From Fibonacci Quilts to Benford’s Law through Zeckendorf Decompositions

Steven J. Miller (sjm1@williams.edu)
http://www.williams.edu/Mathematics/sjmiller/public_html

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Introduction
Goals of the Talk

- Generalize Zeckendorf decompositions
- Analyze gaps (in the bulk and longest)
- Patterns and new recurrences
- Some open problems (if time permits)
Collaborators and Thanks

Collaborators:

Kentucky Sequence, Fibonacci Quilt: Joint with Minerva Catral, Pari Ford, Pamela Harris & Dawn Nelson.
Benfordness: Andrew Best, Patrick Dynes, Xixi Edelsbunner, Brian McDonald, Kimsy Tor, Caroline Turnage-Butterbaugh & Madeleine Weinstein.

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Previous Results

Fibonacci Numbers: \( F_{n+1} = F_n + F_{n-1}; \)
\( F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots \)

Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example:
2014 = 1597 + 377 + 34 + 5 + 1 = \( F_{16} + F_{13} + F_8 + F_4 + F_1 \).

Lekkerkerkerker’s Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in \([F_n, F_{n+1})\) tends to \( \frac{n}{\varphi^2 + 1} \approx 0.276n \),
where \( \varphi = \frac{1 + \sqrt{5}}{2} \) is the golden mean.
Old Results

**Central Limit Type Theorem**

As \( n \to \infty \), the distribution of number of summands in Zeckendorf decomposition for \( m \in [F_n, F_{n+1}) \) is Gaussian.

**Figure:** Number of summands in \([F_{2010}, F_{2011}); F_{2010} \approx 10^{420}\).
Benford’s law

Definition of Benford’s Law

A dataset is said to follow Benford’s Law (base $B$) if the probability of observing a first digit of $d$ is

$$\log_B \left( 1 + \frac{1}{d} \right).$$

- More generally probability a significant at most $s$ is $\log_B(s)$, where $x = S_B(x)10^k$ with $S_B(x) \in [1, B)$ and $k \in \mathbb{Z}$.

- Find base 10 about 30.1% of the time start with a 1, only 4.5% start with a 9.
Gaps
Joint with Olivia Beckwith, Amanda Bower, Louis Gaudet, Rachel Insoft, Shiyu Li, Philip Tosteson
Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.
Distribution of Gaps

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Let $P_n(g)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length $g$. 
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Bulk: What is $P(g) = \lim_{n \to \infty} P_n(g)$?
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Individual: Similar questions about gaps for a fixed $m \in [F_n, F_{n+1})$: distribution of gaps, longest gap.
New Results: Bulk Gaps: \( m \in [F_n, F_{n+1}) \) and \( \phi = \frac{1 + \sqrt{5}}{2} \)

\[
m = \sum_{j=1}^{k(m)=n} F_i, \quad \nu_{m;n}(x) = \frac{1}{k(m) - 1} \sum_{j=2}^{k(m)} \delta(x - (i_j - i_{j-1})).
\]

**Theorem (Zeckendorf Gap Distribution)**

Gap measures \( \nu_{m;n} \) converge to average gap measure where \( P(k) = 1/\phi^k \) for \( k \geq 2 \).

**Figure:** Distribution of gaps in \( [F_{2010}, F_{2011}) \); \( F_{2010} \approx 10^{420} \).
New Results: Longest Gap

Fair coin: largest gap tightly concentrated around $\log n / \log 2$.

**Theorem (Longest Gap)**

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$
\text{Prob} (L_n(m) \leq f(n)) \approx e^{-e^{\log n - f(n) \cdot \log \phi}}
$$

- $\mu_n = \frac{\log \left( \frac{\phi^2}{\phi^2 + 1} n \right)}{\log \phi} + \frac{\gamma}{\log \phi} - \frac{1}{2} + \text{Small Error}$.

