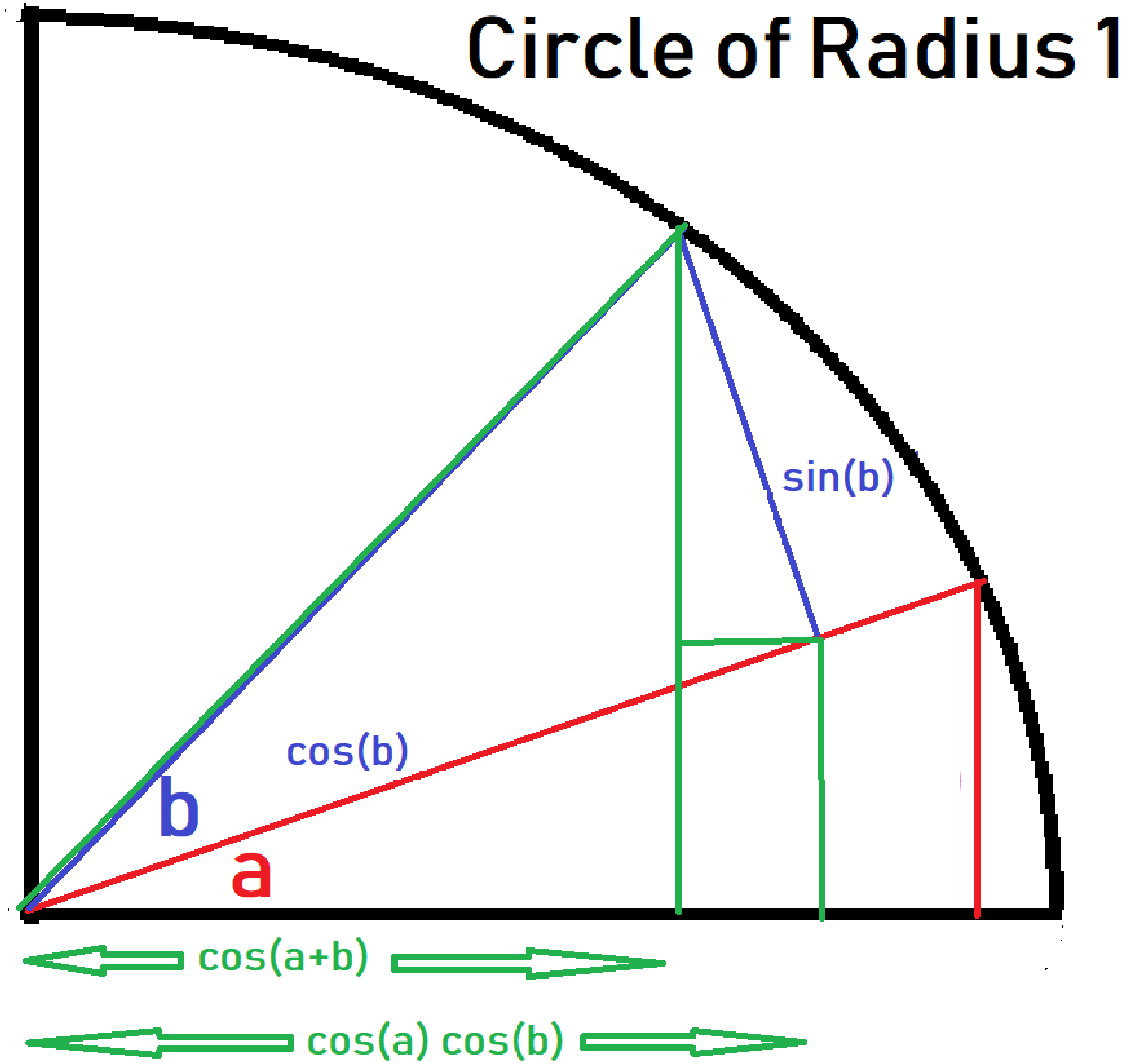
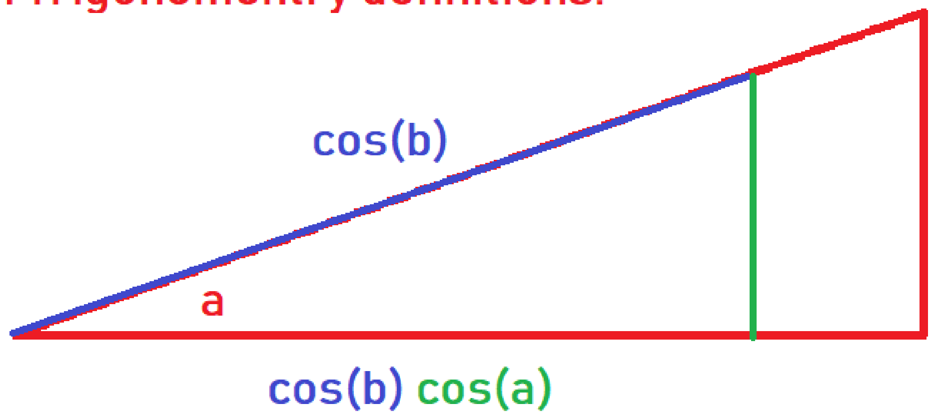


Note this is a right triangle with a smaller right triangle with hypotenuse cos(b) and angle a. Thus the adjacent side is cos(b) cos(a) from Trigonomentry definitions.



