Random Matrix Theory Models for zeros of L-functions near the central point (and applications to elliptic curves)

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Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium (200+ protons / neutrons) even worse!

Info by shooting high-energy neutrons into nucleus.

Fundamental Equation: Quantum Mechanics

\[ H\psi_n = E_n\psi_n \]

Similar to stat mech, leads to considering eigenvalues of ensembles of matrices.

Real Symmetric (GOE), Complex Hermitian (GUE), Classical Compact Groups.
Random Matrix Ensembles

\[
A = \begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\
a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{1N} & a_{2N} & a_{3N} & \cdots & a_N
\end{pmatrix} = A^T
\]

Let \( p(x) \) be a probability density.

\[
p(x) \geq 0 \\
\int_{\mathbb{R}} p(x) \, dx = 1.
\]

Often assume \( p(x) \) has finite moments:

\[
k^{th} \text{-moment} = \int_{\mathbb{R}} x^k p(x) \, dx.
\]

Define

\[
\text{Prob}(A) = \prod_{1 \leq i < j \leq N} p(a_{ij}).
\]
Eigenvalue Distribution

Key to Averaging:

\[ \text{Trace}(A^k) = \sum_{i=1}^{N} \lambda_i^k(A). \]

By the Central Limit Theorem:

\[ \text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}a_{ji} \]
\[ = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \]
\[ \sim N^2 \cdot 1 \]
\[ \sum_{i=1}^{N} \lambda_i^2(A) \sim N^2 \]

Gives \( N \text{Ave}(\lambda_i^2(A)) \sim N^2 \) or \( \lambda_i(A) \sim \sqrt{N}. \)
Eigenvalue Distribution (cont)

\( \delta(x - x_0) \) is a unit point mass at \( x_0 \).

To each \( A \), attach a probability measure:

\[
\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)
\]

Obtain:

\[
k^{th}\text{-moment} = \int x^k \mu_{A,N}(x) \, dx
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \frac{\lambda_i^k(A)}{(2\sqrt{N})^k}
\]

\[
= \frac{\text{Trace}(A^k)}{2^k N^{k+1}}
\]
Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from fixed $p(x)$.

**Semi-Circle Law:** Assume $p$ has mean 0, variance 1, other moments finite. Then

$$\mu_{A,N}(x) \to \frac{2}{\pi} \sqrt{1 - x^2} \text{ with probability } 1$$

Trace formula converts sums over eigenvalues to sums over entries of $A$.

Expected value of $k^{th}$-moment of $\mu_{A,N}(x)$ is

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \frac{\text{Trace}(A^k)}{2^k N^{k/2 + 1}} \prod_{i \leq j} p(a_{ij}) \, da_{ij}$$
Proof: 2nd-Moment

\[
\text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2.
\]

Substituting into expansion gives

\[
\frac{1}{2^2 N^2} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \sum_{i,j=1}^{N} a_{ji}^2 \cdot p(a_{11}) da_{11} \cdots p(a_{NN}) da_{NN}
\]

Integration factors as

\[
\int_{a_{ij} \in \mathbb{R}} a_{ij}^2 \cdot p(a_{ij}) da_{ij} \cdot \prod_{(k,l) \neq (ij), k<l} \int_{a_{kl} \in \mathbb{R}} p(a_{kl}) da_{kl} = 1.
\]

Have \(N^2\) summands, answer is \(\frac{1}{4}\).

**Key:** Averaging Formula, Trace Lemma.
Measures of Spacings: $n$-Level Correlations

$\{\alpha_j\}$ increasing sequence, $B \subset \mathbb{R}^{n-1}$ a compact box. Define the $n$-level correlation by

$$\lim_{N \to \infty} \frac{\# \left\{ \left( \alpha_{j_1} - \alpha_{j_2}, \ldots, \alpha_{j_{n-1}} - \alpha_{j_n} \right) \in B \mid j_i \neq j_k \leq N \right\}}{N}$$

Observations and Results:

1. Normalized spacings of $\zeta(s)$ starting at $10^{20}$ (Odlyzko)

2. Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)

3. $n$-level correlations for all automorphic cuspidal $L$-functions (Rudnick-Sarnak)

4. $n$-level correlations for the classical compact groups (Katz-Sarnak)

5. insensitive to any finite set of zeros
Measures of Spacings: $n$-Level Density and Families

$\phi(x) = \prod_i \phi_i(x_i)$, $\phi_i$ even Schwartz functions, $\hat{\phi}_i$ compactly supported.

$$D_{n,f}(\phi) = \sum_{j_1, \ldots, j_n \text{ distinct}} \phi_1 \left( L_f \gamma_f^{(j_1)} \right) \cdots \phi_n \left( L_f \gamma_f^{(j_n)} \right)$$

$L_f =$ Conductor, the scale factor for low zeros.

1. individual zeros contribute in limit
2. most of contribution is from low zeros
3. average over similar curves (family)

$$D_{n,\mathcal{F}}(\phi) = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} D_{n,f}(\phi).$$
Limiting Behavior

As $N \rightarrow \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{j_1, \ldots, j_n} \prod_{i} \phi_i \left( \frac{\gamma_f^{(j_i)} \log L_f}{2\pi} \right)$$

$$\rightarrow \int \cdots \int \phi(x) W_{n,G(\mathcal{F})}(x) dx$$

$$\rightarrow \int \cdots \int \hat{\phi}(y) \hat{W}_{n,G(\mathcal{F})}(y) dy.$$ 

**Conj**: Distribution of Low Zeros agrees with a classical compact group.
Correspondences

Similarities b/w Nuclear Physics and $L$-Functions:

\[ \text{Zeros} \leftrightarrow \text{Energy Levels} \]
\[ \text{Support} \leftrightarrow \text{Neutron Energy}. \]

**Conjecture:** Zeros near central point in a **family** of $L$-functions behave like eigenvalues near 1 of a classical compact group (Unitary, Symplectic, Orthogonal).
Some Number Theory Results

- **Orthogonal:**
  - Iwaniec-Luo-Sarnak: 1-level density for $H_k^\pm(N)$, $N$ square-free;
  - Hughes-Miller: $n$-level density for $H_k^\pm(N)$, $N$ square-free;
  - Dueñez-Miller: 1, 2-level $\{\phi \times \text{sym}^2 f : f \in H_k(1)\}$, $\phi$ even Maass;
  - Miller: 1, 2-level for one-parameter families of elliptic curves.

- **Symplectic:**
  - Rubinstein: $n$-level densities for $L(s, \chi_d)$;
  - Dueñez-Miller: 1-level for $\{\phi \times f : f \in H_k(1)\}$, $\phi$ even Maass.

- **Unitary:**
  - Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.
Main Tools

- Explicit Formula: Relates sums over zeros to sums over primes.

- Averaging Formulas: Orthogonality of characters, Petersson formula.

- Control of conductors: Monotone.
1-Level Densities

Fourier Transforms for 1-level densities:

\[
\begin{align*}
\hat{W}_{1,SO(\text{even})}(u) &= \delta(u) + \frac{1}{2} \eta(u) \\
\hat{W}_{1,O}(u) &= \delta(u) + \frac{1}{2} \\
\hat{W}_{1,SO(\text{odd})}(u) &= \delta(u) - \frac{1}{2} \eta(u) + 1 \\
\hat{W}_{1,Sp}(u) &= \delta(u) - \frac{1}{2} \eta(u) \\
\hat{W}_{1,U}(u) &= \delta(u)
\end{align*}
\]

where \(\delta(u)\) is the Dirac Delta functional and

\[
\eta(u) = \begin{cases} 
1 & \text{if } |u| < 1 \\
\frac{1}{2} & \text{if } |u| = 1 \\
0 & \text{if } |u| > 1 
\end{cases}
\]
Dirichlet Characters: 
\( m \) Prime

\((\mathbb{Z}/m\mathbb{Z})^*\) is cyclic of order \( m - 1 \) with generator \( g \).

Let \( \zeta_{m-1} = e^{2\pi i/(m-1)} \).

The principal character \( \chi_0 \) is given by

\[
\chi_0(k) = \begin{cases} 
1 & (k, m) = 1 \\
0 & (k, m) > 1.
\end{cases}
\]

The \( m - 2 \) primitive characters are determined (by multiplicativity) by action on \( g \).

