Analytic Number Theory Seminar
Random Matrix Theory and $L$-Functions: I

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/~sjmiller/math/talks/talks.html
Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium (200+ protons / neutrons) even worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

\[ H \psi_n = E_n \psi_n \]

Similar to stat mech, leads to considering eigenvalues of ensembles of matrices.

Real Symmetric, Complex Hermitian, Classical Compact Groups.
Random Matrix Ensembles

Real Symmetric Matrices:

\[
A = \begin{pmatrix}
 a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\
 a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 a_{1N} & a_{2N} & a_{3N} & \cdots & a_{N}
\end{pmatrix}
= A^T
\]

Let \( p(x) \) be a probability density.

\[
p(x) \geq 0
\]

\[
\int_{\mathbb{R}} p(x) \, dx = 1.
\]

Often assume \( p(x) \) has finite moments:

\[
k^{th}\text{-moment} = \int_{\mathbb{R}} x^k p(x) \, dx.
\]

Define

\[
\text{Prob}(A) = \prod_{1 \leq i < j \leq N} p(a_{ij}).
\]
**Eigenvalue Distribution**

Key to Averaging:

\[
\text{Trace}(A^k) = \sum_{i=1}^{N} \lambda_i^k(A).
\]

By the Central Limit Theorem:

\[
\text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \sim N^2 \cdot 1
\]

\[
\sum_{i=1}^{N} \lambda_i^2(A) \sim N^2
\]

Gives \(NAve(\lambda_i^2(A)) \sim N^2\) or \(\lambda_i(A) \sim \sqrt{N}\).
Eigenvalue Distribution (cont)

$\delta(x - x_0)$ is a unit point mass at $x_0$.

To each $A$, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)$$

Obtain:

$$k^{th\text{-}moment} = \int x^k \mu_{A,N}(x) dx$$

$$= \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i^k(A)}{(2\sqrt{N})^k}$$

$$= \frac{\text{Trace}(A^k)}{2^k N^{k/2 + 1}}$$
Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from fixed $p(x)$.

**Semi-Circle Law:** Assume $p$ has mean 0, variance 1, other moments finite. Then

$$\mu_{A,N}(x) \rightarrow \frac{2}{\pi} \sqrt{1 - x^2} \text{ with probability } 1$$

Trace formula converts sums over eigenvalues to sums over entries of $A$.

Expected value of $k^{th}$-moment of $\mu_{A,N}(x)$ is

$$\int \ldots \int \frac{\text{Trace}(A^k)}{2^k N^{k/2 + 1}} \prod_{i<j} p(a_{ij}) da_{ij}$$
Proof: $2^{nd}$-Moment

$$\text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2.$$ 

Substituting into expansion gives

$$\frac{1}{2^2N^2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1}^{N} a_{ji}^2 \cdot p(a_{11})da_{11} \cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij} \in \mathbb{R}} a_{ij}^2 p(a_{ij})da_{ij} \cdot \prod_{(k,l)\neq(ij), k<l}^{(k,l) \neq (ij)} \int_{a_{kl} \in \mathbb{R}} p(a_{kl})da_{kl} = 1.$$ 

Have $N^2$ summands, answer is $\frac{1}{4}$.

Key: Averaging Formula.
**Riemann Zeta Function:** $\zeta(s)$

Riemann Zeta-Function:

$$\zeta(s) = \sum_{n} n^{-s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi(1 - s).$$

Riemann Hypothesis: All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; ie, on the critical line.

Spacings between zeros same as spacings between eigenvalues of Complex Hermitian matrices.
**L-Functions**

*L-functions*: \( \text{Re}(s) > s_0 \):

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s} = \prod_p L_p(p^{-s}, f)^{-1}.
\]

**Functional equation**: \( s \leftrightarrow 1 - s \).

**GRH**: All *L*-functions (after normalization) have their non-trivial zeros on the critical line.
Measures of Spacings:

\( n \)-Level Correlations

\( \{\alpha_j\} \) be an increasing sequence of numbers, \( B \subset \mathbb{R}^{n-1} \) a compact box. Define the \( n \)-level correlation by

\[
\lim_{N \to \infty} \frac{\# \left\{ \left( \alpha_{j_1} - \alpha_{j_2}, \ldots, \alpha_{j_{n-1}} - \alpha_{j_n} \right) \in B, j_i \neq j_k \right\}}{N}
\]

Instead of using a box, can use a smooth test function.