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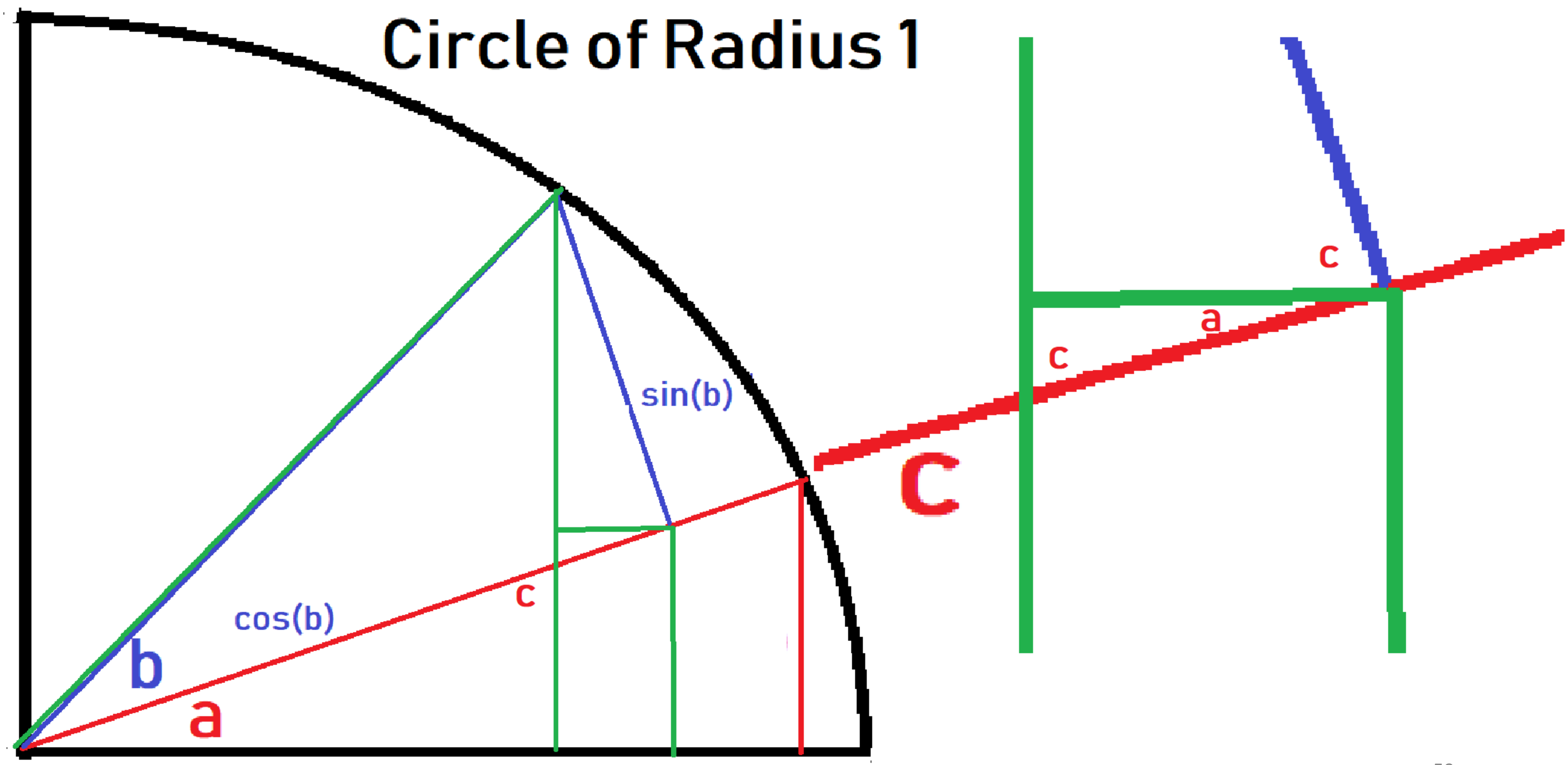