As each \( \chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^* \), for each \( \chi \) there exists an \( l \) such that \( \chi(g) = \zeta_{m-1}^l \). Hence for each \( l, 1 \leq l \leq m - 2 \) we have

\[
\chi_l(k) = \begin{cases} 
\zeta_{m-1}^{la} & k \equiv g^a(m) \\
0 & (k, m) > 0
\end{cases}
\]
Dirichlet $L$-Functions

Let $\chi$ be a primitive character mod $m$. Let

$$c(m, \chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi ik/m}.$$  

c($m, \chi$) is a Gauss sum of modulus $\sqrt{m}$.

\[ L(s, \chi) = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1} \]

\[ \Lambda(s, \chi) = \pi^{-\frac{1}{2}(s+\epsilon)}\Gamma\left(\frac{s + \epsilon}{2}\right) m^{\frac{1}{2}(s+\epsilon)} L(s, \chi), \]

where

\[ \epsilon = \begin{cases} 
0 & \text{if } \chi(-1) = 1 \\
1 & \text{if } \chi(-1) = -1
\end{cases} \]

\[ \Lambda(s, \chi) = (-i)^\epsilon \frac{c(m, \chi)}{\sqrt{m}} \Lambda(1 - s, \bar{\chi}). \]
Explicit Formula

Let $\phi$ be an even Schwartz function with compact support $(-\sigma, \sigma)$.

Let $\chi$ be a non-trivial primitive Dirichlet character of conductor $m$.

\[
\sum \phi \left( \gamma \frac{\log(m/\pi)}{2\pi} \right) = \int_{-\infty}^{\infty} \phi(y) dy - \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) \left[ \chi(p) + \bar{\chi}(p) \right] p^{-\frac{1}{2}} - \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( 2 \frac{\log p}{\log(m/\pi)} \right) \left[ \chi^2(p) + \bar{\chi}^2(p) \right] p^{-1} + O\left( \frac{1}{\log m} \right).
\]
Expansion

\{\chi_0\} \cup \{\chi_l\}_{l \leq m-2} \text{ are all the characters mod } m.

Consider the family of primitive characters mod a prime \(m\) (\(m - 2\) characters):

\[
\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{\gamma} \phi \left( \gamma \chi \frac{\log(m)}{2\pi} \right)
\]

\[
= \int_{-\infty}^{\infty} \phi(y) dy
\]

\[
- \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}}
\]

\[
- \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( 2 \frac{\log p}{\log(m/\pi)} \right) [\chi^2(p) + \bar{\chi}^2(p)] p^{-1}
\]

\[+ O\left( \frac{1}{\log m} \right).\]

Can pass Character Sum through Test Function.
Character Sums

\[ \sum_{\chi} \chi(k) = \begin{cases} 
    m - 1 & k \equiv 1(m) \\
    0 & \text{otherwise}
\end{cases} \]

For any prime \( p \neq m \)

\[ \sum_{\chi \neq \chi_0} \chi(p) = \begin{cases} 
    m - 1 - 1 & p \equiv 1(m) \\
    -1 & \text{otherwise}
\end{cases} \]

Substitute into

\[ \frac{1}{m - 2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) \frac{\chi(p) + \bar{\chi}(p)}{\sqrt{p}} \]
First Sum

\[
\frac{-2}{m-2} \sum_{p} \log p \cdot \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}} + 2 \frac{m-1}{m-2} \sum_{p \equiv 1(m)} \log p \cdot \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) p^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m} \sum_{p} p^{-\frac{1}{2}} + \sum_{p \equiv 1(m)} p^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m} \sum_{k} k^{-\frac{1}{2}} + \sum_{k \equiv 1(m)} k^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m} \sum_{k} k^{-\frac{1}{2}} + \frac{1}{m} \sum_{k \geq m+1} k^{-\frac{1}{2}}
\]

\[
\ll \frac{1}{m^{\sigma/2}}.
\]

No contribution if \( \sigma < 2 \).
Second Sum

\[
\frac{1}{m - 2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p) + \bar{\chi}^2(p)}{p}.
\]

\[
\sum_{\chi \neq \chi_0} [\chi^2(p) + \bar{\chi}^2(p)] = \begin{cases} 
2(m - 2) & p \equiv \pm 1(m) \\
-2 & p \not\equiv \pm 1(m)
\end{cases}
\]

Up to \(O\left(\frac{1}{\log m}\right)\) we find that

\[
\ll \frac{1}{m - 2} \sum_p p^{-1} + \frac{2m - 2}{m - 2} \sum_{p \equiv \pm 1(m)} p^{-1}
\]

\[
\ll \frac{1}{m - 2} \sum_k k^{-1} + \sum_{k \equiv 1(m)} k^{-1} + \sum_{k \equiv -1(m)} k^{-1}
\]

\[
\ll \frac{\log(m^{\sigma/2})}{m - 2} + \frac{1}{m} \sum_{k \geq m+1} k^{-1} + \frac{1}{m} \sum_{k \geq m+1} k^{-1} + O\left(\frac{1}{m}\right)
\]

\[
\ll \frac{\log m}{m} + \frac{\log m}{m} + \frac{\log m}{m}.
\]
Elliptic Curves

Conductors grow rapidly.

Results for small support, where Orthogonal densities indistinguishable.

Study 1 and 2-Level Densities.

\[ D_{n,f}(\phi) = \sum_{j_1,\ldots,j_n \text{ distinct}} \phi_1 \left( L_f \gamma_f^{(j_1)} \right) \cdots \phi_n \left( L_f \gamma_f^{(j_n)} \right) \]

\[ D_{n,\mathcal{F}}(\phi) = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} D_{n,f}(\phi). \]
2-Level Densities

\[ c(\mathcal{G}) = \begin{cases} 
0 & \text{if } \mathcal{G} = \text{SO(even)} \\
\frac{1}{2} & \text{if } \mathcal{G} = \text{O} \\
1 & \text{if } \mathcal{G} = \text{SO(odd)}
\end{cases} \]

For \( \mathcal{G} = \text{SO(even)}, \text{O or SO(odd)} \):

\[
\int \int \widehat{\phi}_1(u_1)\widehat{\phi}_2(u_2)\widehat{W}_{2,\mathcal{G}}(u)du_1du_2 = \left[ \widehat{\phi}_1(0) + \frac{1}{2}\phi_1(0) \right] \left[ \widehat{f}_2(0) + \frac{1}{2}\phi_2(0) \right] \\
+ 2 \int |u|\widehat{\phi}_1(u)\widehat{\phi}_2(u)du \\
- 2\widehat{\phi}_1\phi_2(0) - \phi_1(0)\phi_2(0) \\
+ c(\mathcal{G})\phi_1(0)\phi_2(0).
\]
Elliptic Curves

\[ E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \quad a_i \in \mathbb{Q} \]

Often can write \( E : y^2 = x^3 + Ax + B \).

Let \( N_p \) be the number of solns mod \( p \):

\[
N_p = \sum_{x(p)} \left[ 1 + \left( \frac{x^3 + Ax + B}{p} \right) \right] = p + \sum_{x(p)} \left( \frac{x^3 + Ax + B}{p} \right)
\]

Local data: \( a_E(p) = p - N_p \). Use to build the \( L \)-function:

\[
a_E(p) = -\sum_{x \mod p} \left( \frac{x^3 + Ax + B}{p} \right)
\]
Elliptic Curves: Arithmetic Progression

One-parameter families:

\[ E_t : y^2 = x^3 + A(t)x + B(t), \quad A(t), B(t) \in \mathbb{Z}(t). \]

We have

\[ a_t(p) = - \sum_{x \mod p} \left( \frac{x^3 + A(t)x + B(t)}{p} \right) = a_{t+mp}(p) \]

Can handle sums of \( a_t(p) \) for \( t \) in arithmetic progression.
Elliptic Curves (cont)

\[ L(E, s) = \sum_{n=1}^{\infty} \frac{a_{E}(n)}{n^s} = \prod_p L_p(E, s). \]

By GRH: All zeros on the critical line.