- If $f(n)$ grows slower (resp. faster) than $\log n / \log \phi$, then $\text{Prob}(L_n(m) \leq f(n))$ goes to 0 (resp. 1).
Theorem (Distribution of Bulk Gaps (SMALL 2012))

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ be a positive linear recurrence of length $L$ where $c_i \geq 1$ for all $1 \leq i \leq L$. Then

$$P(g) = \begin{cases} 1 - \left( \frac{a_1}{C_{Lek}} \right) (2 \lambda_1^{-1} + a_1^{-1} - 3) & : g = 0 \\ \lambda_1^{-1} \left( \frac{1}{C_{Lek}} \right) (\lambda_1 (1 - 2a_1) + a_1) & : g = 1 \\ (\lambda_1 - 1)^2 \left( \frac{a_1}{C_{Lek}} \right) \lambda_1^{-g} & : g \geq 2. \end{cases}$$
Main Results

**Theorem (Longest Gap (SMALL 2012))**

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\text{Prob} \left( L_n(m) \leq f(n) \right) \approx e^{-e^{\log n - f(n) \cdot \log \phi}}$$
Kentucky Sequence and Quilts

with Minerva Catral, Pari Ford, Pamela Harris & Dawn Nelson
Kentucky Sequence

Rule: \((s, b)\)-Sequence: Bins of length \(b\), and:

- cannot take two elements from the same bin, and
- if have an element from a bin, cannot take anything from the first \(s\) bins to the left or the first \(s\) to the right.
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**Fibonaccis**: These are \((s, b) = (1, 1)\).

**Kentucky**: These are \((s, b) = (1, 2)\).

\([1, 2], [3, 4], [5, 8], [11, 16], [21, 32], [43, 64], [85, 128]\).
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- $a_{2n} = 2^n$ and $a_{2n+1} = \frac{1}{3}(2^{2+n} - (-1)^n)$:
- $a_{n+1} = a_{n-1} + 2a_{n-3}, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4.$
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- \(a_{2n} = 2^n\) and \(a_{2n+1} = \frac{1}{3}(2^{2+n} - (-1)^n)\):
  - \(a_{n+1} = a_{n-1} + 2a_{n-3}\), \(a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4\).
- \(a_{n+1} = a_{n-1} + 2a_{n-3}\): New as leading term 0.
Gaussian Behavior

Figure: Plot of the distribution of the number of summands for 100,000 randomly chosen \( m \in [1, a_{4000}) = [1, 2^{2000}) \) (so \( m \) has on the order of 602 digits).
Figure: Plot of the distribution of gaps for 10,000 randomly chosen 
\( m \in [1, a_{400}) = [1, 2^{200}) \) (so \( m \) has on the order of 60 digits).
Gaps

**Figure:** Plot of the distribution of gaps for 10,000 randomly chosen $m \in [1, a_{400}) = [1, 2^{200})$ (so $m$ has on the order of 60 digits). Left (resp. right): ratio of adjacent even (resp odd) gap probabilities.

Again find geometric decay, but parity issues so break into even and odd gaps.
The Fibonacci (or Log Cabin) Quilt: Work in Progress

\[ a_{n+1} = a_{n-1} + a_{n-2}, \text{ non-uniqueness (average number of decompositions grows exponentially).} \]

In process of investigating Gaussianity, Gaps, \( K_{\text{min}}, K_{\text{ave}}, K_{\text{max}}, K_{\text{greedy}} \).
**Average Number of Representations**

- \(d_n\): the number of FQ-legal decompositions using only elements of \(\{a_1, a_2, \ldots, a_n\}\).
- \(c_n\) requires \(a_n\) to be used, \(b_n\) requires \(a_n\) and \(a_{n-2}\) to be used.

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<th>(c_n)</th>
<th>(b_n)</th>
<th>(a_n)</th>
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**Table:** First few terms. Find \(d_n = d_{n-1} + d_{n-2} - d_{n-3} + d_{n-5} - d_{n-9}\), implying \(d_{\text{FQ;ave}}(n) \approx C \cdot 1.05459^n\).
Greedy Algorithm

$h_n$: number of integers from 1 to $a_{n+1} - 1$ where the greedy algorithm successfully terminates in a legal decomposition.