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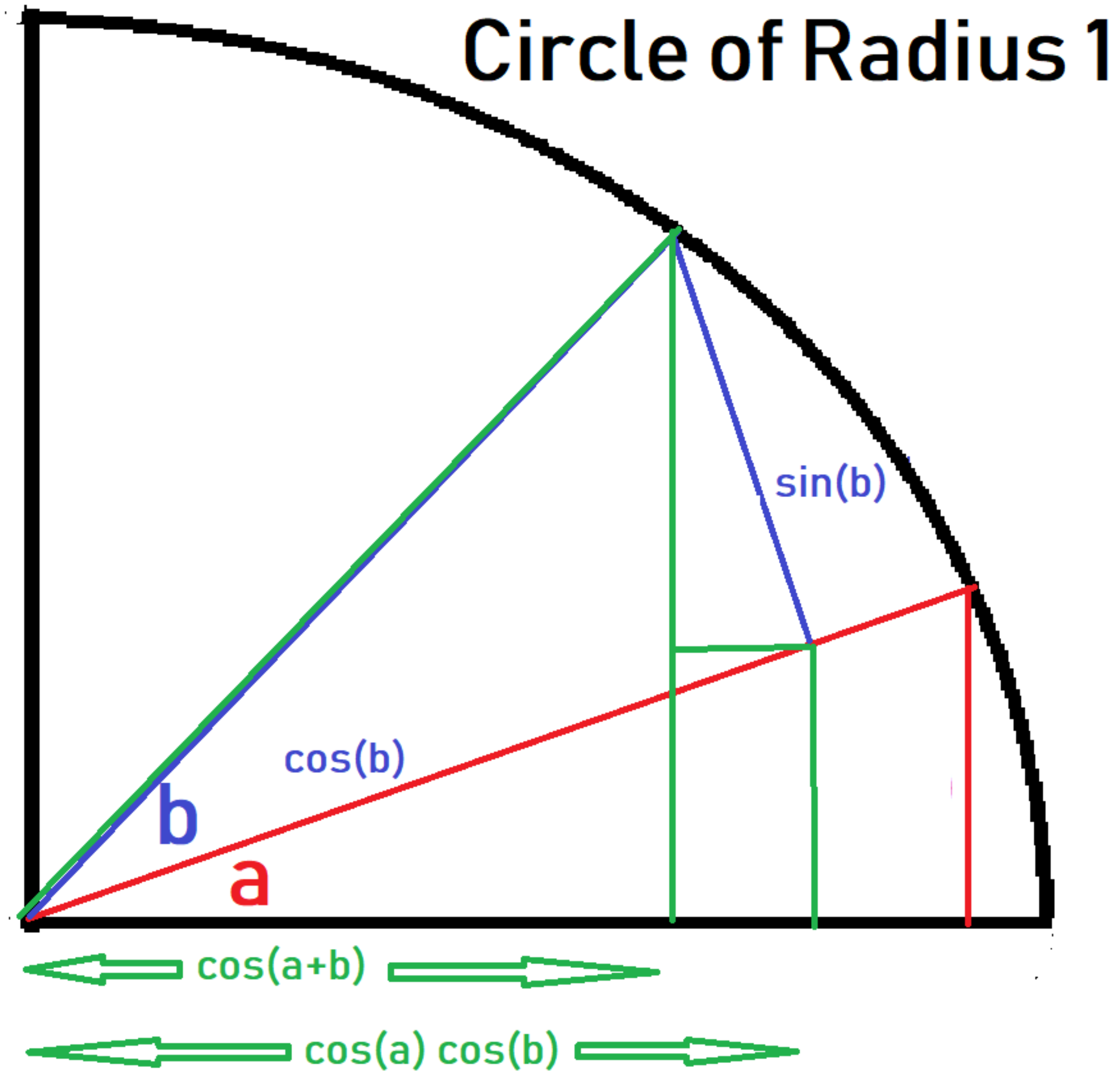
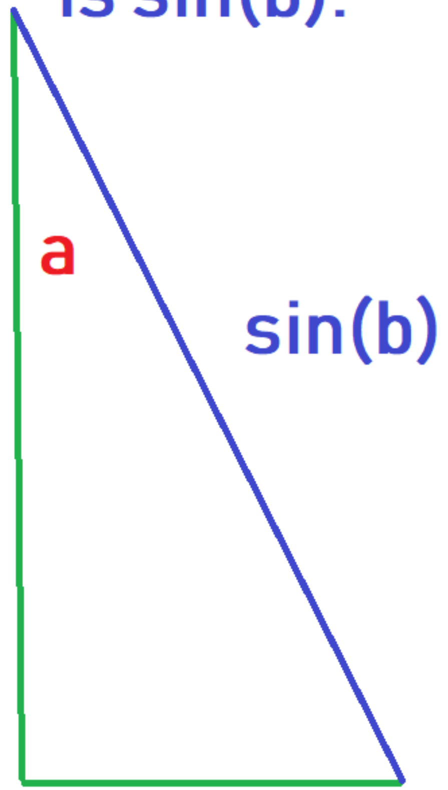


Right Triangle, Hypontenuse is sin(b).