Rational solutions: \( E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T. \)

**Birch and Swinnerton-Dyer Conjecture:**
Geometric rank \( r \) equals analytic rank (order of vanishing at central point).
Comments on Previous Results

• **explicit formula** relating zeros and Fourier coeffs;
• **averaging formulas** for the family;
• **conductors easy to control** (constant or monotone)

Elliptic curve $E_t$: discriminant $\Delta(t)$, conductor $N_{E_t} = C(t)$ is

$$C'(t) = \prod_{p|\Delta(t)} p^{f_p(t)}$$
Normalization of Zeros

Local (hard) vs Global (easy). As $N \to \infty$:

$$\frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} D_{n,E}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} \sum_{j_1, \ldots, j_n} \prod_i \phi_i \left( \frac{\log N_E}{2\pi} \gamma_E^{(j_i)} \right)$$

$$\to \int \cdots \int \phi(x) W_{n,G(\mathcal{F})}(x) dx$$

$$\to \int \cdots \int \hat{\phi}(y) \hat{W}_{n,G(\mathcal{F})}(y) dy.$$ 

**Conj:** Distribution of Low Zeros agrees with Orthogonal Densities.
1-Level Expansion

\[ D_{1,F}(\phi) = \frac{1}{|F|} \sum_{E \in F} \sum_j \phi \left( \frac{\log N_E \gamma_E^{(j)}}{2\pi} \right) \]

\[ = \frac{1}{|F|} \sum_{E \in F} \tilde{\phi}(0) + \phi_i(0) \]

\[ - 2 \frac{1}{|F|} \sum_{E \in F} \sum_p \frac{\log p}{\log N_E p^{2/\phi}} \left( \frac{\log p}{\log N_E} \right) a_E(p) \]

\[ - 2 \frac{1}{|F|} \sum_{E \in F} \sum_p \frac{\log p}{\log N_E p^{2/\phi}} \left( 2 \frac{\log p}{\log N_E} \right) a_E^2(p) \]

\[ + O \left( \frac{\log \log N_E}{\log N_E} \right) \]

Want to move \( \frac{1}{|F|} \sum_{E \in F} \), leads us to study

\[ A_{r,F}(p) = \sum_{t \mod p} a_t^r(p), \quad r = 1 \text{ or } 2. \]
2-Level Expansion

Need to evaluate terms like

\[
\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \prod_{i=1}^{2} \frac{1}{p_i^{r_i}} g_i \left( \frac{\log p_i}{\log N_E} \right) a_{E}^{r_i}(p_i).
\]

Analogue of Petersson / Orthogonality: If \( p_1, \ldots, p_n \) are distinct primes

\[
\sum_{t \mod p_1 \cdots p_n} a_{t}^{r_1}(p_1) \cdots a_{t}^{r_n}(p_n) = A_{r_1,\mathcal{F}}(p_1) \cdots A_{r_n,\mathcal{F}}(p_n).
\]
For many families

\begin{align*}
(1) : A_1, F(p) &= -rp + O(1) \\
(2) : A_2, F(p) &= p^2 + O(p^{3/2})
\end{align*}

Rational Elliptic Surfaces (Rosen and Silverman): If rank $r$ over $\mathbb{Q}(t)$:

\[
\lim_{X \to \infty} \frac{1}{X} \sum_{p \leq X} - \frac{A_1, F(p) \log p}{p} = r
\]

Surfaces with $j(t)$ non-constant (Michel):

\[A_2, F(p) = p^2 + O\left( p^{3/2} \right).\]
DEFINITIONS

\[ D_{n,F}(\phi) = \frac{1}{|F|} \sum_{E \in F} \sum_{j_1, \ldots, j_n} \prod_i \phi_i \left( \frac{\log NE}{2\pi} \gamma_E^{(j_i)} \right) \]

\( D^{(r)}_{n,F}(\phi) \): \( n \)-level density with contribution of \( r \) zeros at central point removed.

\( F_N \): Rational one-parameter family, \( t \in [N, 2N] \), conductors monotone.
ASSUMPTIONS

1-parameter family of Ell Curves, rank \( r \) over \( \mathbb{Q}(t) \), rational surface.
Assume

- GRH;
- \( j(t) \) non-constant;
- Sq-Free Sieve if \( \Delta(t) \) has irr poly factor of \( \deg \geq 4 \).

Pass to positive percent sub-seq where conductors polynomial of degree \( m \).

\( \phi_i \) even Schwartz, support \( \sigma_i \):

- \( \sigma_1 < \min \left( \frac{1}{2}, \frac{2}{3m} \right) \) for 1-level
- \( \sigma_1 + \sigma_2 < \frac{1}{3m} \) for 2-level.
MAIN RESULT

Theorem (M–): Under previous conditions, as $N \to \infty$, $n = 1, 2$:

$$D_{n,F_N}^{(r)}(\phi) \longrightarrow \int \phi(x) W_G(x) \, dx,$$

where

$$G = \begin{cases} 
O & \text{if half odd} \\
\text{SO(even)} & \text{if all even} \\
\text{SO(odd)} & \text{if all odd}
\end{cases}$$

1 and 2-level densities confirm Katz-Sarnak, B-SD predictions for small support.
Examples

Constant-Sign Families:

1. \( y^2 = x^3 + 2^4(-3)^3(9t + 1)^2, \)
   \( 9t + 1 \) Square-Free: all even.

2. \( y^2 = x^3 \pm 4(4t + 2)x, \)
   \( 4t + 2 \) Square-Free: + all odd, − all even.

3. \( y^2 = x^3 + tx^2 - (t + 3)x + 1, \)
   \( t^2 + 3t + 9 \) Square-Free: all odd.

First two rank 0 over \( \mathbb{Q}(t) \), third is rank 1.

Without 2-Level Density, couldn’t say which orthogonal group.
Examples (cont)

Rational Surface of Rank 6 over $\mathbb{Q}(t)$:

$$y^2 = x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x + (2ct - D)(t^2 + 2t - A + 1)^2$$

$$A = 8, 916, 100, 448, 256, 000, 000$$
$$B = -811, 365, 140, 824, 616, 222, 208$$
$$C = 26, 497, 490, 347, 321, 493, 520, 384$$
$$D = -343, 107, 594, 345, 448, 813, 363, 200$$
$$a = 16, 660, 111, 104$$
$$b = -1, 603, 174, 809, 600$$
$$c = 2, 149, 908, 480, 000$$

Need GRH, Sq-Free Sieve to handle sieving.
Sketch of Proof

1. Sieving (Arithmetic Progressions)

2. Partial Summation (Complete Sums)

3. Controlling Conductors (Monotone).
Sieving

\[
\sum_{t=N}^{2N} S(t) = \sum_{d=1}^{N^{k/2}} \mu(d) \sum_{\substack{D(t) \equiv 0(d^2) \\t \in [N,2N]}} S(t)
\]

\[
= \sum_{d=1}^{\log^l N} \mu(d) \sum_{\substack{D(t) \equiv 0(d^2) \\t \in [N,2N]}} S(t) + \sum_{d \geq \log^l N} \mu(d) \sum_{\substack{D(t) \equiv 0(d^2) \\t \in [N,2N]}} S(t).
\]

Handle first by progressions.

Handle second by Cauchy-Schwartz:
The number of \( t \) in the second sum (by Sq-Free Sieve Conj) is \( o(N) \):

37


Sieving (cont)

\[ \log^l N \sum_{d=1}^{\sum \mu(d) \sum_{D(t) \equiv 0 (d^2) \atop t \in [N, 2N]} S(t)} \]

\( t_i(d) \) roots of \( D(t) \equiv 0 \mod d^2 \).

\( t_i(d), t_i(d) + d^2, \ldots, t_i(d) + \left[ \frac{N}{d^2} \right] d^2 \).

If \( (d, p_1 p_2) = 1 \), go through complete set of residue classes \( \frac{N/d^2}{p_1 p_2} \) times.
Partial Summation

\[ \tilde{a}_{d,i,p}(t') = a_{t(d,i,t')}(p), \ G_{d,i,P}(u) \text{ is related to the test functions, } d \text{ and } i \text{ from progressions.} \]

Applying Partial Summation

\[
S(d, i, r, p) = \sum_{t'=0}^{[N/d^2]} \tilde{a}_{d,i,p}^r(t')G_{d,i,p}(t')
\]

\[
= \left( \frac{[N/d^2]}{p} A_{r,F}(p) + O\left(p^R\right) \right) G_{d,i,p}([N/d^2])
\]

\[
- \sum_{u=0}^{[N/d^2]-1} \left( \frac{u}{p} A_{r,F}(p) + O\left(p^R\right) \right) \left( G_{d,i,p}(u) - G_{d,i,p}(u+1) \right)
\]

39
Difficult Piece: Fourth Sum I

\[
\sum_{u=0}^{[N/d^2]-1} O(P^R) \left( \frac{G_{d,i,P}(u) - G_{d,i,P}(u+1)}{d^2 Pr \frac{1}{\log N}} \right)
\]

Taylor \( G_{d,i,P}(u) - G_{d,i,P}(u+1) \) gives \( P^R N \frac{1}{d^2 Pr \frac{1}{\log N}} \).

\[
\frac{1}{|\mathcal{F}|} \sum_{i,d} \text{ gives } O\left(\frac{P^R}{Pr \log N}\right).
\]

Problem is in summing over the primes, as we no longer have \( \frac{1}{|\mathcal{F}|} \).
Fourth Sum: II

If exactly one of the $r_j$’s is non-zero, then

$$\sum_{u=0}^{[N/d^2]-1} \left| G_{d,i,P}(u) - G_{d,i,P}(u + 1) \right|$$

$$= \sum_{u=0}^{[N/d^2]-1} \left| g\left(\frac{\log p}{\log C(t_i(d) + ud^2)}\right) - g\left(\frac{\log p}{\log C(t_i(d) + (u + 1)d^2)}\right) \right|$$

If conductors monotone, for fixed $i$, $d$ and $p$, small independent of $N$ (bounded variation).

