OF THE ORIGIN AND DESIGN OF EIGENVALUES IN GENERAL, WITH CONCISE REMARKS ON THE ENGLISH CONSTITUTION.

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Dearborn 6/27/19 and Williams 7/3/19

Introduction

Historical Preliminaries

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Momentous struggle for Independence, finally realized in 1776.

In CONGRESS, *July* 4, 1776.

A DECLARATION

By the REPRESENTATIVES *of the*

UNITED STATES OF AMERICA,

IN *GENERAL CONGRESS* ASSEMBLED.

WHEN in the Course of human Events, it becomes necessary for one People to dissolve the Political Bands which have connected them with another, and to assume among the Powers of the Earth, the seperate and equal Station to which the Laws of Nature and of Nature's God entitle them, a decent Respect to the Opinions of Mankind requires that they should declare the Causes which impel them to the Seperation.

We hold these Truths to be self-evident, that all Men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty, and the Pursuit of Happiness.—That to secure these Rights, Governments are instituted among

For quartering large Bodies of armed Troops among us:

For protecting them, by a mock Trial, from Punishment for any Murders which they should commit on the Inhabitants of these States:

For cutting off our Trade with all Parts of the World:

For imposing Taxes on us without our Consent:

For depriving us, in many Cases, of the Benefits of Trial by Jury:

For transporting us beyond Seas to be tried for pretended Offences:

For abolishing the free System of English Laws in a neighbouring Province, establishing therein an arbitrary Government, and enlarging its Boundaries, so as to render it at once an Example and fit Instrument for introducing the same absolute Rule into these Colonies:

Historical Preliminaries

Centuries of British Dependence, especially his Excellency the Chancellor of the Exchequer Sir Isaac Newton.

Momentous struggle for Independence, finally realized in 1776, though earlier versions recently uncovered.

A Declaration of Independence

When, in the Course on Probabalistic Events (Math 241a), it becomes necessary and sufficient for one group to dissolve the equations which have connected them with another, and to assume among the probability functions of the field a separate and equal probability to which the Laws of Nature entitle them, a decent respect requires a declaration of the causes which imply the independence.

We hold this theorem to be self-evident: that all events are created equal; that they are endowed by their Observer with certain inalienable probabilities; that among these are "p" and "q"; that to secure these probabilities, probability functions are defined over groups, deriving their powers from the consequences of the theorems; that whenever any form of probability becomes destructive of these ends, it is the right of the group to alter or to abolish it, and to institute new theorems.

We, therefore, the representatives of the Outcome Space, solemnly publish and declare that these events are, and of right ought to be, free and independent states; that they are absolved from all connection to the previous events, and that all connection between them and the Prior States is, and ought to be, totally dissolved; and that as free and independent probabilities, they have full power to do all acts and outcomes which independent probabilities may of right do.

And for the support of this declaration, we mutually pledge to each other our lives, our laws, and our sacred lemmas.

Goals for the Talk

Describe Systems of the World.

See how Independence allows Natural Understanding.

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

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Examples: Spacings between

- Energy Levels of Nuclei.
- Eigenvalues of Matrices.
- Zeros of L-functions.
- Summands in Zeckendorf Decompositions.
- Primes.
- $n^k\alpha \bmod 1$.

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Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.

2. Develop an explicit formula relating what we want to study to something we understand.

3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Background Material: Linear Algebra

Eigenvalue, Eigenvector

Say $\overrightarrow{v} \neq \overrightarrow{0}$ is an eigenvector of A with eigenvalue λ if $A\overrightarrow{v} = \lambda\overrightarrow{v}$.

Example:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Background Material: Probability

Probability Density

A random variable X has a probability density $p(x)$ if

- $p(x) \geq 0$;
- $\int_{-\infty}^{\infty} p(x)dx = 1$;
- $\mathrm{Prob}(X \in [a,b]) = \int_a^b p(x)dx$.

Examples:

1. Exponential: $p(x) = e^{-x/\lambda}/\lambda$ for $x \geq 0$;
2. Normal: $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$;
3. Uniform: $p(x) = \frac{1}{b-a}$ for $a \leq x \leq b$ and 0 otherwise.

Background Material: Probability (cont)

Key Concepts

- Mean (average value): $\mu = \int_{-\infty}^{\infty} x p(x) dx$.
- Variance (how spread out): $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x) dx$.
- k^{th} moment: $\mu_k = \int_{-\infty}^{\infty} x^k p(x) dx$.

Key observation

As a nice function is given by its Taylor series, a nice probability density is determined by its moments.

Classical
Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable: Failure of Newtonian View.

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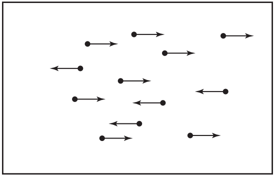
Fundamental Equation:

$$\begin{aligned} H\psi_n &= E_n\psi_n, \quad \text{where} \\ H &: \text{matrix, entries depend on system} \\ E_n &: \text{energy levels} \\ \psi_n &: \text{energy eigenfunctions} \end{aligned}$$

Origins (continued)



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

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Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

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Fix p, define

$$\mathrm{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\mathrm{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

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Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \ = \ \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \ = \ \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$.