Results:

1. Normalized spacings of \( \zeta(s) \) starting at 10^{20} (Odlyzko)

2. Pair and triple correlations of \( \zeta(s) \) (Montgomery, Hejhal)

3. \( n \)-level correlations for all automorphic cuspidal \( L \)-functions (Rudnick-Sarnak)

4. \( n \)-level correlations for the classical compact groups (Katz-Sarnak)

5. insensitive to any finite set of zeros
Measures of Spacings: 
\( n \)-Level Density and Families

Let \( f(x) = \prod_i f_i(x_i) \), \( f_i \) even Schwartz functions whose Fourier Transforms are compactly supported.

\[
D_{n,E}(f) = \sum_{j_1,\ldots,j_n \text{ distinct}} f_1\left(L_E\gamma_E^{(j_1)}\right) \cdots f_n\left(L_E\gamma_E^{(j_n)}\right)
\]

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

To any geometric family, Katz-Sarnak predict the \( n \)-level density depends only on a symmetry group attached to the family.
Correspondences

Similarities b/w Nuclei and $L$-Fns:

Zeros $\longleftrightarrow$ Energy Levels

Support $\longleftrightarrow$ Neutron Energy.
Number Theory Results

- **Orthogonal**: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight $k$ cuspidal newforms of square-free level $N$ (SO(even) and SO(odd) if split by sign).

- **Symplectic**: Rubinstein: $n$-level densities for twists $L(s, \chi_d)$ of the zeta-function.

- **Unitary**: Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.

- **Orthogonal**: Miller: One-parameter families of elliptic curves.
Main Tools

- **Averaging Formulas**: Petersson formula in ILS, Orthogonality of characters in Rubinstein, Miller, Hughes-Rudnick.

- **Explicit Formula**: Relates sums over zeros to sums over primes.

- **Control of conductors**: Monotone.
1-Level Densities

The Fourier Transforms for the 1-level densities are

\[ \hat{W}_{1,O^+}(u) = \delta_0(u) + \frac{1}{2} \eta(u) \]
\[ \hat{W}_{1,O}(u) = \delta_0(u) + \frac{1}{2} \]
\[ \hat{W}_{1,O^-}(u) = \delta_0(u) - \frac{1}{2} \eta(u) + 1 \]
\[ \hat{W}_{1,Sp}(u) = \delta_0(u) - \frac{1}{2} \eta(u) \]
\[ \hat{W}_{1,U}(u) = \delta_0(u) \]

where \( \delta_0(u) \) is the Dirac Delta functional and \( \eta(u) \) is 1, \( \frac{1}{2} \), and 0 for \(|u|\) less than 1, 1, and greater than 1.
Dirichlet $L$-Functions

Let $\chi$ be a primitive character mod $m$. Let

$$c(m, \chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k / m}.$$ 

$c(m, \chi)$ is a Gauss sum of modulus $\sqrt{m}$.

$$L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

$$\Lambda(s, \chi) = \pi^{-\frac{1}{2}(s+\epsilon)} \Gamma \left( \frac{s + \epsilon}{2} \right) m^{\frac{1}{2}(s+\epsilon)} L(s, \chi),$$

where

$$\epsilon = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases}$$

$$\Lambda(s, \chi) = (-i)^\epsilon \frac{c(m, \chi)}{\sqrt{m}} \Lambda(1 - s, \bar{\chi}).$$
Explicit Formula

Let $\phi$ be an even Schwartz function with compact support $(-\sigma, \sigma)$.

Let $\chi$ be a non-trivial primitive Dirichlet character of conductor $m$.

\[
\sum \phi \left( \frac{\log \left( \frac{m}{\pi} \right)}{2\pi} \right) = \int_{-\infty}^{\infty} \phi(y) dy - \sum_{p} \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) \left[ \chi(p) + \bar{\chi}(p) \right] p^{-\frac{1}{2}} - \sum_{p} \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( 2 \frac{\log p}{\log(m/\pi)} \right) \left[ \chi^2(p) + \bar{\chi}^2(p) \right] p^{-1} + O \left( \frac{1}{\log m} \right).
\]
Dirichlet Characters:

$m$ Prime

$(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m - 1$ with generator $g$.