<table>
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<th>$a_n$</th>
<th>$h_n$</th>
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</tr>
<tr>
<td>17</td>
<td>151</td>
<td>184</td>
<td>92.4623</td>
</tr>
</tbody>
</table>

**Table:** First few terms, yields $h_n = h_{n-1} + h_{n-5} + 1$ and percentage converges to about 0.92627.
Benfordness in Interval
Joint with Andrew Best, Patrick Dynes, Xixi Edelsbunner, Brian McDonald, Kimsy Tor, Caroline Turnage-Butterbaugh and Madeleine Weinstein
Theorem (SMALL 2014): Benfordness in Interval

The distribution of the summands in the Zeckendorf decompositions, averaged over the entire interval \([F_n, F_{n+1})\), follows Benford’s Law.
Theorem (SMALL 2014): Benfordness in Interval

The distribution of the summands in the Zeckendorf decompositions, averaged over the entire interval \([F_n, F_{n+1})\), follows Benford’s Law.

Example

Looking at the interval \([F_5, F_6) = [8, 13)\)

\[
\begin{align*}
8 &= 8 &= F_5 \\
9 &= 8 + 1 &= F_5 + F_1 \\
10 &= 8 + 2 &= F_5 + F_2 \\
11 &= 8 + 3 &= F_5 + F_3 \\
12 &= 8 + 3 + 1 &= F_5 + F_3 + F_1
\end{align*}
\]
Preliminaries for Proof

**Density of $S$**

For a subset $S$ of the Fibonacci numbers, define the density $q(S, n)$ of $S$ over the interval $[1, F_n]$ by

$$q(S, n) = \frac{\#\{F_j \in S \mid 1 \leq j \leq n\}}{n}.$$ 

**Asymptotic Density**

If $\lim_{n \to \infty} q(S, n)$ exists, define the *asymptotic density* $q(S)$ by

$$q(S) = \lim_{n \to \infty} q(S, n).$$
Let $S_d$ be the subset of the Fibonacci numbers which share a fixed digit $d$ where $1 \leq d < B$.

**Theorem: Fibonacci Numbers Are Benford**

$$q(S_d) = \lim_{n \to \infty} q(S_d, n) = \log_B \left( 1 + \frac{1}{d} \right).$$

**Proof:** Binet’s formula, Kronecker’s theorem on equidistribution of $n\alpha \mod 1$ for $\alpha \notin \mathbb{Q}$.
Random Variables

Random Variable from Decompositions

Let $X(I_n)$ be a random variable whose values are the the Fibonacci numbers in $[F_1, F_n)$ and probabilities are how often they occur in decompositions of $m \in I_n$:

$$P\{X(I_n) = F_k\} := \begin{cases} \frac{F_{k-1}F_{n-k-2}}{\mu_n F_{n-1}}, & \text{if } 1 \leq k \leq n - 2 \\ \frac{1}{\mu_n}, & \text{if } k = n \\ 0, & \text{otherwise,} \end{cases}$$

where $\mu_n$ is the average number of summands in Zeckendorf decompositions of integers in the interval $[F_n, F_{n+1})$. 
Approximations

**Estimate for** $P\{X(I_n) = F_k\}$

$$P\{X(I_n) = F_k\} = \frac{1}{\mu_n \phi \sqrt{5}} + O\left(\phi^{-2k} + \phi^{-2n+2k}\right).$$

**Constant Fringes Negligible**

For any $r$ (which may depend on $n$):

$$\sum_{r<k<n-r} P\{X(I_n) = F_k\} = 1 - r \cdot O\left(\frac{1}{n}\right).$$
Estimating $P\{X(I_n) \in S\}$

Set $r := \left\lfloor \frac{\log n}{\log \phi} \right\rfloor$.

Density of $S$ over Zeckendorf Summands

We have

$$P\{X(I_n) \in S\} = \frac{nq(S)}{\mu n \phi \sqrt{5}} + o(1) \rightarrow q(s).$$
Remark

- Stronger result than Benfordness of Zeckendorf summands.

- Global property of the Fibonacci numbers can be carried over locally into the Zeckendorf summands.

