Expected Moments of the Spectral Measure for 2-Level Bipartite Graph Adjacency Matrices

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Motivation

- Introduced by Wigner to study spacings between energy levels of heavy nuclei

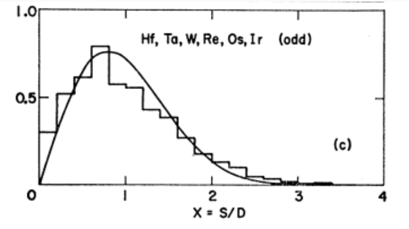

Figure 1: Energy levels of heavy nuclei

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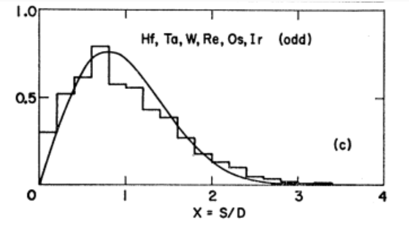

Figure 1: Energy levels of heavy nuclei

- Random matrices can be applied to a variety of spectra: zeroes of the Riemann Zeta function, heavy nuclei, bus arrivals.

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Real, Symmetric Matrices

- Wigner showed that the eigenvalues of a **real, symmetric matrix** i.i.d.r.v entries converges to a semicircular distribution.

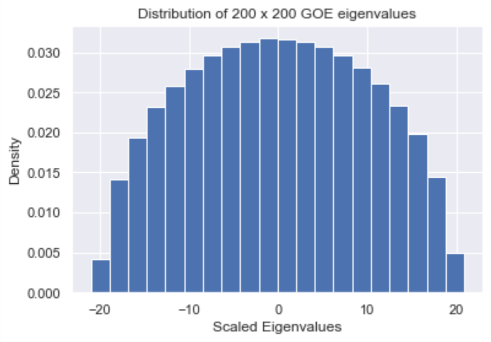

Figure 2: Distribution of eigenvalues

The McKay Ensemble

- Let A be the adjacency matrix of a random d-regular graph on n vertices.

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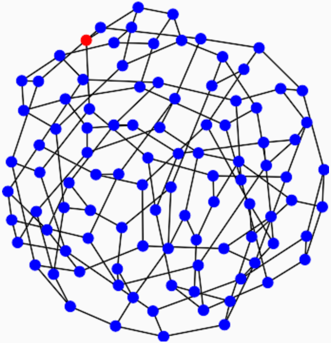

Figure 3: Random 3-regular graph on 100 vertices.

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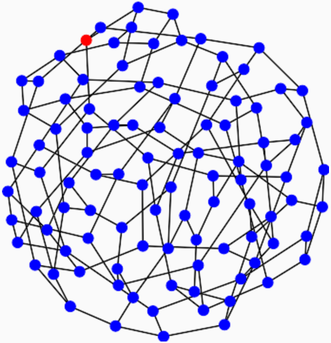

Figure 3: Random 3-regular graph on 100 vertices.

- Aside from trivial eigenvalue d, remaining eigenvalues admit a limiting distribution as $n \to \infty$, the Kesten-McKay distribution.

The McKay Law

- McKay: excluding trivial eigenvalue d, the empirical spectral distribution converges to

$$f(x) := \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} \quad \text{for } |x| \le 2\sqrt{d-1}.$$

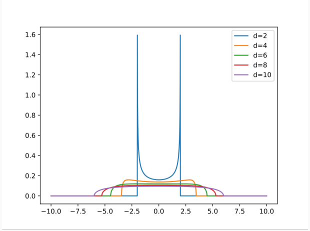

Figure 4: Kesten McKay Law

- Regular-graph analogue of semicircle law. Can create a new random matrix ensemble, similar to the ensemble of checkerboard matrices.

Our Set Up

- McKay gives us the limiting moments of the d-regular graphs. $(m_0 = 1, m_1 = 0, m_2 = d, m_3 = 0, m_4 = 2d^2 - d, \ldots)$

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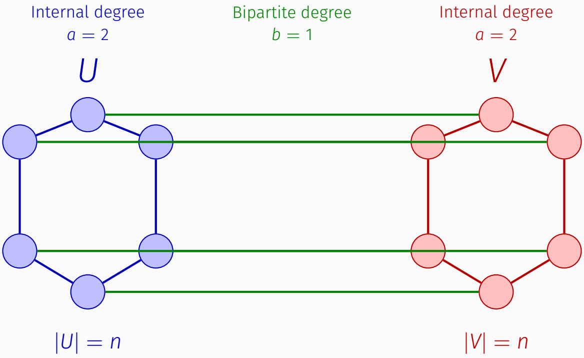
- We considered a specific subset of the d-regular graphs.
- We impose a bipartite structure within the graph to observe how connectivity within the graph impacts the lower order terms of the moments.

Our Set Up

Consider two sets of vertices U, V, each with n vertices.
Consider a b-regular bipartite graph on U and V. Now within each of U and V impose $a = d - b$-regularity.