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To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

$$\text{k}^{\text{th}}\text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Density of States

Wigner's Semi-Circle Law

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\mathrm{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives $N\,\mathrm{Ave}(\lambda_i(A)^2) \;\sim\; N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

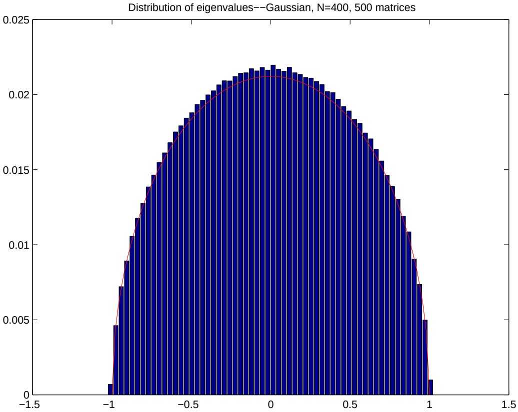
$$\frac{1}{2^2N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N}a_{ji}^2\cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty}a_{ij}^2p(a_{ij})da_{ij}\quad\cdot\quad\prod_{\substack{(k,l)\neq(i,j)\\ k<l}}\int_{a_{kl}=-\infty}^{\infty}p(a_{kl})da_{kl}\;=\;1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

Numerical example: Gaussian density



500 Matrices: Gaussian 400×400

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Numerical example: Cauchy density $p(x) = 1/(\pi(1 + x^2))$

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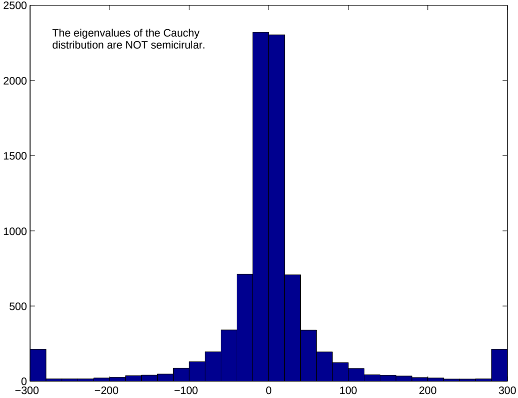


Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

Real Symmetric Toeplitz Matrices

Chris Hammond and Steven J. Miller

Toeplitz Ensembles

Toeplitz matrix is of the form

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N-2} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1-N} & b_{2-N} & b_{3-N} & \cdots & b_0 \end{pmatrix}$$

- Will consider Real Symmetric Toeplitz matrices.
- Main diagonal zero, $N-1$ independent parameters.
- Normalize Eigenvalues by $\sqrt{N}$.

Eigenvalue Density Measure

$$\mu_{A,N}(x)dx \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta\left(x-\frac{\lambda_i(A)}{\sqrt{N}}\right)dx.$$

The k^{th} moment of $\mu_{A,N}(x)$ is

$$M_k(A,N) \;=\; \frac{1}{N^{\frac{k}{2}+1}}\sum_{i=1}^{N}\lambda_i^k(A) \;=\; \frac{\mathrm{Trace}(A^k)}{N^{\frac{k}{2}+1}}.$$

Let

$$M_k \;=\; \lim_{N\to\infty}\mathbb{E}_A\left[M_k(A,N)\right];$$

have $M_2 = 1$ and $M_{2k+1} = 0$.

Even Moments

$$M_{2k}(N) \;=\; \frac{1}{N^{k+1}} \sum_{1 \le i_1, \cdots, i_{2k} \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} \cdots b_{|i_{2k} - i_1|}).$$

Main Term: b_j's matched in pairs, say

$$b_{|i_m - i_{m+1}|} \;=\; b_{|i_n - i_{n+1}|}, \qquad x_m \;=\; |i_m - i_{m+1}| \;=\; |i_n - i_{n+1}|.$$

Two possibilities:

$$i_m - i_{m+1} \;=\; i_n - i_{n+1} \quad \text{or} \quad i_m - i_{m+1} \;=\; -(i_n - i_{n+1}).$$

$(2k-1)!!$ ways to pair, 2^k choices of sign.

