Analytic Number Theory Seminar

Random Matrix Theory and L-Functions: I

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Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium ($200+$ protons / neutrons) even worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

$$H\psi_n \ = \ E_n\psi_n$$

Similar to stat mech, leads to considering eigenvalues of ensembles of matrices.

Real Symmetric, Complex Hermitian, Classical Compact Groups.

Random Matrix Ensembles

Real Symmetric Matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{N} \end{pmatrix} = A^T$$

Let $p(x)$ be a probability density.

$$\begin{aligned} p(x) &\geq 0 \\ \int_{\mathbb{R}} p(x)dx &= 1. \end{aligned}$$

Often assume $p(x)$ has finite moments:

$$k^{th}\text{-moment} = \int_{\mathbb{R}} x^k p(x)dx.$$

Define

$$\text{Prob}(A) = \prod_{1 \leq i < j \leq N} p(a_{ij}).$$

Eigenvalue Distribution

Key to Averaging:

$$\text{Trace}(A^k) \;=\; \sum_{i=1}^{N} \lambda_i^k(A).$$

By the Central Limit Theorem:

$$\begin{aligned}
\text{Trace}(A^2) \;&=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \\
&=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \\
&\sim\; N^2 \cdot 1 \\
\sum_{i=1}^{N} \lambda_i^2(A) \;&\sim\; N^2
\end{aligned}$$

Gives $N\text{Ave}(\lambda_i^2(A)) \;\sim\; N^2$ or $\lambda_i(A) \sim \sqrt{N}$.

Eigenvalue Distribution (cont)

$\delta(x - x_0)$ is a unit point mass at x_0.

To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

Obtain:

$$\begin{aligned} k^{th}\text{-moment} &= \int x^k \mu_{A,N}(x) dx \\ &= \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i^k(A)}{(2\sqrt{N})^k} \\ &= \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}} \end{aligned}$$

Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from fixed $p(x)$.

Semi-Circle Law: Assume p has mean 0, variance 1, other moments finite. Then

$$\mu_{A,N}(x) \ \to\ \frac{2}{\pi}\sqrt{1-x^2} \text{ with probability } 1$$

Trace formula converts sums over eigenvalues to sums over entries of A.

Expected value of k^{th}-moment of $\mu_{A,N}(x)$ is

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}} \prod_{i<j} p(a_{ij}) da_{ij}$$

Proof: 2^{nd}-Moment

$$\text{Trace}(A^2) = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2.$$

Substituting into expansion gives

$$\frac{1}{2^2N^2}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}}\sum_{i,j=1}^{N} a_{ji}^2 \cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}\in\mathbb{R}} a_{ij}^2 p(a_{ij})da_{ij} \quad\cdot \prod_{\substack{(k,l)\neq(ij)\\ k<l}} \int_{a_{kl}\in\mathbb{R}} p(a_{kl})da_{kl} = 1.$$

Have N^2 summands, answer is $\frac{1}{4}$.

Key: Averaging Formula.

Riemann Zeta Function: $\zeta(s)$

Riemann Zeta-Function:

$$\zeta(s) = \sum_n n^{-s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis: All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; ie, on the critical line.

Spacings between zeros same as spacings between eigenvalues of Complex Hermitian matrices.

L-Functions

L-functions: $\mathrm{Re}(s) > s_0$:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s} = \prod_p L_p(p^{-s}, f)^{-1}.$$

Functional equation: $s \longleftrightarrow 1 - s$.

GRH: All L-functions (after normalization) have their non-trivial zeros on the critical line.

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ be an increasing sequence of numbers, $B \subset \mathbf{R}^{n-1}$ a compact box. Define the n-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1} - \alpha_{j_2}, \dots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k\right\}}{N}$$

Instead of using a box, can use a smooth test function.

Results:

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko)
2. Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak)
4. n-level correlations for the classical compact groups (Katz-Sarnak)
5. insensitive to any finite set of zeros

Measures of Spacings: n-Level Density and Families

Let $f(x) = \prod_i f_i(x_i)$, f_i even Schwartz functions whose Fourier Transforms are compactly supported.

$$D_{n,E}(f) \;=\; \sum_{\substack{j_1,\dots,j_n \\ distinct}} f_1\left(L_E\gamma_E^{(j_1)}\right)\cdots f_n\left(L_E\gamma_E^{(j_n)}\right)$$

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

To any geometric family, Katz-Sarnak predict the n-level density depends only on a symmetry group attached to the family.

Correspondences

Similarities b/w Nuclei and L-Fns:

$$\text{Zeros} \quad \longleftrightarrow \quad \text{Energy Levels}$$

$$\text{Support} \quad \longleftrightarrow \quad \text{Neutron Energy.}$$

Number Theory Results

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign).