Let $\zeta_{m-1} = e^{2\pi i / (m-1)}$.

The principal character $\chi_0$ is given by

$$\chi_0(k) = \begin{cases} 1 & (k, m) = 1 \\ 0 & (k, m) > 1. \end{cases}$$

The $m-2$ primitive characters are determined (by multiplicativity) by action on $g$.

As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each $\chi$ there exists an $l$ such that $\chi(g) = \zeta_{m-1}^l$. Hence for each $l, 1 \leq l \leq m - 2$ we have

$$\chi_l(k) = \begin{cases} \zeta_{m-1}^{la} & k \equiv g^a(m) \\ 0 & (k, m) > 0 \end{cases}$$
Expansion

\( \{ \chi_0 \} \cup \{ \chi_l \}_{1 \leq l \leq m-2} \) are all the characters mod \( m \).

Consider the family of primitive characters mod a prime \( m \) (\( m - 2 \) character):

\[
\int_{-\infty}^{\infty} \phi(y) dy \\
- \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}} \\
- \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( 2 \frac{\log p}{\log(m/\pi)} \right) [\chi^2(p) + \bar{\chi}^2(p)] p^{-1} \\
+ O \left( \frac{1}{\log m} \right).
\]

Note can pass Character Sum through Test Function.
Character Sums

\[ \sum_{\chi} \chi(k) = \begin{cases} m - 1 & k \equiv 1(m) \\ 0 & \text{otherwise} \end{cases} \]

For any prime \( p \neq m \)

\[ \sum_{\chi \neq \chi_0} \chi(p) = \begin{cases} m - 1 - 1 & p \equiv 1(m) \\ -1 & \text{otherwise} \end{cases} \]

Substitute into

\[ \frac{1}{m - 2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \phi \left( \frac{\log p}{\log(m/\pi)} \right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}} \]
First Sum

\[
-2 \frac{m^\sigma}{m-2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}} \\
+ \frac{2(m-1)}{m-2} \sum_{p \equiv 1(m)} m^\sigma \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m} \sum_p m^\sigma p^{-\frac{1}{2}} + \sum_{p \equiv 1(m)} m^\sigma p^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m} \sum_k m^\sigma k^{-\frac{1}{2}} + \sum_{k \equiv 1(m)} \sum_{k \geq m+1} m^\sigma k^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m} \sum_k m^\sigma k^{-\frac{1}{2}} + \frac{1}{m} \sum_k m^\sigma k^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m} 
\]

No contribution if \( \sigma < 2 \).
Second Sum

\[
\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}
\left(2 \frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p) + \bar{\chi}^2(p)}{p}.
\]

\[
\sum_{\chi \neq \chi_0} [\chi^2(p) + \bar{\chi}^2(p)] = \begin{cases} 
2(m-2) & p \equiv \pm 1(m) \\
-2 & p \not\equiv \pm 1(m)
\end{cases}
\]

Up to \(O\left(\frac{1}{\log m}\right)\) we find that

\[
\ll \frac{1}{m-2} \sum_p p^{-1} + \frac{2m-2}{m-2} \sum_{p=\pm 1(m)} p^{-1}
\]

\[
\ll \frac{1}{m-2} \sum_k k^{-1} + \sum_{k=1(m)}^{m/2} k^{-1} + \sum_{k=m+1}^{m/2} k^{-1}
\]

\[
\ll \frac{\log(m^{\sigma/2})}{m-2} + \frac{1}{m} \sum_k k^{-1} + \frac{1}{m} \sum_{k=m/2+1}^{m/2} k^{-1} + O\left(\frac{1}{m}\right)
\]

\[
\ll \frac{\log m}{m} + \frac{\log m}{m} + \frac{\log m}{m}.
\]
**Dirichlet Characters: \( m \) Square-free**

Fix an \( r \) and let \( m_1, \ldots, m_r \) be distinct odd primes.

\[
\begin{align*}
  m &= m_1 m_2 \cdots m_r \\
  M_1 &= (m_1 - 1)(m_2 - 1) \cdots (m_r - 1) = \phi(m) \\
  M_2 &= (m_1 - 2)(m_2 - 2) \cdots (m_r - 2).
\end{align*}
\]

\( M_2 \) is the number of primitive characters mod \( m \), each of conductor \( m \).