Main Term: All Signs Negative (else lower order contribution)

$$M_{2k}(N) = \frac{1}{N^{k+1}} \sum_{1\le i_1,\cdots,i_{2k}\le N} \mathbb{E}(b_{|i_1-i_2|}b_{|i_2-i_3|}\cdots b_{|i_{2k}-i_1|}).$$

Let $x_1,\dots,x_k$ be the values of the $|i_j - i_{j+1}|$'s, $\epsilon_1,\dots,\epsilon_k$ the choices of sign. Define $\widetilde{x}_1 = i_1 - i_2$, $\widetilde{x}_2 = i_2 - i_3, \dots$.

$$\begin{aligned}
i_2 &= i_1 - \widetilde{x}_1 \\
i_3 &= i_1 - \widetilde{x}_1 - \widetilde{x}_2 \\
&\vdots \\
i_1 &= i_1 - \widetilde{x}_1 - \cdots - \widetilde{x}_{2k} \\
\widetilde{x}_1 + \cdots + \widetilde{x}_{2k} &= \sum_{j=1}^{k}(1+\epsilon_j)\eta_j x_j \ = \ 0, \quad \eta_j \ = \ \pm 1.
\end{aligned}$$

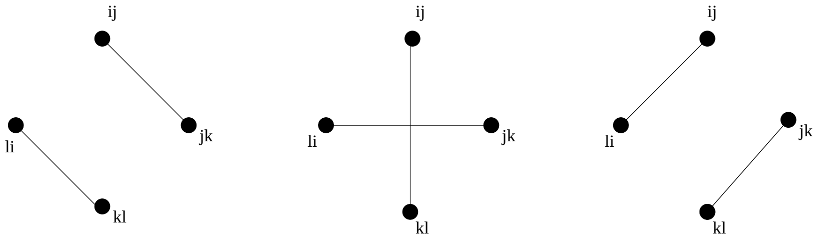
Even Moments: Summary

Main Term: paired, all signs negative.

$$M_{2k}(N) \ \leq \ (2k-1)!! + O_k\left(\frac{1}{N}x\right).$$

Bounded by Gaussian.

The Fourth Moment



$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} b_{|i_3 - i_4|} b_{|i_4 - i_1|})$$

Let $x_j = |i_j - i_{j+1}|$.

The Fourth Moment

Case One: $x_1 = x_2, x_3 = x_4$**:**

$$i_1 - i_2 \ = \ -(i_2 - i_3) \quad \text{and} \quad i_3 - i_4 \ = \ -(i_4 - i_1).$$

Implies

$$i_1 \ = \ i_3, \quad i_2 \text{ and } i_4 \text{ arbitrary.}$$

Left with $\mathbb{E}[b_{x_1}^2 b_{x_3}^2]$:

$$N^3 - N \text{ times get } 1, \ N \text{ times get } p_4 \ = \ \mathbb{E}[b_{x_1}^4].$$

Contributes 1 in the limit.

The Fourth Moment

$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} b_{|i_3 - i_4|} b_{|i_4 - i_1|})$$

Case Two: Diophantine Obstruction: $x_1 = x_3$ and $x_2 = x_4$.

$$i_1 - i_2 \ = \ -(i_3 - i_4) \quad \text{and} \quad i_2 - i_3 \ = \ -(i_4 - i_1).$$

This yields

$$i_1 \ = \ i_2 + i_4 - i_3, \quad i_1, i_2, i_3, i_4 \in \{1, \dots, N\}.$$

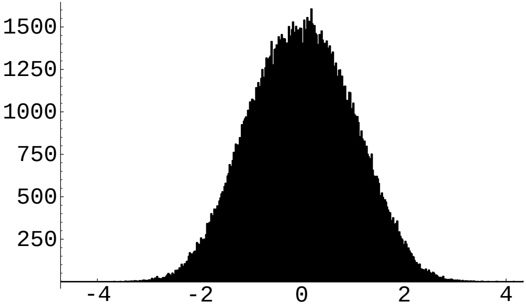
If $i_2, i_4 \geq \frac{2N}{3}$ and $i_3 < \frac{N}{3}$, $i_1 > N$: at most $(1 - \frac{1}{27})N^3$ valid choices.

The Fourth Moment

Theorem: Fourth Moment: Let p_4 be the fourth moment of p. Then

$$M_4(N) \;=\; 2\frac{2}{3} + O_{p_4}\left(\frac{1}{N}\right).$$

500 Toeplitz Matrices, 400×400.



Main Result

Theorem: HM ’05

For real symmetric Toeplitz matrices, the limiting spectral measure converges in probability to a unique measure of unbounded support which is not the Gaussian. If p is even have strong convergence).

Massey, Miller and Sinsheimer ’07 proved that if first row is a palindrome converges to a Gaussian.

Block Circulant Ensemble

With Murat Koloğlu, Gene Kopp, Fred Strauch and Wentao Xiong.

The Ensemble of m-Block Circulant Matrices

Symmetric matrices periodic with period m on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & c_1 & \color{red}{c_2} & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & \color{blue}{d_2} & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & \color{red}{c_2} & c_3 & c_4 & d_3 \\
c_3 & d_2 & c_1 & d_0 & d_1 & \color{blue}{d_2} & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & \color{red}{c_2} & c_3 \\
d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & \color{blue}{d_2} \\
\hline
\color{red}{c_2} & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & \color{blue}{d_2} & d_3 & d_4 & c_3 & d_2 & c_1 & d_0
\end{array}\right).$$

Choose distinct entries i.i.d.r.v.