- Symplectic: Rubinstein: n-level densities for twists $L(s, \chi_d)$ of the zeta-function.

- Unitary: Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.

- Orthogonal: Miller: One-parameter families of elliptic curves.

Main Tools

- Averaging Formulas: Petersson formula in ILS, Orthogonality of characters in Rubinstein, Miller, Hughes-Rudnick.

- Explicit Formula: Relates sums over zeros to sums over primes.

- Control of conductors: Monotone.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,O^+}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,O}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,O^-}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and $\eta(u)$ is 1, $\frac{1}{2}$, and 0 for $|u|$ less than 1, 1, and greater than 1.

Dirichlet L-Functions

Let χ be a primitive character mod m. Let

$$c(m,\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}.$$

$c(m,\chi)$ is a Gauss sum of modulus $\sqrt{m}$.

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1} \\ \Lambda(s,\chi) &= \pi^{-\frac{1}{2}(s+\epsilon)} \Gamma\left(\frac{s+\epsilon}{2}\right) m^{\frac{1}{2}(s+\epsilon)} L(s,\chi), \end{aligned}$$

where

$$\epsilon = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases}$$

$$\Lambda(s,\chi) = (-i)^{\epsilon} \frac{c(m,\chi)}{\sqrt{m}} \Lambda(1-s,\bar{\chi}).$$

Explicit Formula

Let ϕ be an even Schwartz function with compact support $(-\sigma, \sigma)$.

Let χ be a non-trivial primitive Dirichlet character of conductor m.

$$\sum \phi\left(\gamma \frac{\log(\frac{m}{\pi})}{2\pi}\right)$$
$$= \int_{-\infty}^{\infty} \phi(y)dy$$
$$- \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}}$$
$$- \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) [\chi^2(p) + \bar{\chi}^2(p)] p^{-1}$$
$$+ O\left(\frac{1}{\log m}\right).$$

Dirichlet Characters: m Prime

$(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$ with generator g.

Let $\zeta_{m-1} = e^{2\pi i/(m-1)}$.

The principal character χ_0 is given by

$$\chi_0(k) = \begin{cases} 1 & (k,m) = 1 \\ 0 & (k,m) > 1. \end{cases}$$

The $m-2$ primitive characters are determined (by multiplicativity) by action on g.

As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each χ there exists an l such that $\chi(g) = \zeta_{m-1}^l$. Hence for each $l, 1 \le l \le m-2$ we have

$$\chi_l(k) \ = \ \begin{cases} \zeta_{m-1}^{la} & k \equiv g^a (m) \\ 0 & (k,m) > 0 \end{cases}$$

Expansion

$\{\chi_0\} \cup \{\chi_l\}_{1 \leq l \leq m-2}$ are all the characters mod m.

Consider the family of primitive characters mod a prime m ($m - 2$ character):

$$\begin{aligned}
& \int_{-\infty}^{\infty} \phi(y) dy \\
- & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}} \\
- & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big) [\chi^2(p) + \bar{\chi}^2(p)] p^{-1} \\
+ & O\Big(\frac{1}{\log m}\Big).
\end{aligned}$$

Note can pass Character Sum through Test Function.

Character Sums

$$\sum_{\chi} \chi(k) = \begin{cases} m-1 & k \equiv 1(m) \\ 0 & \text{otherwise} \end{cases}$$

For any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) \;=\; \begin{cases} m-1-1 & p \equiv 1(m) \\ -1 & \text{otherwise} \end{cases}$$

Substitute into

$$\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}}$$

First Sum

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}$$

$$+\ 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac{1}{2}}$$

$$\ll\ \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}}\ +\ \sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac{1}{2}}$$

$$\ll\ \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}}\ +\ \sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma}k^{-\frac{1}{2}}$$

$$\ll\ \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}}\ +\ \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}}$$

$$\ll\ \frac{1}{m}m^{\sigma/2}.$$

No contribution if $\sigma < 2$.

Second Sum

$$\frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)\frac{\chi^2(p)+\bar{\chi}^2(p)}{p}.$$

$$\sum_{\chi\neq\chi_0}[\chi^2(p)+\bar{\chi}^2(p)] = \begin{cases} 2(m-2) & p\equiv\pm 1(m) \\ -2 & p\not\equiv\pm 1(m) \end{cases}$$