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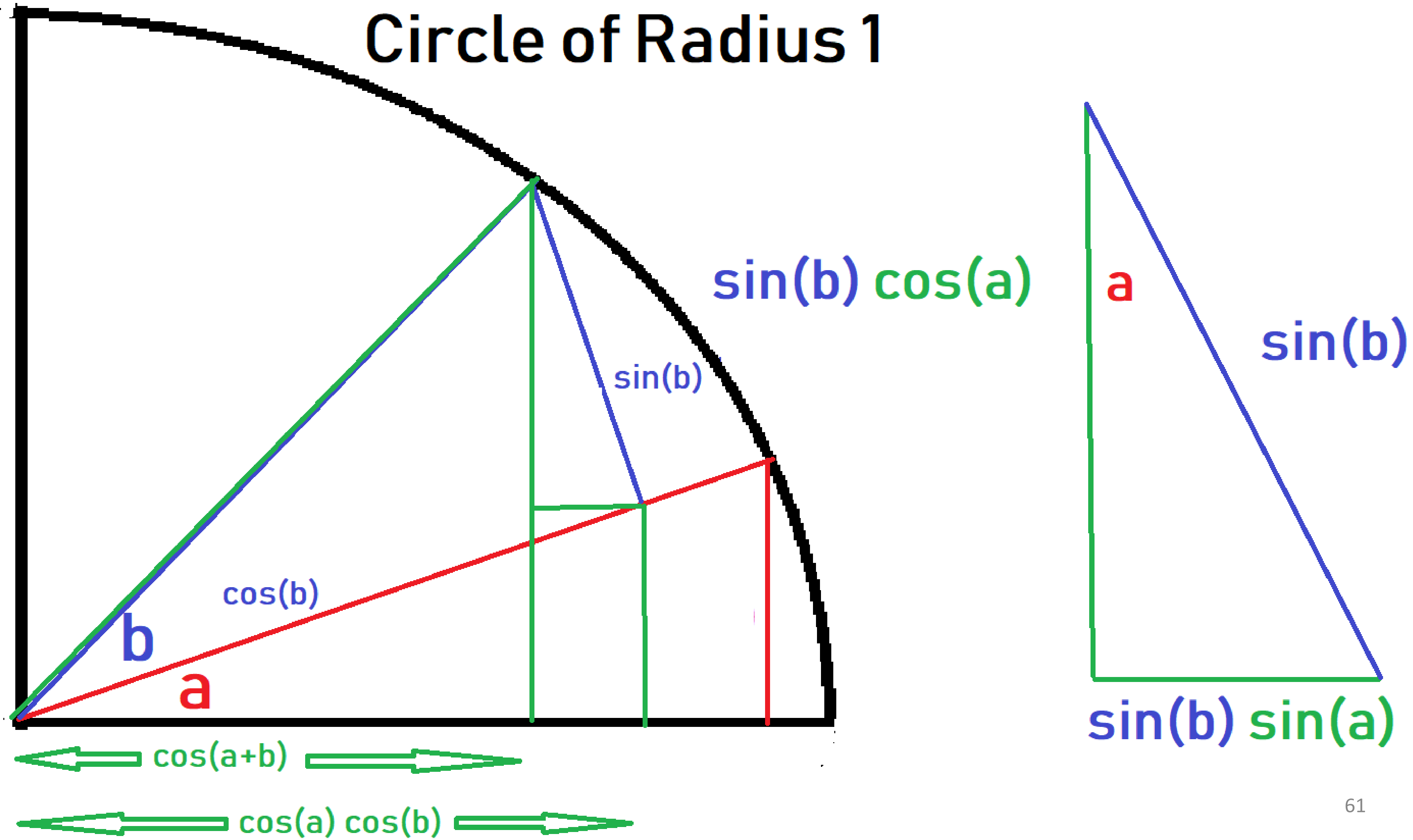
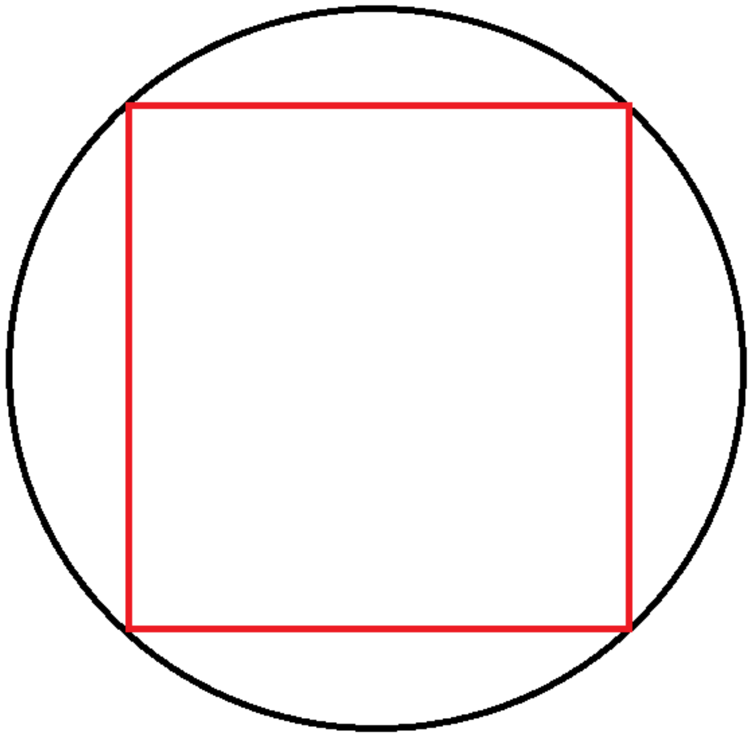
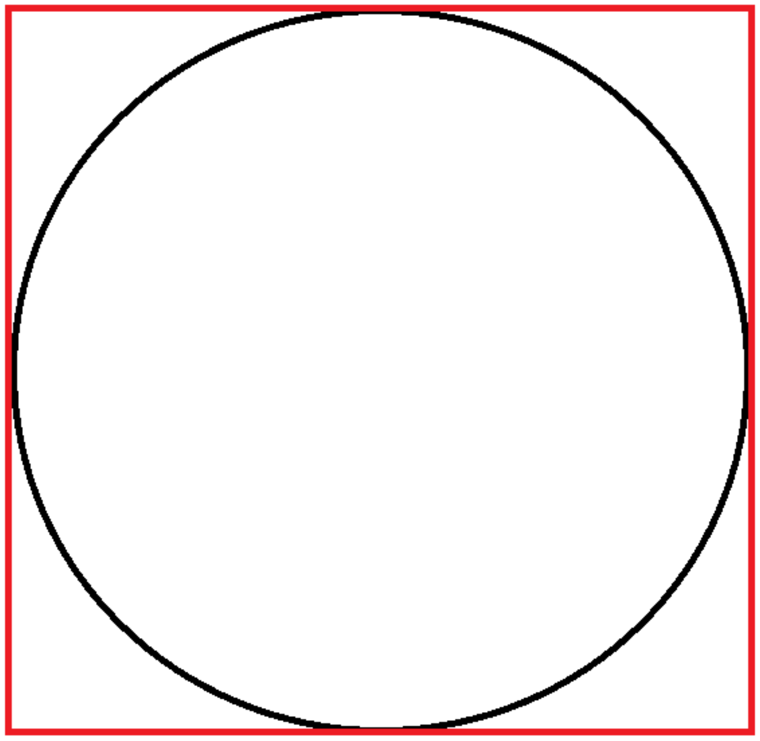


How can we use these trig identities to get formulas for π?

Look at inscribed and circumscribed circles.

Easiest to use a square and then keep bisecting sides.