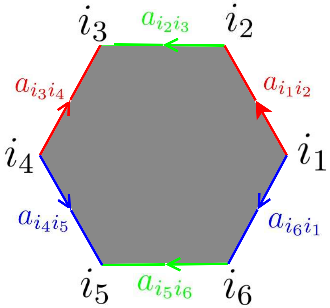
Oriented Matchings and Dualization

Compute moments of eigenvalue distribution (as m stays fixed and $N \to \infty$) using the combinatorics of pairings. Rewrite:

$$\begin{aligned} M_n(N) &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{1 \le i_1, \dots, i_n \le N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}. \end{aligned}$$

where the sum is over oriented matchings on the edges $\{(1,2),(2,3),\dots,(n,1)\}$ of a regular n-gon.

Oriented Matchings and Dualization

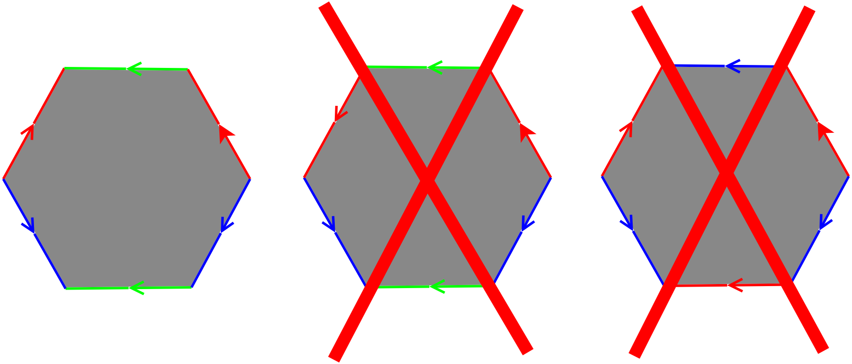


$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
\hline
c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0
\end{array}\right)$$

Figure: An oriented matching in the expansion for $M_n(N) = M_6(8)$.

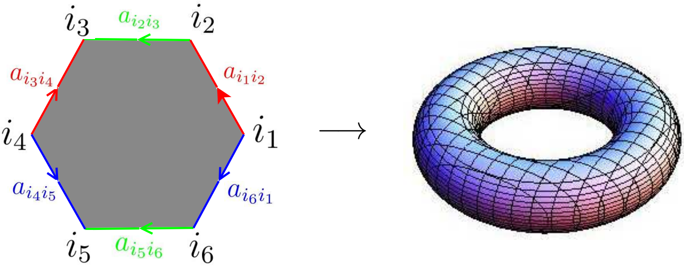
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



Only Topology Matters

Think of pairings as topological identifications; the contributing ones give rise to orientable surfaces.



Contribution from such a pairing is m^{-2g}, where g is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.

Computing the Even Moments

Theorem: Even Moment Formula

$$M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus g surface.

J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$.

Harer and Zagier

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) r^{k+1-2g} \ = \ (2k-1)!!\, c(k,r)$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,r) x^{k+1} \ = \ \left(\frac{1+x}{1-x}\right)^r.$$

Thus, we write

$$M_{2k} \ = \ m^{-(k+1)} (2k-1)!!\, c(k,m).$$

A multiplicative convolution and Cauchy's residue formula yield the characteristic function of the distribution.

$$\begin{aligned}\phi(t) &= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} = \frac{1}{m}\sum_{k=0}^{\infty} \frac{(-t^2/2m)^k}{k!} c(k,m) \\ &= \frac{1}{2\pi i m} \oint_{|z|=2} \frac{1}{2z^{-1}} \left(\left(\frac{1+z^{-1}}{1-z^{-1}} \right)^m - 1 \right) e^{-t^2 z/2m} \frac{dz}{z} \\ &= \frac{1}{m} e^{\frac{-t^2}{2m}} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{1}{(\ell-1)!} \left(\frac{-t^2}{m} \right)^{\ell-1}.\end{aligned}$$

Results

Fourier transform and algebra yields

Theorem: Koloğlu, Kopp and Miller

The limiting spectral density function $f_m(x)$ of the real symmetric m-block circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

Results (continued)

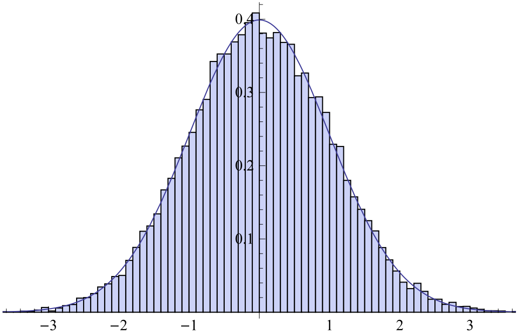


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results (continued)

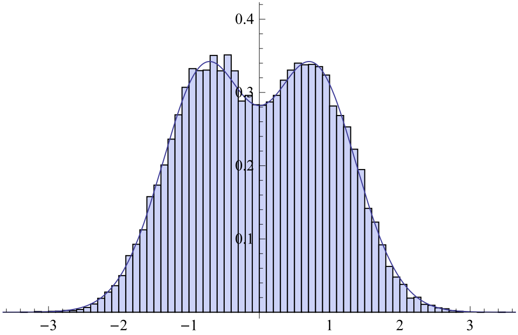


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-block circulant matrices of size 400×400.