Up to $O\left(\frac{1}{\log m}\right)$ we find that

$$\begin{aligned}
&\ll \frac{1}{m-2}\sum_p^{m^{\sigma/2}} p^{-1} + \frac{2m-2}{m-2}\sum_{p\equiv\pm1(m)}^{m^{\sigma/2}} p^{-1} \\
&\ll \frac{1}{m-2}\sum_k^{m^{\sigma/2}} k^{-1} + \sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^{\sigma/2}} k^{-1} + \sum_{\substack{k\equiv -1(m)\\ k\geq m-1}}^{m^{\sigma/2}} k^{-1} \\
&\ll \frac{\log(m^{\sigma/2})}{m-2} + \frac{1}{m}\sum_k^{m^{\sigma/2}} k^{-1} + \frac{1}{m}\sum_k^{m^{\sigma/2}} k^{-1} + O\left(\frac{1}{m}\right) \\
&\ll \frac{\log m}{m} + \frac{\log m}{m} + \frac{\log m}{m}.
\end{aligned}$$

Dirichlet Characters: m Square-free

Fix an r and let $m_1, \dots, m_r$ be distinct odd primes.

$$\begin{aligned} m &= m_1 m_2 \cdots m_r \\ M_1 &= (m_1 - 1)(m_2 - 1) \cdots (m_r - 1) = \phi(m) \\ M_2 &= (m_1 - 2)(m_2 - 2) \cdots (m_r - 2). \end{aligned}$$

M_2 is the number of primitive characters mod m, each of conductor m.

A general primitive character mod m is given by $\chi(u) = \chi_{l_1}(u)\chi_{l_2}(u) \cdots \chi_{l_r}(u)$.

Let $\mathcal{F} = \{\chi : \chi = \chi_{l_1}\chi_{l_2} \cdots \chi_{l_r}\}$.

$$\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big) p^{-\frac{1}{2}} \sum_{\chi \in \mathcal{F}} [\chi(p) + \bar{\chi}(p)]$$

$$\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\Big(2\frac{\log p}{\log(m/\pi)}\Big) p^{-1} \sum_{\chi \in \mathcal{F}} [\chi^2(p) + \bar{\chi}^2(p)]$$

Characters Sums:

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \begin{cases} m_i - 1 - 1 & p \equiv 1(m_i) \\ -1 & \text{otherwise} \end{cases}$$

Define

$$\delta_{m_i}(p, 1) = \begin{cases} 1 & p \equiv 1(m_i) \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \sum_{\chi \in \mathcal{F}} \chi(p) &= \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}(p) \cdots \chi_{l_r}(p) \\ &= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) \\ &= \prod_{i=1}^{r} \left(-1 + (m_i - 1)\delta_{m_i}(p, 1) \right). \end{aligned}$$

Expansion Preliminaries:

$k(s)$ is an s-tuple $(k_1, k_2, \ldots, k_s)$ with $k_1 < k_2 < \cdots < k_s$.

This is just a subset of $(1, 2, \ldots, r)$, 2^r possible choices for $k(s)$.

$$\delta_{k(s)}(p, 1) = \prod_{i=1}^{s} \delta_{m_{k_i}}(p, 1).$$

If $s = 0$ we define $\delta_{k(0)}(p, 1) = 1 \; \forall p$.

Then

$$\begin{aligned} & \prod_{i=1}^{r} \Big(-1 + (m_i - 1)\delta_{m_i}(p, 1) \Big) \\ = \; & \sum_{s=0}^{r} \sum_{k(s)} (-1)^{r-s} \delta_{k(s)}(p, 1) \prod_{i=1}^{s} (m_{k_i} - 1) \end{aligned}$$

First Sum:

$$\ll \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \frac{1}{M_2}\Big(1 + \sum_{s=1}^{r} \sum_{k(s)} \delta_{k(s)}(p,1) \prod_{i=1}^{s} (m_{k_i} - 1)\Big).$$

As $m/M_2 \leq 3^r$, $s = 0$ sum contributes

$$S_{1,0} = \frac{1}{M_2} \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2}\sigma - 1},$$

hence negligible for $\sigma < 2$. Now we study

$$\begin{aligned} S_{1,k(s)} &= \frac{1}{M_2} \prod_{i=1}^{s} (m_{k_i} - 1) \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \delta_{k(s)}(p,1) \\ &\ll \frac{1}{M_2} \prod_{i=1}^{s} (m_{k_i} - 1) \sum_{n \equiv 1 (m_{k(s)})}^{m^\sigma} n^{-\frac{1}{2}} \\ &\ll \frac{1}{M_2} \prod_{i=1}^{s} (m_{k_i} - 1) \frac{1}{\prod_{i=1}^{s} (m_{k_i})} \sum_{n}^{m^\sigma} n^{-\frac{1}{2}} \\ &\ll 3^r m^{\frac{1}{2}\sigma - 1}. \end{aligned}$$

First Sum (cont):

There are 2^r choices, yielding

$$S_1 \ll 6^r m^{\frac{1}{2}\sigma - 1},$$

which is negligible as m goes to infinity for fixed r if $\sigma < 2$.