If two of the $r_j$’s are non-zero:

$$|a_1a_2 - b_1b_2| = |a_1a_2 - b_1a_2 + b_1a_2 - b_1b_2|$$

$$\leq |a_1a_2 - b_1a_2| + |b_1a_2 - b_1b_2|$$

$$= |a_2| \cdot |a_1 - b_1| + |b_1| \cdot |a_2 - b_2|$$
Handling the Conductors: I

\[ y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t) \]

\[ C(t) = \prod_{p|\Delta(t)} p^{f_p(t)} \]

\[ D_1(t) = \text{primitive irred poly factors of } \Delta(t), c_4(t) \text{ share} \]

\[ D_2(t) = \text{remaining primitive irred poly factors of } \Delta(t) \]

\[ D(t) = D_1(t)D_2(t) \]

\[ D(t) \text{ sq-free, } C(t) \text{ like } D_1^2(t)D_2(t) \text{ except for a finite set of bad primes.} \]
Handling the Conductors: II

\[ y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t) \]

Let \( P \) be the product of the bad primes.

Tate’s Algorithm gives \( f_p(t) \), depend only on \( a_i(t) \) mod powers of \( p \).

Apply Tate’s Algorithm to \( E_{t_1} \). Get \( f_p(t_1) \) for \( p|P \). For \( m \) large, \( p|P \),

\[ f_p(\tau) = f_p(P^mt + t_1) = f_p(t_1), \]

and order of \( p \) dividing \( D(P^mt + t_1) \) is independent of \( t \).

Get integers st \( C'(\tau) = c_{bad} \frac{D_2^2(\tau)}{c_1} \frac{D_2(\tau)}{c_2} \), \( D(\tau) \) sq-free.
Excess Rank

One-parameter family, rank $r$ over $\mathbb{Q}(t)$, RMT $\implies$ 50% rank $r$, $r+1$.

For many families, observe

- Percent with rank $r$ = 32%
- Percent with rank $r+1$ = 48%
- Percent with rank $r+2$ = 18%
- Percent with rank $r+3$ = 2%

Problem: small data sets, sub-families, convergence rate $\log(\text{conductor})$?
Data on Excess Rank

\[ y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 \]

Family: \( a_1 : 0 \text{ to } 10, \text{ rest } -10 \text{ to } 10. \)

- Percent with rank 0 = 28.60%
- Percent with rank 1 = 47.56%
- Percent with rank 2 = 20.97%
- Percent with rank 3 = 2.79%
- Percent with rank 4 = 0.08%

14 Hours, 2,139,291 curves (2,971 singular, 248,478 distinct).
Data on Excess Rank

\[ y^2 + y = x^3 + tx \]

Each data set 2000 curves from start.

<table>
<thead>
<tr>
<th>t-Start</th>
<th>Rk 0</th>
<th>Rk 1</th>
<th>Rk 2</th>
<th>Rk 3</th>
<th>Time (hrs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1000</td>
<td>39.4</td>
<td>47.8</td>
<td>12.3</td>
<td>0.6</td>
<td>&lt;1</td>
</tr>
<tr>
<td>1000</td>
<td>38.4</td>
<td>47.3</td>
<td>13.6</td>
<td>0.6</td>
<td>&lt;1</td>
</tr>
<tr>
<td>4000</td>
<td>37.4</td>
<td>47.8</td>
<td>13.7</td>
<td>1.1</td>
<td>1</td>
</tr>
<tr>
<td>8000</td>
<td>37.3</td>
<td>48.8</td>
<td>12.9</td>
<td>1.0</td>
<td>2.5</td>
</tr>
<tr>
<td>24000</td>
<td>35.1</td>
<td>50.1</td>
<td>13.9</td>
<td>0.8</td>
<td>6.8</td>
</tr>
<tr>
<td>50000</td>
<td>36.7</td>
<td>48.3</td>
<td>13.8</td>
<td>1.2</td>
<td>51.8</td>
</tr>
</tbody>
</table>

Last set has conductors of size \(10^{11}\), but on logarithmic scale still small.
Excess Rank Calculations
Families with $y^2 = f_t(x); \ D(t) \ \text{SqFree}$

<table>
<thead>
<tr>
<th>Family</th>
<th>$t$ Range</th>
<th>Num $t$</th>
<th>$r$</th>
<th>$r + 1$</th>
<th>$r + 2$</th>
<th>$r + 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+4(4t + 2)$</td>
<td>[2, 2002]</td>
<td>1622</td>
<td>0</td>
<td>95.44</td>
<td>4.56</td>
<td></td>
</tr>
<tr>
<td>$-4(4t + 2)$</td>
<td>[2, 2002]</td>
<td>1622</td>
<td>0</td>
<td>70.53</td>
<td>29.35</td>
<td></td>
</tr>
<tr>
<td>$9t + 1$</td>
<td>[2, 247]</td>
<td>169</td>
<td>0</td>
<td>71.01</td>
<td>28.99</td>
<td></td>
</tr>
<tr>
<td>$t^2 + 9t + 1$</td>
<td>[2, 272]</td>
<td>169</td>
<td>1</td>
<td>71.60</td>
<td>27.81</td>
<td></td>
</tr>
<tr>
<td>$t(t - 1)$</td>
<td>[2, 2002]</td>
<td>643</td>
<td>0</td>
<td>40.44</td>
<td>10.26</td>
<td>0.62</td>
</tr>
<tr>
<td>$(6t + 1)x^2$</td>
<td>[2, 101]</td>
<td>93</td>
<td>1</td>
<td>34.41</td>
<td>17.20</td>
<td>1.08</td>
</tr>
<tr>
<td>$(6t + 1)x$</td>
<td>[2, 77]</td>
<td>66</td>
<td>2</td>
<td>30.30</td>
<td>16.67</td>
<td>3.03</td>
</tr>
</tbody>
</table>

1. $x^3 + 4(4t + 2)x, 4t + 2$ Sq-Free, odd.
2. $x^3 - 4(4t + 2)x, 4t + 2$ Sq-Free, even.
3. $x^3 + 2^4(-3)^3(9t + 1)^2, 9t + 1$ Sq-Free, even.
4. $x^3 + tx^2 - (t + 3)x + 1, t^2 + 3t + 9$ Sq-Free, odd.
5. $x^3 + (t + 1)x^2 + tx, t(t - 1)$ Sq-Free, rank 0.
6. $x^3 + (6t + 1)x^2 + 1, 4(6t + 1)^3 + 27$ Sq-Free, rank 1.
7. $x^3 - (6t + 1)^2x + (6t + 1)^2, (6t + 1)[4(6t + 1)^2 - 27]$ Sq-Free, rank 2.
Excess Rank Calculations
Families with $y^2 = f_t(x)$; All $D(t)$

<table>
<thead>
<tr>
<th>Family</th>
<th>$t$ Range</th>
<th>Num</th>
<th>$t$ Range</th>
<th>$r$</th>
<th>$r + 1$</th>
<th>$r + 2$</th>
<th>$r + 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+4(4t + 2)$</td>
<td>[2, 2002]</td>
<td>2001</td>
<td></td>
<td>0</td>
<td>6.45</td>
<td>85.76</td>
<td>3.95</td>
</tr>
<tr>
<td>$-4(4t + 2)$</td>
<td>[2, 2002]</td>
<td>2001</td>
<td></td>
<td>0</td>
<td>63.52</td>
<td>9.90</td>
<td>25.99</td>
</tr>
<tr>
<td>$9t + 1$</td>
<td>[2, 247]</td>
<td>247</td>
<td></td>
<td>0</td>
<td>55.28</td>
<td>23.98</td>
<td>20.73</td>
</tr>
<tr>
<td>$t^2 + 9t + 1$</td>
<td>[2, 272]</td>
<td>271</td>
<td>1</td>
<td>73.80</td>
<td></td>
<td></td>
<td>25.83</td>
</tr>
<tr>
<td>$t(t - 1)$</td>
<td>[2, 2002]</td>
<td>2001</td>
<td></td>
<td>0</td>
<td>42.03</td>
<td>48.43</td>
<td>9.25</td>
</tr>
<tr>
<td>$(6t + 1)x^2$</td>
<td>[2, 101]</td>
<td>100</td>
<td>1</td>
<td>32.00</td>
<td>50.00</td>
<td>17.00</td>
<td>1.00</td>
</tr>
<tr>
<td>$(6t + 1)x$</td>
<td>[2, 77]</td>
<td>76</td>
<td>2</td>
<td>32.89</td>
<td>50.00</td>
<td>14.47</td>
<td>2.63</td>
</tr>
</tbody>
</table>