Results (continued)

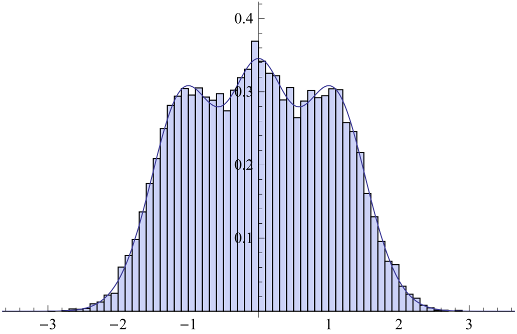


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-block circulant matrices of size 402×402.

Results (continued)

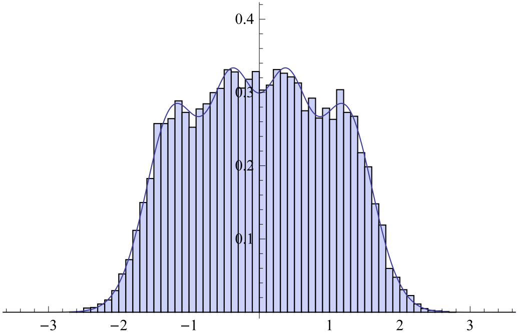


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-block circulant matrices of size 400×400.

Results (continued)

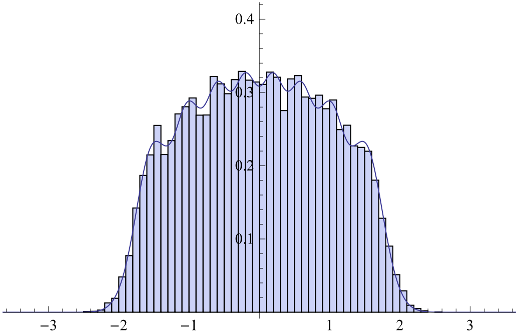


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-block circulant matrices of size 400×400.

Results (continued)

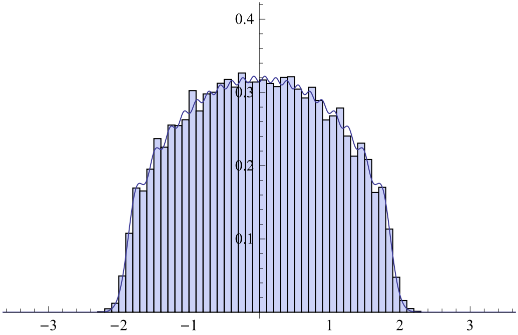


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-block circulant matrices of size 400×400.

Results (continued)

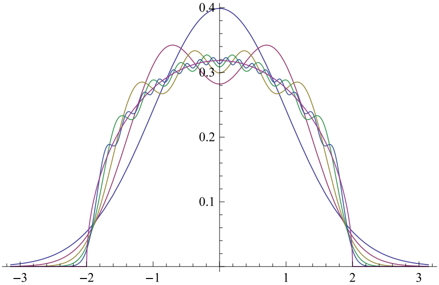


Figure: Plot of convergence to the semi-circle.

The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices (with Murat Kologlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812

Checkerboard Matrices

Checkerboard Matrices: $N \times N$ (k, w)-checkerboard ensemble

Matrices $M = (m_{ij}) = M^T$ with a_{ij} iidrv, mean 0, variance 1, finite higher moments, w fixed and

$$m_{ij} = \begin{cases} a_{ij} & \text{if } i \not\equiv j \bmod k \\ w & \text{if } i \equiv j \bmod k. \end{cases}$$

Example: $(3, w)$-checkerboard matrix:

$$\begin{pmatrix} w & a_{0,1} & a_{0,2} & w & a_{0,4} & \cdots & a_{0,N-1} \\ a_{1,0} & w & a_{1,2} & a_{1,3} & w & \cdots & a_{1,N-1} \\ a_{2,0} & a_{2,1} & w & a_{2,3} & a_{2,4} & \cdots & w \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0,N-1} & a_{1,N-1} & w & a_{3,N-1} & a_{4,N-1} & \cdots & w \end{pmatrix}$$

Split Eigenvalue Distribution

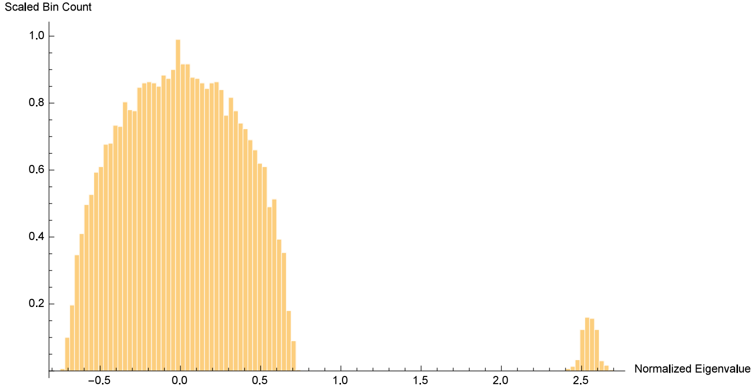


Figure: Histogram of normalized eigenvalues: 2-checkerboard 100×100 matrices, 100 trials. Peaceful Independence of Colony as it pursues its own destiny.