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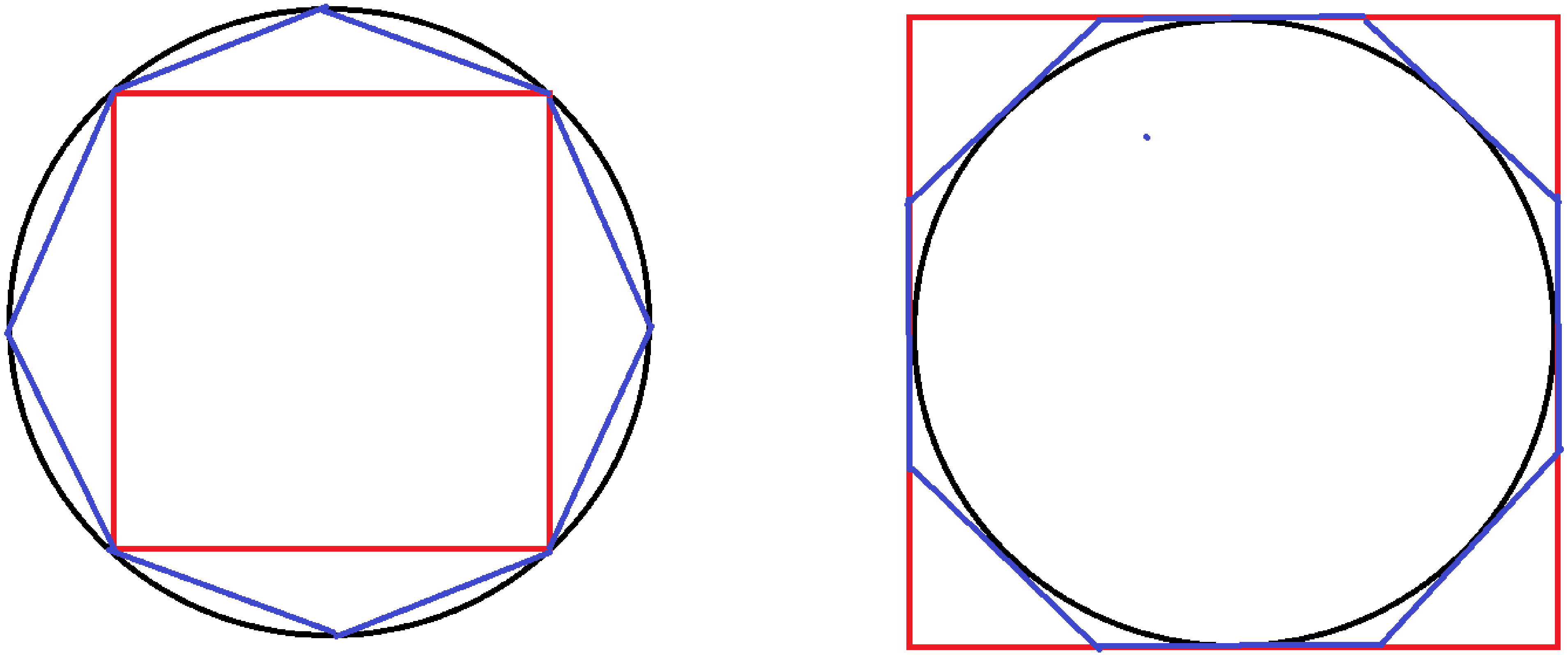
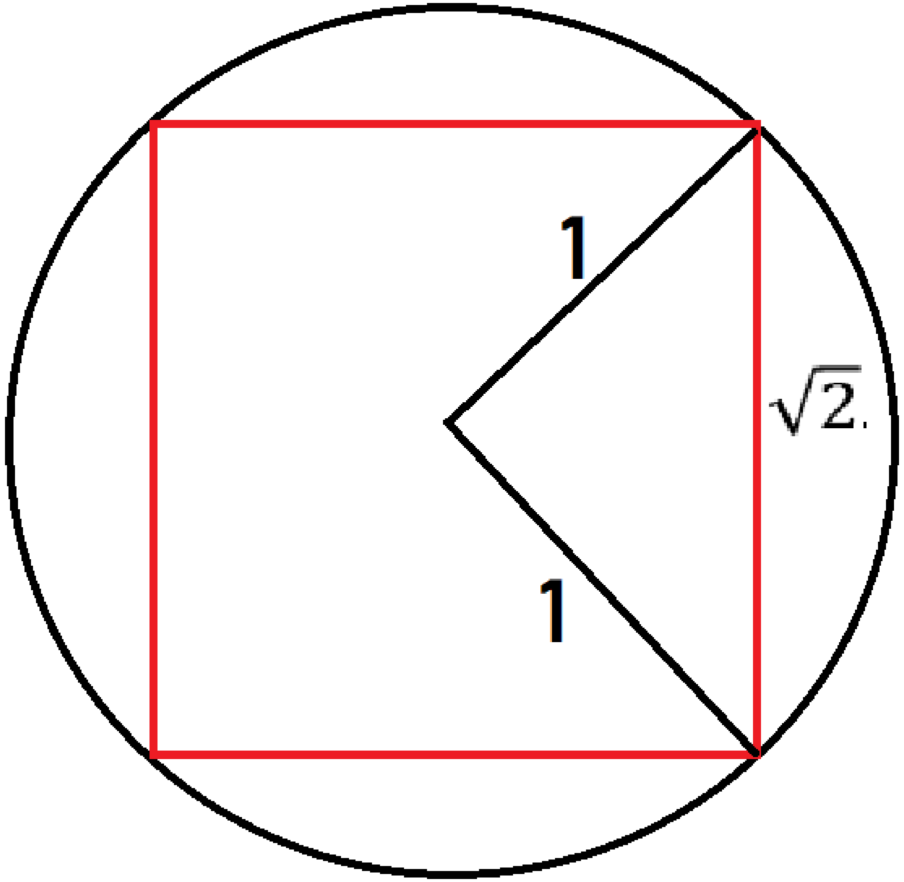


Approximation from a Square:

Have four sides, by the Pythagorean Theorem each side is of length $\sqrt{2}$.