1. $x^3 + 4(4t + 2)x, 4t + 2$ Sq-Free, odd.
2. $x^3 - 4(4t + 2)x, 4t + 2$ Sq-Free, even.
3. $x^3 + 2^4(-3)^3(9t + 1)^2, 9t + 1$ Sq-Free, even.
4. $x^3 + tx^2 - (t + 3)x + 1, t^2 + 3t + 9$ Sq-Free, odd.
5. $x^3 + (t + 1)x^2 + tx, t(t - 1)$ Sq-Free, rank 0.
6. $x^3 + (6t + 1)x^2 + 1, 4(6t + 1)^3 + 27$ Sq-Free, rank 1.
7. $x^3 - (6t + 1)^2x + (6t + 1)^2, (6t + 1)[4(6t + 1)^2 - 27]$ Sq-Free, rank 2.
Orthogonal Random Matrix Model

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0, \pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k} (\cos \theta_k - \cos \theta_j)^2 \prod_j d\theta_j$$

Model: forced zeros independent (suggested by Function Field analogue)

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} g & \cr I_{2r} & \end{pmatrix} : g \in SO(2N - 2r) \right\}$$
Orthogonal Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0, \pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k} (\cos \theta_k - \cos \theta_j)^2 \prod_j d\theta_j$$

Interaction Model: NOT SUGGESTED BY FUNCTION FIELD

Sub-ensemble of $SO(2N)$ with the last $2n$ of the $2N$ eigenvalues equal $+1$:

$$d\epsilon_{2n}(\theta) \propto \prod_{j<k} (\cos \theta_k - \cos \theta_j)^2 \prod_j (1 - \cos \theta_j)^{2n} \prod_j d\theta_j,$$

with $1 \leq j, k \leq N - n$.

Independent Model: SUGGESTED BY FUNCTION FIELD

$$A_{2N,2n} = \left\{ \begin{pmatrix} g & \cr & I_{2n} \end{pmatrix} : g \in SO(2N - 2n) \right\}$$
Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2} \eta(u).$$

Fourier transform of 1-level density (Rank 2, Independent):

$$\hat{\rho}_{2, \text{Ind}}(u) = \left[ \delta(u) + \frac{1}{2} \eta(u) + 2 \right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2, \text{Int}}(u) = \left[ \delta(u) + \frac{1}{2} \eta(u) + 2 \right] + 2(|u| - 1) \eta(u).$$
Testing RMT Model

For small support, 1-level densities for Elliptic Curves agree with \( \rho_{r,\text{Indep}} \).

Curve \( E \), conductor \( N_E \), expect first zero \( \frac{1}{2} + i\gamma_E^{(1)} \) with \( \gamma_E^{(1)} \approx \frac{1}{\log N_E} \).

If \( r \) zeros at central point, if repulsion of zeros is of size \( \frac{c_r}{\log N_E} \), might detect in 1-level density:

\[
\frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} \sum_{j} \phi \left( \frac{\gamma_E^{(j)} \log N_E}{2\pi} \right).
\]

Corrections of size

\[
\phi(x_0 + c_r) - \phi(x_0) \approx \phi'(x(x_0, c_r)) \cdot c_r.
\]
Theoretical Distribution of First Normalized Zero

First normalized eigenvalue: 230,400 from $\text{SO}(6)$ with Haar Measure

First normalized eigenvalue: 322,560 from $\text{SO}(7)$ with Haar Measure
Rank 0 Curves: 1st Normalized Zero
(Far left and right bins just for formatting)

750 curves, $\log(\text{cond}) \in [3.2, 12.6]$; mean = 1.04

750 curves, $\log(\text{cond}) \in [12.6, 14.9]$; mean = .88
Rank 2 Curves: 1st Normalized Zero

665 curves, log(cond) ∈ [10, 10.3125]; mean = 2.30

665 curves, log(cond) ∈ [16, 16.5]; mean = 1.82
Rank 2 Curves: $[0, 0, 0, -t^2, t^2]$ 1st Normalized Zero

35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, mean = 2.24

34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, mean = 2.00
Summary

• Similar behavior in different systems.

• Find correct scale.

• Average over similar elements.

• Need an Explicit Formula.

• Different statistics tell different stories.

• Evidence for B-SD, RMT interpretation of zeros

• Need more data.
Appendices

The first two appendices list various standard conjectures. The second provides (at least conjecturally) when a family should have equidistribution of signs of functional equations. Experimental evidence is provided in the third appendix, which is on the distribution of signs of elliptic curves in a one-parameter family. Testing whether or not a generic family is equidistributed in sign. We looked at 1000 consecutive elliptic curves, and calculated the excess of positive over negative. We did this many times, and created a histogram plot. The fluctuations look Gaussian! The third appendix gives the formula to numerically approximate the analytic rank of an elliptic curve. For a curve of conductor $N_E$, one needs about $\sqrt{N_E} \log N_E$ Fourier coefficients. The fourth appendix gives some estimates on bounding the number of curves in a family with given rank.
Appendix I: Standard Conjectures

**Generalized Riemann Hypothesis (for Elliptic Curves)** Let \( L(s, E) \) be the (normalized) \( L \)-function of the elliptic curve \( E \). Then the non-trivial zeros of \( L(s, E) \) satisfy \( \text{Re}(s) = \frac{1}{2} \).

**Birch and Swinnerton-Dyer Conjecture** [BSD1], [BSD2] Let \( E \) be an elliptic curve of geometric rank \( r \) over \( \mathbb{Q} \) (the Mordell-Weil group is \( \mathbb{Z}^r \oplus T \), \( T \) is the subset of torsion points). Then the analytic rank (the order of vanishing of the \( L \)-function at the central point) is also \( r \).

**Tate’s Conjecture for Elliptic Surfaces** [Ta] Let \( \mathcal{E}/\mathbb{Q} \) be an elliptic surface and \( L_2(\mathcal{E}, s) \) be the \( L \)-series attached to \( H^2_{\text{ét}}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l) \). Then \( L_2(\mathcal{E}, s) \) has a meromorphic continuation to \( \mathbb{C} \) and satisfies \( -\text{ord}_{s=2} L_2(\mathcal{E}, s) = \text{rank} \ NS(\mathcal{E}/\mathbb{Q}) \), where \( NS(\mathcal{E}/\mathbb{Q}) \) is the \( \mathbb{Q} \)-rational part of the Néron-Severi group of \( \mathcal{E} \). Further, \( L_2(\mathcal{E}, s) \) does not vanish on the line \( \text{Re}(s) = 2 \).

Most of the 1-param families we investigate are rational surfaces, where Tate’s conjecture is known. See [RSi].
Appendix II: Equidistribution of Signs

**ABC Conjecture** Fix $\epsilon > 0$. For co-prime positive integers $a$, $b$ and $c$ with $c = a + b$ and $N(a, b, c) = \prod_{p|abc} p$, $c \ll_{\epsilon} N(a, b, c)^{1+\epsilon}$.

The full strength of ABC is never needed; rather, we need a consequence of ABC, the Square-Free Sieve (see [Gr]):

**Square-Free Sieve Conjecture** Fix an irreducible polynomial $f(t)$ of degree at least 4. As $N \to \infty$, the number of $t \in [N, 2N]$ with $f(t)$ divisible by $p^2$ for some $p > \log N$ is $o(N)$.

For irreducible polynomials of degree at most 3, the above is known, complete with a better error than $o(N)$ ([Ho], chapter 4).

**Restricted Sign Conjecture (for the Family $\mathcal{F}$)** Consider a one-parameter family $\mathcal{F}$ of elliptic curves. As $N \to \infty$, the signs of the curves $E_t$ are equidistributed for $t \in [N, 2N]$.

The Restricted Sign conjecture often fails. First, there are families with constant $j(E_t)$ where all curves have the same sign. Helfgott [He] has recently related the Restricted Sign conjecture to the Square-Free Sieve conjecture and standard conjectures on sums of Moebius:

**Polynomial Moebius** Let $f(t)$ be a non-constant polynomial such that no fixed square divides $f(t)$ for all $t$. Then $\sum_{t=N}^{2N} \mu(f(t)) = o(N)$.