Split Eigenvalue Distribution

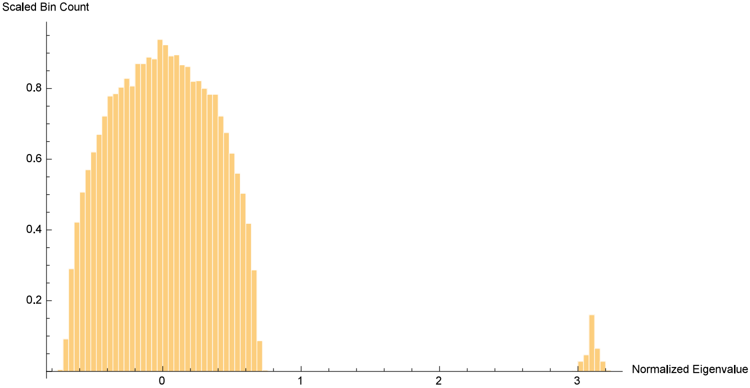


Figure: Histogram of normalized eigenvalues: 2-checkerboard 150×150 matrices, 100 trials. Peaceful Independence of Colony as it pursues its own destiny.

Split Eigenvalue Distribution

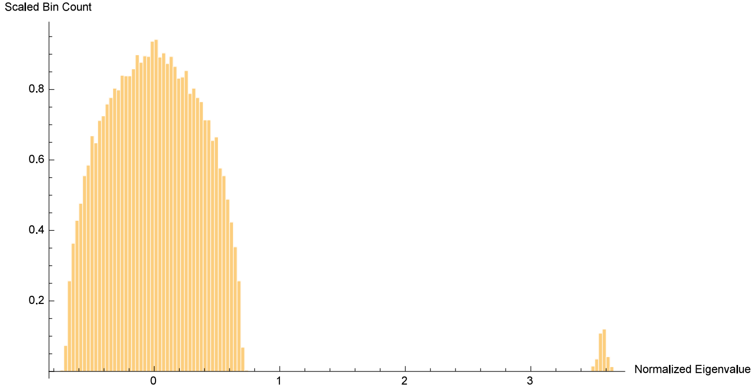


Figure: Histogram of normalized eigenvalues: 2-checkerboard 200×200 matrices, 100 trials. Peaceful Independence of Colony as it pursues its own destiny.

Split Eigenvalue Distribution

Figure: Histogram of normalized eigenvalues: 2-checkerboard 250×250 matrices, 100 trials. Peaceful Independence of Colony as it pursues its own destiny.

Split Eigenvalue Distribution

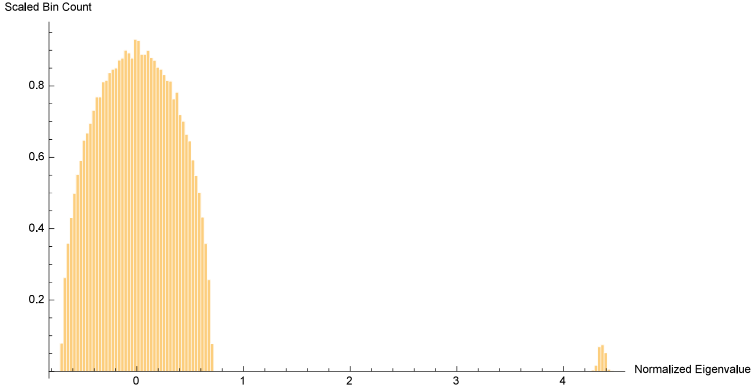


Figure: Histogram of normalized eigenvalues: 2-checkerboard 300×300 matrices, 100 trials. Peaceful Independence of Colony as it pursues its own destiny.

Split Eigenvalue Distribution

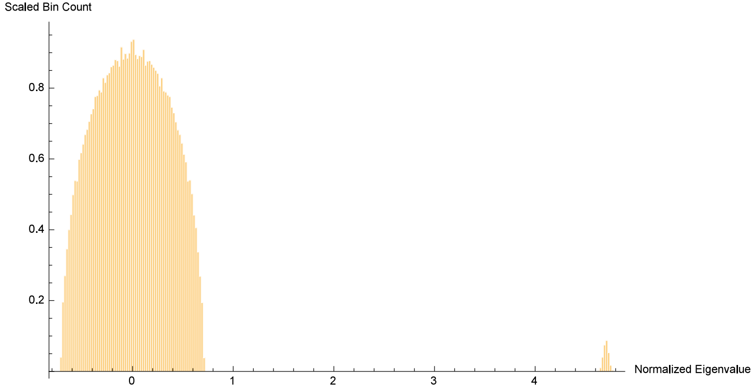


Figure: Histogram of normalized eigenvalues: 2-checkerboard 350×350 matrices, 100 trials. Peaceful Independence of Colony as it pursues its own destiny.