Sqrt(2) is about 1.414.

So 4 Sqrt(2) is about 5.656,
while 2 pi is about 6.283.

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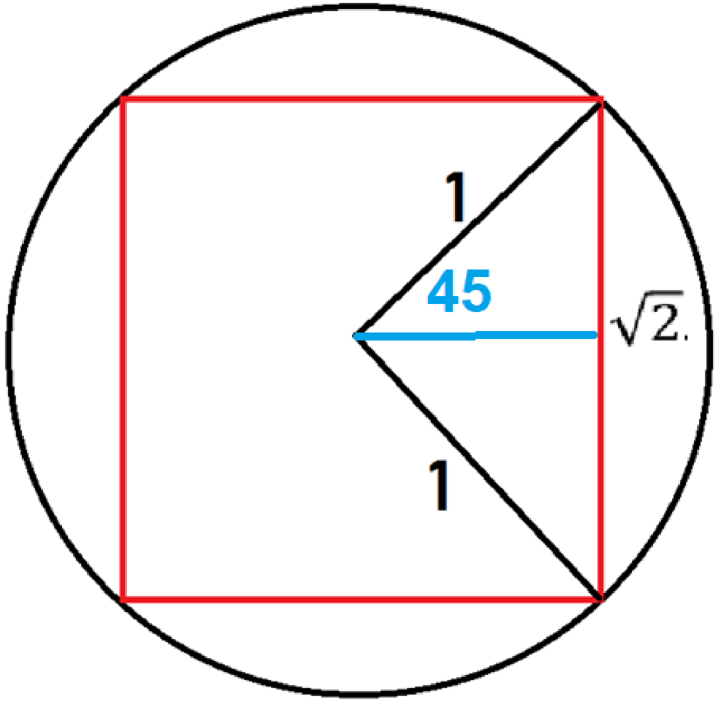

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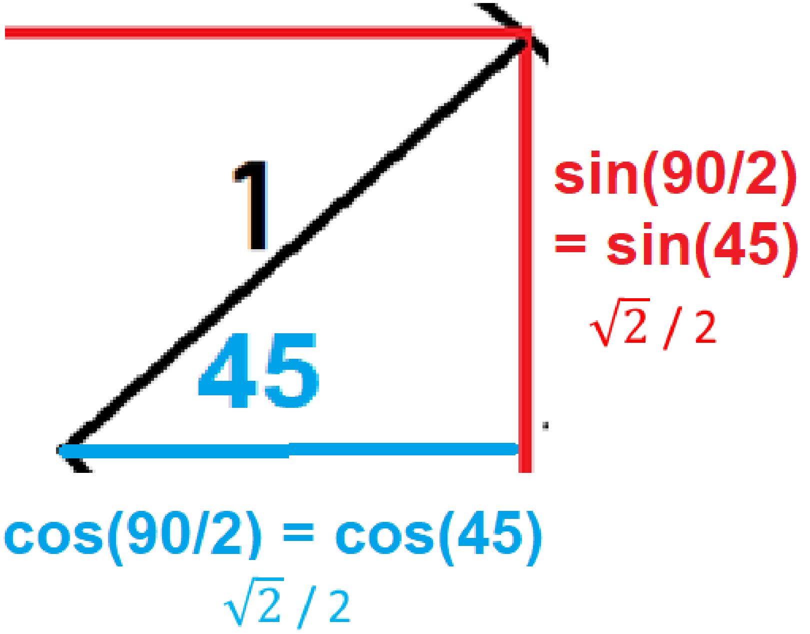
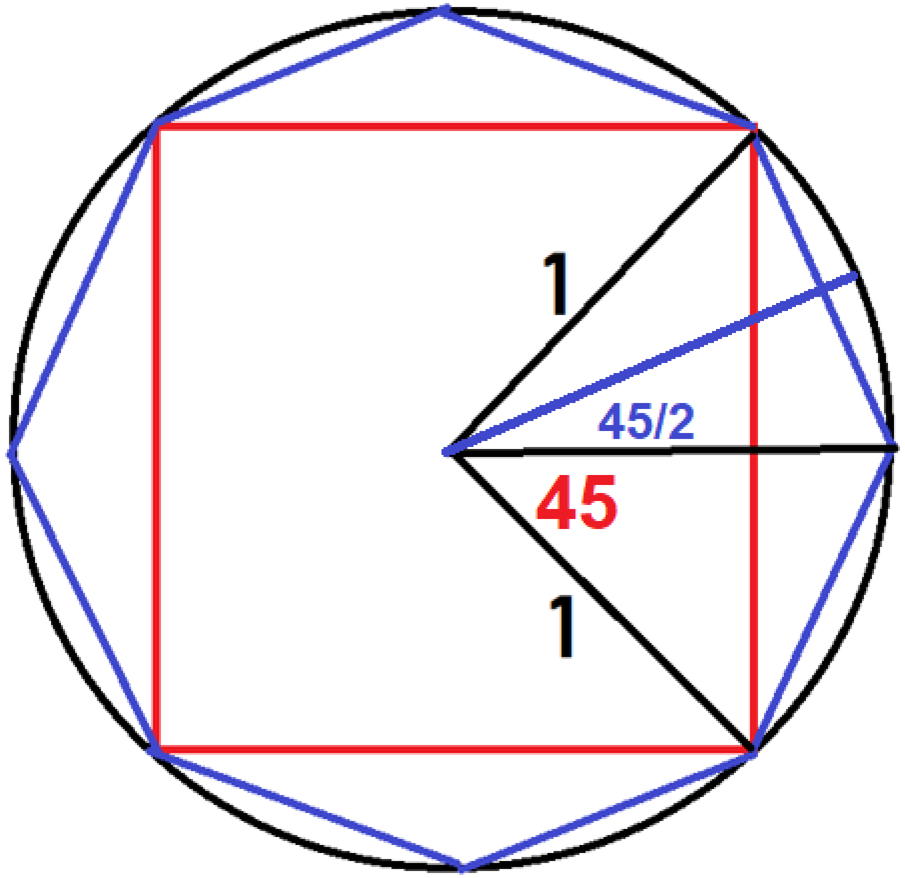