A general primitive character mod \( m \) is given by \( \chi(u) = \chi_{l_1}(u)\chi_{l_2}(u) \cdots \chi_{l_r}(u) \).

Let \( \mathcal{F} = \{ \chi : \chi = \chi_{l_1}\chi_{l_2} \cdots \chi_{l_r} \} \).

\[
\begin{align*}
  &\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}} \sum_{\chi \in \mathcal{F}} [\chi(p) + \overline{\chi}(p)] \\
  &\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( 2 \frac{\log p}{\log(m/\pi)} \right) p^{-1} \sum_{\chi \in \mathcal{F}} [\chi^2(p) + \overline{\chi}^2(p)]
\end{align*}
\]
Characters Sums:

\[
\sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \begin{cases} 
  m_i - 1 - 1 & p \equiv 1(m_i) \\
  -1 & \text{otherwise}
\end{cases}
\]

Define

\[
\delta_{m_i}(p, 1) = \begin{cases} 
  1 & p \equiv 1(m_i) \\
  0 & \text{otherwise}
\end{cases}
\]

Then

\[
\sum_{\chi \in \mathcal{F}} \chi(p) = \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}(p) \cdots \chi_{l_r}(p)
\]

\[
= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}(p)
\]

\[
= \prod_{i=1}^{r} \left( -1 + (m_i - 1)\delta_{m_i}(p, 1) \right).
\]
Expansion Preliminaries:

$k(s)$ is an $s$-tuple $(k_1, k_2, \ldots, k_s)$ with $k_1 < k_2 < \cdots < k_s$.

This is just a subset of $(1, 2, \ldots, r)$, $2^r$ possible choices for $k(s)$.

$$\delta_{k(s)}(p, 1) = \prod_{i=1}^{s} \delta_{m_{k_i}}(p, 1).$$

If $s = 0$ we define $\delta_{k(0)}(p, 1) = 1 \ \forall p$.

Then

$$\prod_{i=1}^{r} \left( -1 + (m_i - 1)\delta_{m_i}(p, 1) \right)$$

$$= \sum_{s=0}^{r} \sum_{k(s)} (-1)^{r-s} \delta_{k(s)}(p, 1) \prod_{i=1}^{s} (m_{k_i} - 1)$$

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First Sum:

$$\ll \sum_{p} \frac{m^{\sigma}}{p^{\frac{1}{2}}} \frac{1}{M_{2}} \left(1 + \sum_{s=1}^{r} \sum_{k(s)} \delta_{k(s)}(p, 1) \prod_{i=1}^{s} (m_{k_{i}} - 1)\right).$$

As $m/M_{2} \leq 3^{r}$, $s = 0$ sum contributes

$$S_{1,0} = \frac{1}{M_{2}} \sum_{p} \frac{m^{\sigma}}{p^{\frac{1}{2}}} \ll 3^{r} m^{\frac{1}{2} \sigma - 1},$$

hence negligible for $\sigma < 2$. Now we study

$$S_{1,k(s)} = \frac{1}{M_{2}} \prod_{i=1}^{s} (m_{k_{i}} - 1) \sum_{p} \frac{m^{\sigma}}{p^{\frac{1}{2}}} \delta_{k(s)}(p, 1)$$

$$\ll \frac{1}{M_{2}} \prod_{i=1}^{s} (m_{k_{i}} - 1) \sum_{n=1(m_{k(s)})}^{\infty} n^{-\frac{1}{2}}$$

$$\ll \frac{1}{M_{2}} \prod_{i=1}^{s} (m_{k_{i}} - 1) \frac{1}{\prod_{i=1}^{s} (m_{k_{i}})} \sum_{n} n^{-\frac{1}{2}}$$

$$\ll 3^{r} m^{\frac{1}{2} \sigma - 1}.$$
First Sum (cont):

There are $2^r$ choices, yielding

$$S_1 \ll 6^r m^{\frac{1}{2\sigma}-1},$$

which is negligible as $m$ goes to infinity for fixed $r$ if $\sigma < 2$.

Cannot let $r$ go to infinity.