Cannot let r go to infinity.

If m is the product of the first r primes,

$$\begin{aligned}\log m &= \sum_{k=1}^{r} \log p_k \\ &= \sum_{p \leq r} \log p \approx r\end{aligned}$$

Therefore

$$6^r \approx m^{\log 6} \approx m^{1.79}.$$

Second Sum Expansions:

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) \ = \begin{cases} m_i - 1 - 1 & p \equiv \pm 1 (m_i) \\ -1 & \text{otherwise} \end{cases}$$

$$\begin{aligned} & \sum_{\chi \in \mathcal{F}} \chi^2(p) \\ = & \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}^2(p) \cdots \chi_{l_r}^2(p) \\ = & \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) \\ = & \prod_{i=1}^{r} \Big(-1 + (m_i - 1)\delta_{m_i}(p,1) + (m_i - 1)\delta_{m_i}(p,-1) \Big) \end{aligned}$$

Second Sum Bounds:

Handle similarly as before. Say

$$\begin{aligned} p &\equiv \quad 1 \bmod m_{k_1}, \ldots, m_{k_a} \\ p &\equiv -1 \bmod m_{k_{a+1}}, \ldots, m_{k_b} \end{aligned}$$

How small can p be?

$+1$ congruences imply $p \geq m_{k_1} \cdots m_{k_a} + 1$.

-1 congruences imply $p \geq m_{k_{a+1}} \cdots m_{k_b} - 1$.

Since the product of these two lower bounds is greater than $\prod_{i=1}^{b}(m_{k_i} - 1)$, at least one must be greater than $\left(\prod_{i=1}^{b}(m_{k_i} - 1)\right)^{\frac{1}{2}}$.

There are 3^r pairs, yielding

$$\text{Second Sum} = \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} S_{2,k(s),j(s)} \ll 9^r m^{-\frac{1}{2}}.$$

Summary:

Agrees with Unitary for $\sigma < 2$.

We proved:

Lemma:

- m square-free odd integer with $r = r(m)$ factors;
- $m = \prod_{i=1}^{r} m_i$;
- $M_2 = \prod_{i=1}^{r}(m_i - 2)$.

Consider the family $\mathcal{F}_m$ of primitive characters mod m. Then

$$\text{First Sum} \ll \frac{1}{M_2} 2^r m^{\frac{1}{2}\sigma}$$
$$\text{Second Sum} \ll \frac{1}{M_2} 3^r m^{\frac{1}{2}}.$$

Dirichlet Characters: $m \in [N, 2N]$ Square-free

$\mathcal{F}_N$ all primitive characters with conductor odd square-free integer in $[N, 2N]$.

At least $N/\log^2 N$ primes in the interval.

At least $N \cdot \frac{N}{\log^2 N} = N^2 \log^{-2} N$ primitive characters:

$$M \ \geq \ N^2 \log^{-2} N \quad \Rightarrow \quad \frac{1}{M} \leq \frac{\log^2 N}{N^2}.$$

Bounds

$$S_{1,m} \ll \frac{1}{M} 2^{r(m)} m^{\frac{1}{2}\sigma}$$
$$S_{2,m} \ll \frac{1}{M} 3^{r(m)} m^{\frac{1}{2}}.$$

$2^{r(m)} = \tau(m)$, the number of divisors of m, and $3^{r(m)} \leq \tau^2(m)$.

While it is possible to prove

$$\sum_{n \leq x} \tau^l(n) \ll x(\log x)^{2^l - 1}$$

the crude bound

$$\tau(n) \leq c(\epsilon) n^\epsilon$$

yields the same region of convergence.

First Sum Bound

$$
\begin{aligned}
S_1 &= \sum_{\substack{m=N \\ m\ squarefree}}^{2N} S_{1,m} \\
&\ll \sum_{m=N}^{2N} \frac{1}{M} 2^{r(m)} m^{\frac{1}{2}\sigma} \\
&\ll \frac{1}{M} N^{\frac{1}{2}\sigma} \sum_{m=N}^{2N} \tau(m) \\
&\ll \frac{1}{M} N^{\frac{1}{2}\sigma} c(\epsilon) N^{1+\epsilon} \\
&\ll \frac{\log^2 N}{N^2} N^{\frac{1}{2}\sigma} c(\epsilon) N^{1+\epsilon} \\
&\ll c(\epsilon) N^{\frac{1}{2}\sigma+\epsilon-1} \log^2 N.
\end{aligned}
$$

No contribution if $\sigma < 2$.

Second sum handled similarly.

Summary

- Similar behavior in different systems.
- Find correct scale.
- Average over similar elements.
- Need an Explicit Formula.
- Different statistics tell different stories.