Approximation from an Octogon:

Have eight sides, so the perimeter should be 8 times the length of one side.



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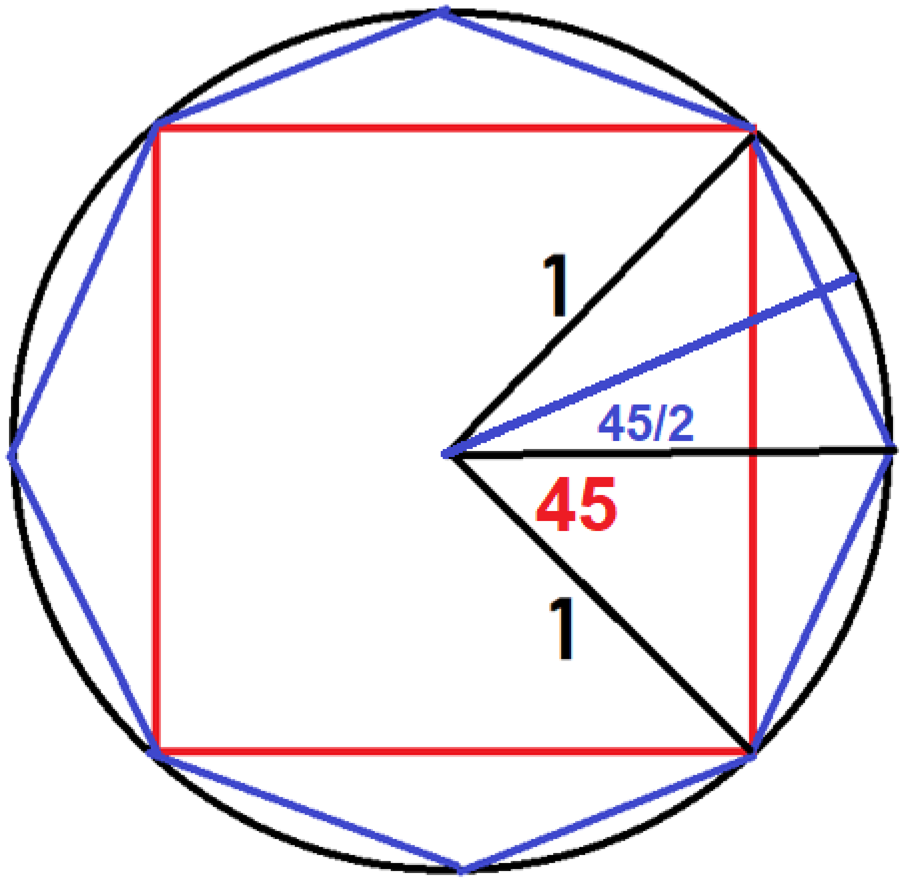

Notice the octogon has 8 sides, and these come from halving the angles of the square. For the square, half a side was sin(45). Now for the octogon half the side will be sin(45/2). Fortunately we have our angle formulas from before.

$$\cos(x/2) = \sqrt{\frac{\cos(x)+1}{2}} \text{ and } \sin(x/2) = \sin(x) / \sqrt{2(\cos(x)+1)}.$$