If $m$ is the product of the first $r$ primes,

$$\log m = \sum_{k=1}^{r} \log p_k = \sum_{p \leq r} \log p \approx r$$

Therefore

$$6^r \approx m^{\log 6} \approx m^{1.79}. $$
Second Sum Expansions:

\[
\sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \begin{cases} 
  m_i - 1 - 1 & p \equiv \pm 1(m_i) \\
  -1 & \text{otherwise}
\end{cases}
\]

\[
\sum_{\chi \in \mathcal{F}} \chi^2(p) = \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}(p) \cdots \chi_{l_r}(p)
\]

\[
= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p)
\]

\[
= \prod_{i=1}^{r} \left( -1 + (m_i - 1)\delta_{m_i}(p, 1) + (m_i - 1)\delta_{m_i}(p, -1) \right)
\]
**Second Sum Bounds:**

Handle similarly as before. Say

\[
p \equiv 1 \mod m_{k_1}, \ldots, m_{k_a}
\]

\[
p \equiv -1 \mod m_{k_{a+1}}, \ldots, m_{k_b}
\]

How small can \( p \) be?

+1 congruences imply \( p \geq m_{k_1} \cdots m_{k_a} + 1 \).

−1 congruences imply \( p \geq m_{k_{a+1}} \cdots m_{k_b} - 1 \).

Since the product of these two lower bounds is greater than \( \prod_{i=1}^{b} (m_{k_i} - 1) \), at least one must be greater than \( \left( \prod_{i=1}^{b} (m_{k_i} - 1) \right)^{\frac{1}{2}} \).

There are \( 3^r \) pairs, yielding

\[
\text{Second Sum} = \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} S_{2,k(s),j(s)} \ll 9^r m^{-\frac{1}{2}}.
\]
Summary:

Agrees with Unitary for $\sigma < 2$.

We proved:

Lemma:

- $m$ square-free odd integer with $r = r(m)$ factors;
- $m = \prod_{i=1}^{r} m_i$;
- $M_2 = \prod_{i=1}^{r} (m_i - 2)$.

Consider the family $\mathcal{F}_m$ of primitive characters mod $m$. Then

$$\text{First Sum} \ll \frac{1}{M_2} 2^r m_2^{1\sigma}$$
$$\text{Second Sum} \ll \frac{1}{M_2} 3^r m_2^{\frac{1}{2}}.$$
Dirichlet Characters:
\( m \in [N, 2N] \) Square-free

\( \mathcal{F}_N \) all primitive characters with conductor odd square-free integer in \([N, 2N]\).

At least \( N/\log^2 N \) primes in the interval.

At least \( N \cdot \frac{N}{\log^2 N} = N^2 \log^{-2} N \) primitive characters:

\[
M \geq N^2 \log^{-2} N \implies \frac{1}{M} \leq \frac{\log^2 N}{N^2}.
\]
Bounds

\[ S_{1,m} \ll \frac{1}{M} 2^{r(m)} m^{\frac{1}{2}\sigma} \]
\[ S_{2,m} \ll \frac{1}{M} 3^{r(m)} m^{\frac{1}{2}}. \]

\( 2^{r(m)} = \tau(m) \), the number of divisors of \( m \), and \( 3^{r(m)} \leq \tau^2(m) \).

While it is possible to prove

\[ \sum_{n \leq x} \tau^l(n) \ll x (\log x)^{2^l - 1} \]

the crude bound

\[ \tau(n) \leq c(\epsilon)n^\epsilon \]

yields the same region of convergence.
First Sum Bound

\[ S_1 = \sum_{m=N \text{ square free}}^{2N} S_{1,m} \]

\[ \ll \sum_{m=N}^{2N} \frac{1}{M} 2^{r(m)} m^{\frac{1}{2}\sigma} \]

\[ \ll \frac{1}{M} N^{\frac{1}{2}\sigma} \sum_{m=N}^{2N} \tau(m) \]

\[ \ll \frac{1}{M} N^{\frac{1}{2}\sigma} c(\epsilon) N^{1+\epsilon} \]

\[ \ll \frac{\log^2 N}{N^2} N^{\frac{1}{2}\sigma} c(\epsilon) N^{1+\epsilon} \]

\[ \ll c(\epsilon) N^{\frac{1}{2}\sigma+\epsilon-1} \log^2 N. \]

No contribution if \( \sigma < 2 \).

Second sum handled similarly.
Summary

• Similar behavior in different systems.

• Find correct scale.

• Average over similar elements.

• Need an Explicit Formula.

• Different statistics tell different stories.