The Polynomial Moebius conjecture is known for linear $f(t)$. Helfgott shows the Square-Free Sieve and Polynomial Moebius imply the Restricted Sign conjecture for many families. More precisely, let $M(t)$ be the product of the irreducible polynomials dividing $\Delta(t)$ and not $c_4(t)$.

**Theorem: Equidistribution of Sign in a Family [He]**: Let $\mathcal{F}$ be a one-parameter family with $a_i(t) \in \mathbb{Z}[t]$. If $j(E_t)$ and $M(t)$ are non-constant, then the signs of $E_t$, $t \in [N, 2N]$, are equidistributed as $N \to \infty$. Further, if we restrict to good $t$, $t \in [N, 2N]$ such that $D(t)$ is good (usually square-free), the signs are still equidistributed in the limit.
Distribution of Signs: \( y^2 = x^3 + (t + 1)x^2 + tx \)

Histogram plot: \( D(t) \) sq-free, first \( 2 \cdot 10^6 \) such \( t \).

Histogram plot: All \( t \in [2, 2 \cdot 10^6] \).
Distribution of signs: \( y^2 = x^3 + (t+1)x^2 + tx \)

The observed behavior agrees with the predicted behavior. Note as the number of curves increase (comparing the plot of \(5 \cdot 10^7\) points to \(2 \cdot 10^6\) points), the fit to the Gaussian improves.

Graphs by Atul Pokharel
Appendix III: Numerically Approximating Ranks: Preliminaries

Cusp form \( f \), level \( N \), weight 2:

\[

dfrac{f(-1/Nz)}{Nz^2} = -\epsilon N z^2 f(z), \\
\dfrac{f(i/y\sqrt{N})}{y^2} = \epsilon y^2 f(iy/\sqrt{N}).
\]

Define

\[
L(f, s) = (2\pi)^s \Gamma(s)^{-1} \int_0^{\infty} (-iz)^s f(z) \frac{dz}{z}, \\
\Lambda(f, s) = (2\pi)^{-s} N^{s/2} \Gamma(s) L(f, s) = \int_0^{\infty} f(iy/\sqrt{N}) y^{s-1} dy.
\]

Get

\[
\Lambda(f, s) = \epsilon \Lambda(f, 2 - s), \quad \epsilon = \pm 1.
\]

To each \( E \) corresponds an \( f \), write \( \int_0^\infty = \int_0^1 + \int_1^\infty \) and use transformations.
Algorithm for $L^r(s, E)$: I

\[ \Lambda(E, s) = \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy \]
\[ = \int_0^1 f(iy/\sqrt{N})y^{s-1}dy + \int_1^\infty f(iy/\sqrt{N})y^{s-1}dy \]
\[ = \int_1^\infty f(iy/\sqrt{N})(y^{s-1} + \epsilon y^{1-s})dy. \]

Differentiate $k$ times with respect to $s$:

\[ \Lambda^{(k)}(E, s) = \int_1^\infty f(iy/\sqrt{N})(\log y)^k(y^{s-1} + \epsilon(-1)^ky^{1-s})dy. \]

At $s = 1$,

\[ \Lambda^{(k)}(E, 1) = (1 + \epsilon(-1)^k) \int_1^\infty f(iy/\sqrt{N})(\log y)^kdy. \]

Trivially zero for half of $k$; let $r$ be analytic rank.
Algorithm for $L^r(s, E)$: II

\[
\Lambda^{(r)}(E, 1) = 2 \int_{1}^{\infty} f(iy/\sqrt{N})(\log y)^r \, dy \\
= 2 \sum_{n=1}^{\infty} a_n \int_{1}^{\infty} e^{-2\pi ny/\sqrt{N}} (\log y)^r \, dy.
\]

Integrating by parts

\[
\Lambda^{(r)}(E, 1) = \frac{\sqrt{N}}{\pi} \sum_{n=1}^{\infty} a_n \frac{n}{n} \int_{1}^{\infty} \frac{e^{-2\pi ny/\sqrt{N}} (\log y)^{r-1} \, dy}{y}.
\]

We obtain

\[
L^{(r)}(E, 1) = 2r! \sum_{n=1}^{\infty} \frac{a_n}{n} G_r \left( \frac{2\pi n}{\sqrt{N}} \right),
\]

where

\[
G_r(x) = \frac{1}{(r - 1)!} \int_{1}^{\infty} e^{-xy} (\log y)^{r-1} \, dy.
\]
Expansion of $G_r(x)$

$$G_r(x) = P_r \left( \log \frac{1}{x} \right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-r}}{n^r \cdot n!} x^n$$

$P_r(t)$ is a polynomial of degree $r$, $P_r(t) = Q_r(t - \gamma)$.

$$Q_1(t) = t;$$
$$Q_2(t) = \frac{1}{2} t^2 + \frac{\pi^2}{12};$$
$$Q_3(t) = \frac{1}{6} t^3 + \frac{\pi^2}{12} t - \frac{\zeta(3)}{3};$$
$$Q_4(t) = \frac{1}{24} t^4 + \frac{\pi^2}{24} t^2 - \frac{\zeta(3)}{3} t + \frac{\pi^4}{160};$$
$$Q_5(t) = \frac{1}{120} t^5 + \frac{\pi^2}{72} t^3 - \frac{\zeta(3)}{6} t^2 + \frac{\pi^4}{160} t - \frac{\zeta(5)}{5} - \frac{\zeta(3) \pi^2}{36}.$$ 

For $r = 0$,

$$\Lambda(E, 1) = \frac{\sqrt{N}}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-2\pi ny/\sqrt{N}}.$$ 

Need about $\sqrt{N}$ or $\sqrt{N} \log N$ terms.
Appendix IV: Bounding Excess Rank

\[ D_{1,\mathcal{F}}(\phi_1) = \hat{\phi}_1(0) + \frac{1}{2} \phi_1(0) + r\phi_1(0). \]

To estimate the percent with rank at least \( r + R, P_R, \) we get

\[ R\phi_1(0)P_R \leq \hat{\phi}_1(0) + \frac{1}{2} \phi_1(0), \quad R > 1. \]

Note the family rank \( r \) has been cancelled from both sides.

The 2-level density gives \textit{squares} of the rank on the left, get a cross term \( rR. \)

The disadvantage is our support is smaller.

Once \( R \) is large, the 2-level density yields better results. We now give more details.
For $n = 1$ and 2, consider the test functions

$$\hat{f}_i(u) = \frac{1}{2} \left( \frac{1}{2} \sigma_n - \frac{1}{2} |u| \right), \quad |u| \leq \sigma$$

$$f_i(x) = \frac{\sin^2(2\pi \frac{1}{2} \sigma_n x)}{(2\pi x)^2}.$$

Expect $\sigma_2 = \frac{\sigma_1}{2}$; only able to prove for $\sigma_2 = \frac{\sigma_1}{4}$.

Note $f_i(0) = \frac{\sigma_n^2}{4}$, $\hat{f}_i(0) = f_i(0) \frac{1}{\sigma_n}$.

Assume B-SD, Equidistribution of Sign
Notation

Family with rank $r$, $D_{1,F}(f) = \hat{f}(0) + \frac{1}{2} f(0) + rf(0)$.

By even (odd) we mean a curve whose rank $r_E$ has $r_E - r$ even (odd).

$P_0$: probability even curve has rank $\geq r + 2a_0$.

$P_1$: probability odd curve has rank $\geq r + 1 + 2b_0$.

$$D_{1,F}(f) = \frac{1}{|F|} \sum_{E \in F} \sum_{\gamma_E} f \left( \frac{\log N_E}{2\pi} \gamma_E \right),$$

$\gamma_E$ is the imaginary part of the zeros.
**Average Rank: 1-Level Bounds**

\[
\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} r_E f(0) \leq \widehat{f}_1(0) + \frac{1}{2} f_1(0) + r f_1(0)
\]

\[
\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} r_E \leq \frac{1}{\sigma_1} + \frac{1}{2} + r.
\]

- **All Curves:** \( r = 0, \sigma = \frac{4}{7}, \) giving 2.25 (Brumer, Heath-Brown: [Br], [BHB3], [BHB5])

- **1-Parameter Families:** \( \left( \text{deg}(N(t)) + r + \frac{1}{2} \right) \cdot (1 + o(1)) \) (Silverman [Si3]).

Hope 1-Level Density true for \( \sigma \to \infty \).