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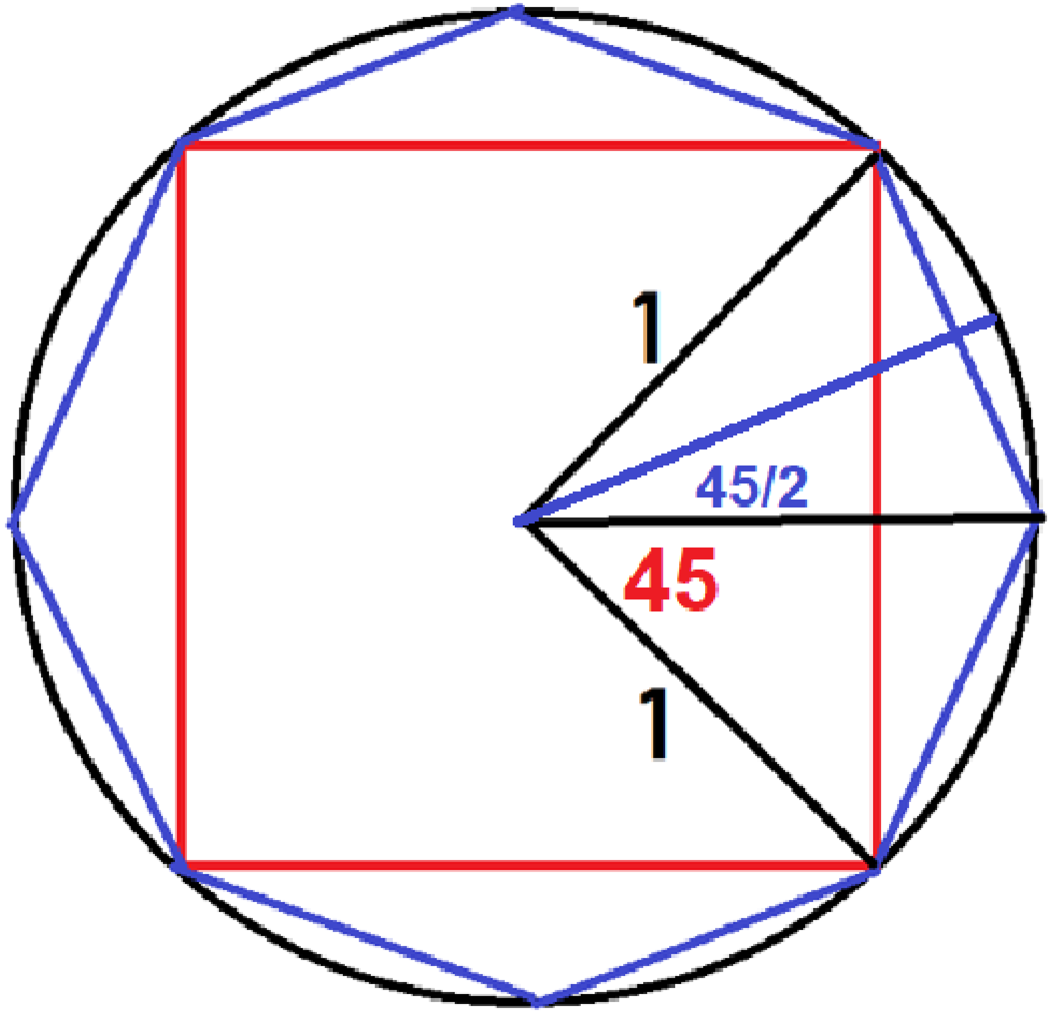

$\cos(x/2) = \sqrt{\frac{\cos(x)+1}{2}}$ and $\sin(x/2) = \sin(x) / \sqrt{2\,(\cos(x)+1)}$.

$\cos(45) = \sin(45) = \sqrt{2} / 2$

$\cos(45/2) = \sqrt{\frac{1}{2}\left(1+\frac{1}{\sqrt{2}}\right)}$

$\sin(45/2) = \frac{1}{2\sqrt{1+\frac{1}{\sqrt{2}}}}$

So $8 \sin(45/2) = \frac{4}{\sqrt{1+\frac{1}{\sqrt{2}}}}$ which is about 3.06147

$$\text{NumSides} = 8,\ \text{SemiPerim} = \frac{4}{\sqrt{1+\frac{1}{\sqrt{2}}}}, \text{ or about } 3.06147$$

$$\text{NumSides} = 16,\ \text{SemiPerim} = 8\sqrt{\frac{2}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)}}, \text{ or about } 3.12145$$

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$$\text{NumSides} = 64,\ \text{SemiPerim} = 32\sqrt{\frac{2}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)}}, \text{ or about } 3.14033$$

It is worth noting that different expressions can be the same. The example here is from an earlier version of this talk where I had a mistake; I urge you to do a similar analysis for the n=8 case.

We start with 4 – 2 $\sqrt{2}$. What can we multiply any number by without changing it?

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The answer is 1; it is not always easy to figure out how we should write 1!

From the later terms in the pattern, we think the first answer should be $\frac{4}{2+\sqrt{2}}$.

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$$\frac{4-2\sqrt{2}}{1} * \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{\left(4-2\sqrt{2}\right)(2+\sqrt{2})}{2+\sqrt{2}} =$$

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$$\frac{4 - 2\sqrt{2}}{1} * \frac{2+\sqrt{2}}{2+\sqrt{2}} = \frac{\left(4 - 2\sqrt{2}\right)(2+\sqrt{2})}{2+\sqrt{2}} = \frac{4 * 2 - 2\sqrt{2} * \sqrt{2} \; - 2\sqrt{2} * 2 + 4 * \sqrt{2}}{2+\sqrt{2}}$$

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NumSides = 8, SemiPerim = $\dfrac{4}{\sqrt{1+\frac{1}{\sqrt{2}}}}$, or about 3.06147

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NumSides = 128, SemiPerim =

$64\sqrt{\dfrac{2}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right)}}$, or about 3.14128

NumSides = 8, SemiPerim = $\frac{4}{\sqrt{1+\frac{1}{\sqrt{2}}}}$, or about 3.06147

NumSides = 16, SemiPerim = $8\sqrt{\frac{2}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)}}$, or about 3.12145

NumSides = 32, SemiPerim = $16\sqrt{\frac{2}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)}}$, or about 3.13655

NumSides = 64, SemiPerim = $32\sqrt{\frac{2}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)}}$, or about 3.14033

NumSides = 128, SemiPerim =

$$64\sqrt{\frac{2}{\left(2+\sqrt{2}\right)\left(2+\sqrt{2+\sqrt{2}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}\right)\left(2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}\right)}}$$, or about 3.14128

Note that outside if we have 2^n sides we have a factor of 2^{n-1} and inside the square-root the denominator has n-1 factors. We can thus bring the 2^{n-1} inside the square-root as a 4^{n-1}, and note that we can write the above as a product of 4's divided by nested square-roots. If c(n) is the largest factor from the n^{th} level, in the $(n+1)^{st}$ level the new term in the denominator is c(n+1) = 2 + sqrt(c(n)), and with some work we can see that the sequence c(n) converges to 4 from below. Thus as we always add inside the square-root the factor 4/c(n) to what we had before, we increase slowly (and we do see each approximation is larger than the previous).