- If we have a subset of the Fibonacci numbers $S$ with asymptotic density $q(S)$, then the density of the set $S$ over the Zeckendorf summands will converge to this asymptotic density.
Benfordness of Random and Zeckendorf Decompositions
Joint with Andrew Best, Patrick Dynes, Xixi Edelsbunner, Brian McDonald, Kimsy Tor, Caroline Turnage-Butterbaugh and Madeleine Weinstein
Random Decompositions

**Theorem 2 (SMALL 2014): Random Decomposition**

If we choose each Fibonacci number with probability $q$, disallowing the choice of two consecutive Fibonacci numbers, the resulting sequence follows Benford’s law.

**Example: $n = 10$**

$$
F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9 + F_{10} = 2 + 8 + 21 + 89 = 120
$$
Choosing a Random Decomposition

Select a random subset $A$ of the Fibonaccis as follows:

- Fix $q \in (0, 1)$.
- Let $A_0 := \emptyset$.
- For $n \geq 1$, if $F_{n-1} \in A_{n-1}$, let $A_n := A_{n-1}$, else
  
  $$A_n = \begin{cases} 
  A_{n-1} \cup \{F_n\} & \text{with probability } q \\
  A_{n-1} & \text{with probability } 1 - q.
  \end{cases}$$

- Let $A := \bigcup_n A_n$. 
Main Result

**Theorem**

*With probability 1, A (chosen as before) is Benford.*

**Stronger claim:** For any subset $S$ of the Fibonacci numbers with density $d$ in the Fibonacci numbers, $S \cap A$ has density $d$ in $A$ with probability 1.
Preliminaries

Lemma
The probability that $F_k \in A$ is

$$p_k = \frac{q}{1 + q} + O(q^k).$$

Using elementary techniques, we get

Lemma
Define $X_n := \#A_n$. Then

$$E[X_n] = \frac{nq}{1 + q} + O(1)$$

$$\text{Var}(X_n) = O(n).$$
Expected Value of $Y_n$

Define $Y_{n,S} := \#A_n \cap S$. Using standard techniques, we get

\[
\mathbb{E}[Y_n] = \frac{nqd}{1 + q} + o(n).
\]

\[
\text{Var}(Y_{n,S}) = o(n^2).
\]
Expected Value of $Y_n$

Define $Y_{n,S} := \#A_n \cap S$. Using standard techniques, we get

**Lemma**

\[
\mathbb{E}[Y_n] = \frac{nqd}{1 + q} + o(n).
\]

\[
\text{Var}(Y_{n,S}) = o(n^2).
\]

Immediately implies with probability $1 + o(1)$

\[
Y_{n,S} = \frac{nqd}{1 + q} + o(n), \quad \lim_{n \to \infty} \frac{Y_{n,S}}{X_n} = d.
\]

Hence $A \cap S$ has density $d$ in $A$, completing the proof.
Zeckendorf Decompositions and Benford’s Law

Theorem (SMALL 2014): Benfordness of Decomposition

If we pick a random integer in \([0, F_{n+1})\), then with probability 1 as \(n \to \infty\) its Zeckendorf decomposition converges to Benford’s Law.
Proof of Theorem

1. Choose integers randomly in $[0, F_{n+1})$ by random decomposition model from before.

2. Choose $m = F_{a_1} + F_{a_2} + \cdots + F_{a_{\ell}} \in [0, F_{n+1})$ with probability

$$p_m = \begin{cases} q^{\ell}(1 - q)^{n-2\ell} & \text{if } a_{\ell} \leq n \\ q^{\ell}(1 - q)^{n-2\ell+1} & \text{if } a_{\ell} = n. \end{cases}$$

3. **Key idea:** Choosing $q = 1/\varphi^2$, the previous formula simplifies to

$$p_m = \begin{cases} \varphi^{-n} & \text{if } m \in [0, F_n) \\ \varphi^{-n-1} & \text{if } m \in [F_n, F_{n+1}), \end{cases}$$

use earlier results.
References


Generalizations
Positive Linear Recurrence Sequences

This method can be greatly generalized to **Positive Linear Recurrence Sequences**: linear recurrences with non-negative coefficients:

\[ H_{n+1} = c_1 H_{n-j_1} + c_2 H_{n-j_2} + \cdots + c_L H_{n-j_L}. \]