Would yield average rank is \( r + \frac{1}{2} \).
**Excess Rank: 1-Level Bounds**

Assume half even, half odd.

Even curves: $1 - P_0$ have rank $\leq r + 2a_0 - 2$; replace ranks with $r$. $P_0$ have rank $\geq r + 2a_0$; replace with $r + 2a_0$.

Odd curves: $1 - P_1$ contributing $r + 1$. $P_1$ contributing $r + 1 + 2b_0$.

$$
\frac{1}{\sigma_1} + \frac{1}{2} + r \geq \frac{1}{2} \left[ (1 - P_0) r + P_0 (r + 2a_0) \right] \\
\quad + \frac{1}{2} \left[ (1 - P_1) (r + 1) + P_1 (r + 1 + 2b_0) \right]
$$

$$
\frac{1}{\sigma_1} \geq a_0 P_0 + b_0 P_1.
$$

1-Level Density Bounds for Excess Rank

$$
P_0 \leq \frac{1}{a_0 \sigma_1},
$$

$$
P_1 \leq \frac{1}{b_0 \sigma_1},
$$

$$
\text{Prob}\{\text{rank} \geq r + 2a_0\} \leq \frac{1}{a_0 \sigma_1}.
$$
2-Level Bounds:

\[ D_{2,\mathcal{F}}(f) = D^*_{2,\mathcal{F}}(f) - 2D_{1,\mathcal{F}}(f_1 f_2) + f_1(0) f_2(0) N(\mathcal{F}, -1) \]

\[ D^*_{2,\mathcal{F}}(f) = \prod_{i=1}^{2} \left[ \hat{f}_i(0) + \frac{1}{2} f_i(0) \right] + 2 \int |u| \hat{f}_1(u) \hat{f}_2(u) du \]

\[ + r \hat{f}_1(0) f_2(0) + r f_1(0) \hat{f}_2(0) + (r^2 + r) f_1(0) f_2(0) \]

\[ D_{1,\mathcal{F}}(f) = \hat{f}(0) + \frac{1}{2} f(0) + r f(0). \]

\( D^*_{2,\mathcal{F}}(f) \) is over all zeros. Gives

\[
\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} r_E^2 \leq \frac{1}{\sigma_2^2} + \frac{1}{\sigma_2} + \frac{1}{4} + \frac{1}{3} + \frac{2r}{\sigma_2} + r^2 + r
\]

\[
= \frac{1}{\sigma_2^2} + \frac{2r + 1}{\sigma_2} + \frac{1}{12} + r^2 + r + \frac{1}{2}.
\]
**Excess Rank: 2-Level Bounds:**

Similar proof yields

**Theorem: First 2-Level Density Bounds**

\[
P_0 \leq \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r + \frac{1}{2}}{\sigma_2}}{a_0(a_0 + r)}
\]

\[
P_1 \leq \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r + \frac{1}{2}}{\sigma_2}}{b_0(b_0 + r + 1)}.
\]

For \(\sigma_2 = \frac{\sigma_1}{4}\), \(r = 0\), \(a_1 = 1\): **worse** than 1-level density.

For fixed \(\sigma_2 = \frac{\sigma_1}{4}\) and \(r\), as we increase \(a_0\) we eventually do get a better bound.

Proportional to \(\frac{1}{(a_0\sigma_1)^2}\) instead of \(\frac{1}{a_0\sigma_1}\).
Excess Rank: 2-Level Bounds: II

Use \( D_{2,\mathcal{F}}(f) \) instead of \( D^*_{2,\mathcal{F}}(f) \).

\[ r_E = \text{number of zeros of curve } E. \text{ Sum over } j_1 \neq j_2. \]

\( r_E \) even, get \( r_E(r_E - 2) \) (each zero matched with \( r_E - 2 \) others).

\( r_E \) odd: \( (r_E - 1)(r_E - 2) + (r_E - 1) = r_E(r_E - 2) + 1. \)

Theorem: Second 2-Level Density Bounds

\[
P_0 \leq \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r}{\sigma_2} - \frac{1}{6\sigma_2}}{a_0(a_0 + r - 1)}
\]

\[
P_1 \leq \frac{\frac{1}{2\sigma_2^2} + \frac{1}{24} + \frac{r}{\sigma_2} - \frac{1}{6\sigma_2}}{b_0(b_0 + r)}
\]

where \( a_0 \neq 1 \) if \( r = 0. \)

\( \sigma_2 = \frac{\sigma_1}{4} \) and \( r = 0, \) better for \( a_0 > \frac{\sigma_1^2 + 8\sigma_1 + 192}{24\sigma_1}. \)

\( r = 1, \) better for \( a_0 > \frac{\sigma_1^2 + 80\sigma_1 + 192}{24\sigma_1}. \)

Decay is proportional to \( \frac{1}{(a_0\sigma_1)^2}. \)

Note the numerator is never negative; at least \( \frac{1}{18}. \)
**Excess Rank: 2-Level Bounds: IIIa**

\[ r_E = r + z_E. \]

\[ \sum_{j_1} \sum_{j_2} f_1(L\gamma_{Ej_1})f_2(L\gamma_{Ej_2}). \] Let \( j_1 \) be one of the \( r \) family zeros, varying \( j_2 \) gives \( f_1(0)D_{1,E}(f_2). \) Interchanging \( j_1 \) and \( j_2 \) we get a contribution of \( D_{1,E}(f_1)f_2(0) \) for each of the \( r \) family.

Only double counting when \( j_1 \) and \( j_2 \) are both a family zero. Subtract off \( r^2f_1(0)f_2(0). \) For the other \( z_E \) zeros: already taken into account contribution from \( j_1 \) one of the \( z_E \) zeros and \( j_2 \) one of the \( r \) family zeros (and vice-versa).

Thus, for a given curve, a lower bound of the contribution from all pairs \((j_1, j_2)\) is

\[ rf_1(0)D_{1,E}(f_2) + rD_{1,E}(f_1)f_2(0) - r^2f_1(0)f_2(0) + z_E^2. \]
Excess Rank: 2-Level Bounds: IIIb

Summing over all $E \in \mathcal{F}$ and simplifying gives

$$\frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} z_E^2 \leq \frac{1}{\sigma_2^2} + \frac{1}{\sigma_1^2} + \frac{1}{12} + \frac{1}{2}.$$ 

Similar calculation gives

**Theorem: Third 2-Level Density Bounds**

$$P_0 \leq \frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_1^2} + \frac{1}{24} a_0^2$$

$$P_1 \leq \frac{1}{2\sigma_2^2} + \frac{1}{2\sigma_1^2} + \frac{1}{24} b_0 + b_0^2$$

$\sigma_2 = \frac{\sigma_1}{4}$: beats 1-level for $a_0 > \frac{\sigma_1^2 + 48\sigma_1 + 192}{24\sigma_1}$.

$r \neq 0$: beats first 2-level once $a_0 > \frac{\sigma_1^2 + 48\sigma_1 + 192}{96\sigma_1}$.

$r \geq 1$: beats second 2-level once $a_0 > \frac{3(r-1)\sigma_1^2 + 48\sigma_1 + 192}{3r-2 \cdot 96\sigma_1}$.
Heath-Brown & Brumer

Family of all elliptic curves $E_{a,b}$:

$$\mathcal{F}_T = \{ y^2 = x^3 + ax + b; |a| \leq T^{\frac{3}{4}}, |b| \leq T^{\frac{3}{2}} \}.$$  

From 1-Level Expansion, get

$$r(E_{a,b}) \leq 2 + \frac{\log T}{\log X} - 2 \sum_{p \leq X} a_p(E_{a,b}) \frac{h}{h} \left( \frac{\log p}{\log X} \right) + O \left( \frac{1}{\log X} \right).$$

If $r(E_{a,b}) \geq r \geq 3 + 2 \frac{\log T}{\log X}$, then $|U(E_{a,b}, X)| \geq \frac{\log T}{2}.$

Led to

$$\# \{ E_{a,b} \in \mathcal{F}_T : r(E_{a,b}) \geq r \} \cdot \left( \frac{\log T}{2} \right)^{2k} \leq \sum_{E_{a,b} \in \mathcal{F}} |U(E_{a,b}, X)|^{2k}.$$ 

Find $X = T^{\frac{1}{10k}}, k = \left[ \frac{r-3}{20} \right]$. Yields

$$\text{Prob}(\text{rank}(E_{a,b}) \geq r) \ll (11r)^{-\frac{r}{20}}$$

$$\text{rank}(E_{a,b}) \leq 17 \frac{\log T}{\log \log T}.$$
Appendix V: Dirichlet Characters:  
*m* Square-free

Fix an *r* and let *m_1*, \ldots, *m_r* be distinct odd primes.

\[ m = m_1 m_2 \cdots m_r \]

\[ M_1 = (m_1 - 1)(m_2 - 1) \cdots (m_r - 1) = \phi(m) \]

\[ M_2 = (m_1 - 2)(m_2 - 2) \cdots (m_r - 2). \]

*M_2* is the number of primitive characters mod *m*, each of conductor *m*.