The Weighting Function

Use weighting function $f_n(x) = x^{2n}(x-2)^{2n}$.

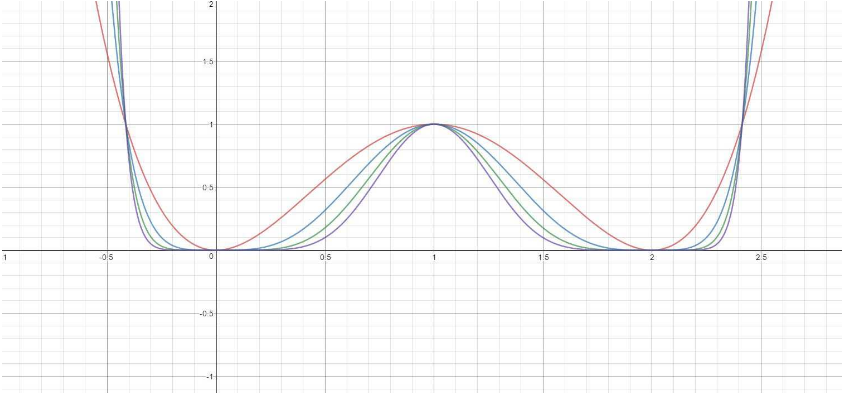


Figure: $f_n(x)$ plotted for $n \in \{1, 2, 3, 4\}$.

The Weighting Function

Use weighting function $f_n(x) = x^{2n}(x-2)^{2n}$.

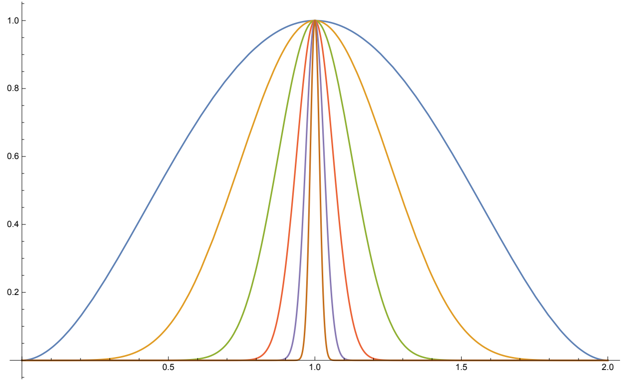


Figure: $f_n(x)$ plotted for $n = 4^m, m \in \{0, 1, \dots, 5\}$.

Spectral distribution of hollow GOE

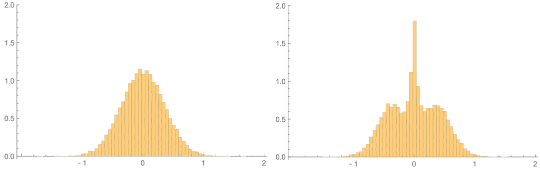


Figure: Hist. of eigenvals of 32000 (Left) 2×2 hollow GOE matrices, (Right) 3×3 hollow GOE matrices.

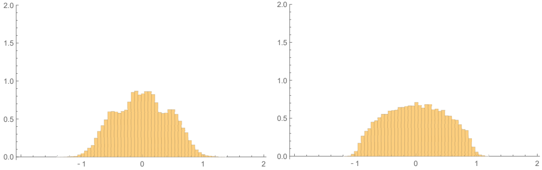


Figure: Hist. of eigenvals of 32000 (Left) 4×4 hollow GOE matrices, (Right) 16×16 hollow GOE matrices.

Introduction to L-Functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

Riemann Zeta Function

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Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right] \left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right] \cdots$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function (cont)

$$\begin{aligned}\zeta(s) &= \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1 \\ \pi(x) &= \#\{p : p \text{ is prime}, p \leq x\}\end{aligned}$$

Properties of $\zeta(s)$ and Primes:

Riemann Zeta Function (cont)

$$\begin{aligned}\zeta(s) &= \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1 \\ \pi(x) &= \#\{p : p \text{ is prime}, p \leq x\}\end{aligned}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.