**Theorem (Zeckendorf’s Theorem for PLRS recurrences)**

*Any* \( b \in \mathbb{N} \) *has a unique* legal *decomposition into sums of* \( H_n \),

\[ b = a_1 H_{i_1} + \cdots + a_k H_{i_k}. \]

*Here* legal *reduces to non-adjacency of summands in the Fibonacci case.*
The number of $b \in [H_n, H_{n+1})$, with longest gap $< f$ is the coefficient of $x^{n-s}$ in the generating function:
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\[
\frac{1}{1 - x} \left( c_1 - 1 + c_2 x^{t_2} + \cdots + c_L x^{t_L} \right) \times \\
\sum_{k \geq 0} \left[ (c_1 - 1)x^{t_1} + \cdots + (c_L - 1)x^{t_L} \right] \left( \frac{x^{s+1} - x^f}{1 - x} \right)^k + \\
x^{t_1} \left( \frac{x^{s+t_2-t_1+1} - x^f}{1 - x} \right) + \cdots + x^{t_{L-1}} \left( \frac{x^{s+t_L-t_{L-1}+1} + 1 - x^f}{1 - x} \right)^k.
\]
The **number** of \( b \in [H_n, H_{n+1}) \), with longest gap \( < f \) is the coefficient of \( x^{n-s} \) in the generating function:

\[
\frac{1}{1 - x} \left( c_1 - 1 + c_2 x^{t_2} + \cdots + c_L x^{t_L} \right) \times \\
\times \sum_{k \geq 0} \left[ \left( (c_1 - 1)x^{t_1} + \cdots + (c_L - 1)x^{t_L} \right) \left( \frac{x^{s+1} - x^f}{1 - x} \right) + \\
+ x^{t_1} \left( \frac{x^{s+t_2-t_1+1} - x^f}{1 - x} \right) + \cdots + x^{t_{L-1}} \left( \frac{x^{s+t_L-t_{L-1}+1} + 1 - x^f}{1 - x} \right) \right]^k.
\]

A geometric series!
Generalized Generating Function

Let \( f > j_L \). The number of \( x \in [H_n, H_{n+1}) \), with longest gap \(< f \) is given by the coefficient of \( s^n \) in the generating function

\[
F(s) = \frac{1 - s^L}{\mathcal{M}(s) + s^f \mathcal{R}(s)},
\]

where

\[
\mathcal{M}(s) = 1 - c_1 s - c_2 s^{j_2+1} - \cdots - c_L s^{j_L+1},
\]

and

\[
\mathcal{R}(s) = c_{j_1+1} s^{j_1} + c_{j_2+1} s^{j_2} + \cdots + (c_{j_L+1} - 1) s^{j_L}.
\]

and \( c_i \) and \( j_i \) are defined as above.
What are the extra obstructions?

The **coefficients** in the **partial fraction** expansion might **blow up** from multiple roots.
What are the extra obstructions?

The **coefficients** in the **partial fraction** expansion might **blow up** from multiple roots.

**Theorem (Mean and Variance for "Most Recurrences")**

For $x$ in the interval $[H_n, H_{n+1})$, the mean longest gap $\mu_n$ and the variance of the longest gap $\sigma^2_n$ are given by

$$
\mu_n = \frac{\log \left( \frac{\mathcal{R}(\frac{1}{\lambda_1})}{\mathcal{G}(\frac{1}{\lambda_1})} \right) n}{\log \lambda_1} + \frac{\gamma}{\log \lambda_1} - \frac{1}{2} + \text{Small Error} + \epsilon_1(n),
$$

and

$$
\sigma^2_n = \frac{\pi^2}{6 \log \lambda_1} - \frac{1}{12} + \text{Small Error} + \epsilon_2(n),
$$

where $\epsilon_i(n)$ tends to zero in the limit, and Small Error comes from the Euler-Maclaurin Formula.