A general primitive character mod *m* is given by

\[ \chi(u) = \chi_{l_1}(u)\chi_{l_2}(u) \cdots \chi_{l_r}(u). \]

Let \( \mathcal{F} = \{ \chi : \chi = \chi_{l_1}\chi_{l_2} \cdots \chi_{l_r} \} \).

\[
\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} \sum_{\chi \in \mathcal{F}} [\chi(p) + \bar{\chi}(p)]
\]

\[
\frac{1}{M_2} \sum_p \frac{\log p}{\log(m/\pi)} \hat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) p^{-1} \sum_{\chi \in \mathcal{F}} [\chi^2(p) + \bar{\chi}^2(p)]
\]
Characters Sums:

\[ \sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \begin{cases} m_i - 1 - 1 & p \equiv 1(m_i) \\ -1 & \text{otherwise} \end{cases} \]

Define

\[ \delta_{m_i}(p, 1) = \begin{cases} 1 & p \equiv 1(m_i) \\ 0 & \text{otherwise} \end{cases} \]

Then

\[ \sum_{\chi \in \mathcal{F}} \chi(p) = \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}(p) \cdots \chi_{l_r}(p) \]

\[ = \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) \]

\[ = \prod_{i=1}^{r} \left( -1 + (m_i - 1) \delta_{m_i}(p, 1) \right). \]
Expansion Preliminaries:

$k(s)$ is an $s$-tuple $(k_1, k_2, \ldots, k_s)$ with $k_1 < k_2 < \cdots < k_s$.

This is just a subset of $(1, 2, \ldots, r)$, $2^r$ possible choices for $k(s)$.

$$\delta_{k(s)}(p, 1) = \prod_{i=1}^{s} \delta_{m_{k_i}}(p, 1).$$

If $s = 0$ we define $\delta_{k(0)}(p, 1) = 1 \forall p$.

Then

$$\prod_{i=1}^{r} \left( -1 + (m_i - 1)\delta_{m_i}(p, 1) \right) = \sum_{s=0}^{r} \sum_{k(s)} (-1)^{r-s} \delta_{k(s)}(p, 1) \prod_{i=1}^{s} (m_{k_i} - 1)$$
First Sum:

\[ \ll \sum_{p} p^{-\frac{1}{2}} \frac{1}{M_2} \left( 1 + \sum_{s=1}^{r} \sum_{k(s)} \delta_{k(s)}(p, 1) \prod_{i=1}^{s} (m_{ki} - 1) \right). \]

As \( m/M_2 \leq 3^r \), \( s = 0 \) sum contributes

\[ S_{1,0} = \frac{1}{M_2} \sum_{p} p^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2} \sigma - 1}, \]

hence negligible for \( \sigma < 2 \). Now we study

\[ S_{1,k(s)} = \frac{1}{M_2} \prod_{i=1}^{s} (m_{ki} - 1) \sum_{p} p^{-\frac{1}{2}} \delta_{k(s)}(p, 1) \]

\[ \ll \frac{1}{M_2} \prod_{i=1}^{s} (m_{ki} - 1) \sum_{n \equiv 1(m_{k(s)})} n^{-\frac{1}{2}} \]

\[ \ll \frac{1}{M_2} \prod_{i=1}^{s} (m_{ki} - 1) \frac{1}{\prod_{i=1}^{s} (m_{ki})} \sum_{n} n^{-\frac{1}{2}} \]

\[ \ll 3^r m^{\frac{1}{2} \sigma - 1}. \]
First Sum (cont):

There are $2^r$ choices, yielding

$$S_1 \ll 6^r m^{1/2\sigma - 1},$$

which is negligible as $m$ goes to infinity for fixed $r$ if $\sigma < 2$.

Cannot let $r$ go to infinity.

If $m$ is the product of the first $r$ primes,

$$\log m = \sum_{k=1}^{r} \log p_k$$

$$= \sum_{p \leq r} \log p \approx r$$

Therefore

$$6^r \approx m^{\log 6} \approx m^{1.79}.$$
Second Sum Expansions:

\[
\sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) = \begin{cases} 
  m_i - 1 - 1 & p \equiv \pm 1(m_i) \\
  -1 & \text{otherwise}
\end{cases}
\]

\[
\sum_{\chi \in \mathcal{F}} \chi^2(p) = \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}^2(p) \cdots \chi_{l_r}^2(p) \\
= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) \\
= \prod_{i=1}^{r} \left( -1 + (m_i - 1)\delta_{m_i}(p, 1) + (m_i - 1)\delta_{m_i}(p, -1) \right)
\]
Second Sum Bounds:

Handle similarly as before. Say

\[ p \equiv 1 \mod m_{k_1}, \ldots, m_{k_a} \]
\[ p \equiv -1 \mod m_{k_{a+1}}, \ldots, m_{k_b} \]

How small can \( p \) be?

+1 congruences imply \( p \geq m_{k_1} \cdots m_{k_a} + 1 \).

−1 congruences imply \( p \geq m_{k_{a+1}} \cdots m_{k_b} - 1 \).

Since the product of these two lower bounds is greater than \( \prod_{i=1}^{b}(m_{k_i} - 1) \), at least one must be greater than \( \left( \prod_{i=1}^{b}(m_{k_i} - 1) \right)^{\frac{1}{2}} \).

There are \( 3^r \) pairs, yielding

\[ \text{Second Sum} = \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} S_{2,k(s),j(s)} \ll 9^r m^{-\frac{1}{2}}. \]
Summary:

Agrees with Unitary for $\sigma < 2$.

We proved:

Lemma:

• $m$ square-free odd integer with $r = r(m)$ factors;
• $m = \prod_{i=1}^{r} m_i$;
• $M_2 = \prod_{i=1}^{r} (m_i - 2)$.

Consider the family $\mathcal{F}_m$ of primitive characters mod $m$. Then

First Sum $\ll \frac{1}{M_2^2} 2^r m^{\frac{1}{2} \sigma}$

Second Sum $\ll \frac{1}{M_2} 3^r m^{\frac{1}{2}}$. 

85
Dirichlet Characters:

\[ m \in [N, 2N] \textbf{Square-free} \]

\[ \mathcal{F}_N \text{ all primitive characters with conductor odd square-free integer in } [N, 2N]. \]

At least \( \frac{N}{\log^2 N} \) primes in the interval.

At least \( N \cdot \frac{N}{\log^2 N} = N^2 \log^{-2} N \) primitive characters:

\[ M \geq N^2 \log^{-2} N \Rightarrow \frac{1}{M} \leq \frac{\log^2 N}{N^2}. \]
Bounds

\[ S_{1,m} \ll \frac{1}{M} 2^{r(m)} m^{\frac{1}{2} \sigma} \]

\[ S_{2,m} \ll \frac{1}{M} 3^{r(m)} m^{\frac{1}{2}}. \]

\[ 2^{r(m)} = \tau(m), \text{ the number of divisors of } m, \text{ and } \]

\[ 3^{r(m)} \leq \tau^2(m). \]

While it is possible to prove

\[ \sum_{n \leq x} \tau^l(n) \ll x (\log x)^{2^l - 1} \]

the crude bound

\[ \tau(n) \leq c(\epsilon) n^\epsilon \]

yields the same region of convergence.
First Sum Bound

\[ S_1 = \sum_{m=1}^{2N} S_{1,m} \]

\[ \ll \sum_{m=1}^{2N} \frac{1}{M} 2^{r(m)} m^{\frac{1}{2} \sigma} \]

\[ \ll \frac{1}{M} N^{\frac{1}{2} \sigma} \sum_{m=1}^{2N} \tau(m) \]

\[ \ll \frac{1}{M} N^{\frac{1}{2} \sigma} c(\epsilon) N^{1+\epsilon} \]

\[ \ll \frac{\log^2 N}{N^2} N^{\frac{1}{2} \sigma} c(\epsilon) N^{1+\epsilon} \]

\[ \ll c(\epsilon) N^{\frac{1}{2} \sigma + \epsilon - 1} \log^2 N. \]

No contribution if \( \sigma < 2 \).

Second sum handled similarly.
Bibliography