Riemann Zeta Function (cont)

$$\begin{aligned} \zeta(s) &= \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1 \\ \pi(x) &= \#\{p : p \text{ is prime}, p \leq x\} \end{aligned}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.
- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

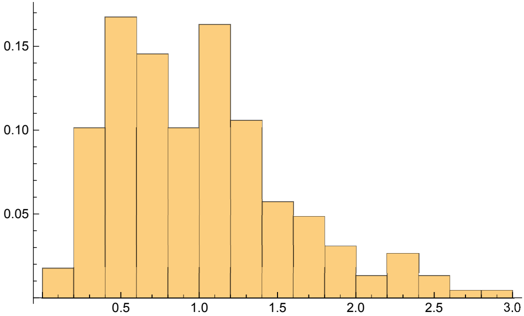
$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

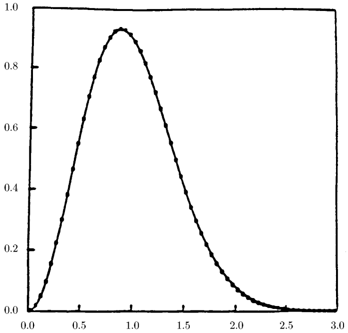
Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Nuclear spacings: Thorium



227 spacings b/w adjacent energy levels of Thorium.

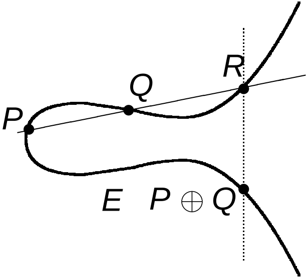
Zeros of $\zeta(s)$ vs GUE



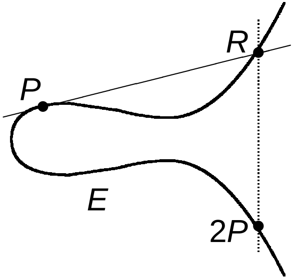
70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Elliptic Curves: Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Elliptic curve L-function

$E: y^2 = x^3 + ax + b$, associate L-function

$$L(s,E) \; = \; \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \; = \; \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

Elliptic curve L-function

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where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s, E)$ at $s = 1/2$.

Properties of zeros of L-functions

- infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \quad = \quad -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) \;=\; -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
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\end{aligned}
$$

Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.
$$

Explicit Formula (Contour Integration)

$$\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
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&= \sum_p\frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.$$

Explicit Formula (Contour Integration)

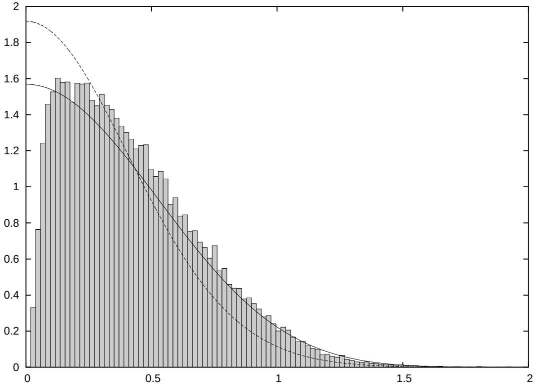
$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1} \\ &= \frac{d}{ds}\sum_p\log\left(1-p^{-s}\right) \\ &= \sum_p\frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \text{Good}(s). \end{aligned}$$

Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p\int\phi(s)e^{-\sigma\log p}e^{-it\log p}ds.$$

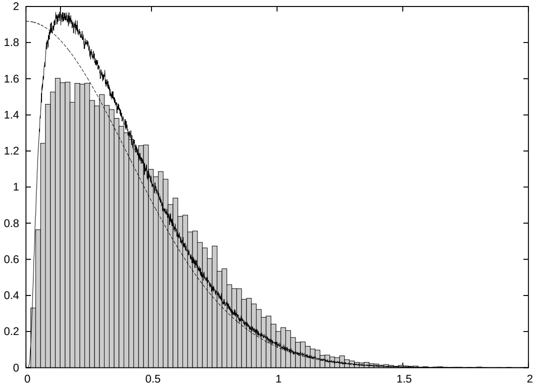
Knowledge of zeros gives info on coefficients.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



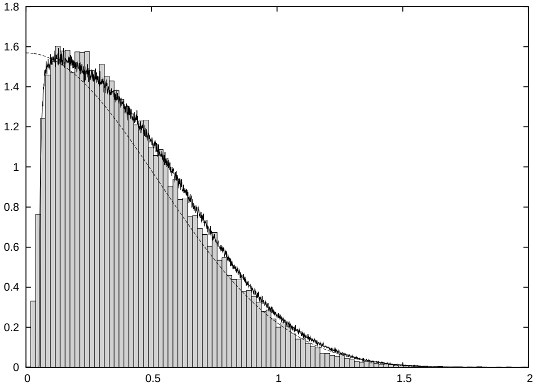
Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with $N_0 = 12$ (solid) with discretisation and with standard $N_0 = 12.26$ (dashed) without discretisation.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) effective N of N_{eff} = 2 (solid) with discretisation and with effective N of $N_{\mathrm{eff}} = 2.32$ (dashed) without discretisation

Correspondences

Similarities between *L*-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longrightarrow$	Neutron
Support of test function	$\longleftrightarrow$	Neutron Energy.

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6. *The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices* (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812

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