Every vertex has degree $a + b = 2 + 1 = 3$

Combinatorial Trace Method

Let $A = (a_{ij})_{i,j=1}^{N}$ denote the adjacency matrix of G, and define the kth moment of its empirical spectral measure by

$$\mu_k(A) \ := \ \frac{1}{n}\sum_{i=1}^{n}\lambda_i^k \ = \ \frac{1}{n}\mathrm{Tr}(A^k).$$

$$\mathrm{Tr}(A^k) = \sum_{i=1}^{n}(A^k)_{ii}.$$

First and second moment

Corollary (1st moment)
For a random (a, b)-regular two-level bipartite graph,

$$\mathbb{E}[\mu_1(A)] = 0.$$

Corollary (2nd moment)
For a random (a, b)-regular two-level bipartite graph,

$$\mathbb{E}[\mu_2(A)] = d = a + b.$$

Proof of Lemma (2)
Because A is symmetric and its entries belong to $\{0, 1\}$,

$$\begin{aligned} \mathrm{Tr}(A^2) &= \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} a_{ji} \\ &= \sum_{i=1}^{N} d. \end{aligned}$$

1/N Theorem

Theorem
The expected moment for the (a,b)-regular two-level bipartite graph is

$$\mathbb{E}[\mu_k] = m_k(a+b) + \frac{R_k(a,b)}{N} + O(N^{-2}),$$

where

- m_k *is the* k^{th} *moment of the Kesten-McKay distribution*
- $R_k(a,b)$ *is the expected number of Betti number* $\beta = 1$ k*-cycles*

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  - Represents a simple core c-cycle ($c \leq k$) decorated with tree tails.
- $\beta \geq 2$ (Multicyclic Diagrams):
  - Scale: $O(1/N^2)$ or smaller
  - Vanishes in the leading-order $1/N$ moment correction.

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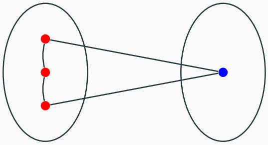
- We count the number of ways to choose vertices and half-edges of the assigned vertices.
- For the probability, we will assume all a half-edges have uniform probability to connect to all of the remaining a half-edges in the same N_i set, and all b half-edges have uniform probability of connecting to the b half-edges in the opposite set.

Example of using half-edge technique for Betti one



Configuration 2.1

Vertex choices:

$$n(n-1)(n-2)\cdot n.$$

Here $n(n-1)(n-2)$ chooses three ordered vertices in N_1, and n chooses one vertex in N_2.

Half-edge choices:

$$ab, \qquad a(a-1), \qquad ab, \qquad b(b-1),$$

so their product is

$$a^3b^3(a-1)(b-1).$$

Contribution to the fourth moment

$$\frac{8n(n-1)(n-2)\,n\,a^3b^3(a-1)(b-1)}{(na-1)(na-2)(nb-1)nb} = 8ab(a-1)(b-1) + O\left(\frac{1}{N}\right).$$

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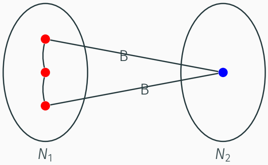

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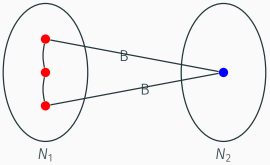

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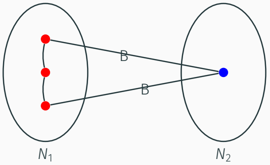

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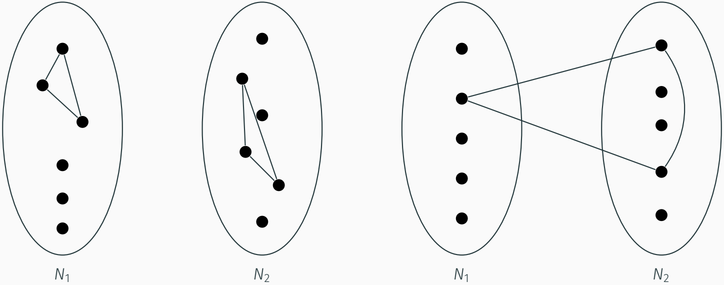
In general, the contribution corresponding to w is

$$a^{n_{BA}(w)}b^{n_{AB}(w)}(a-1)^{n_{AA}(w)}(b-1)^{n_{BB}(w)}.$$

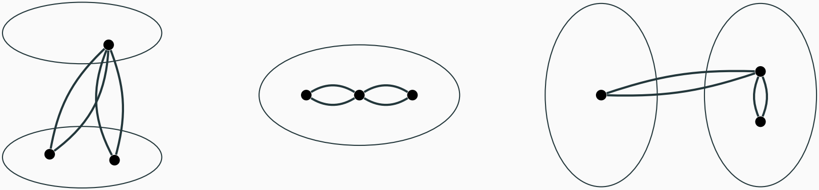
Third moment

We divide the triangles into three classes:

1. triangles whose three vertices lie in N_1;
2. triangles whose three vertices lie in N_2;
3. triangles having two vertices in one level and one vertex in the other level.



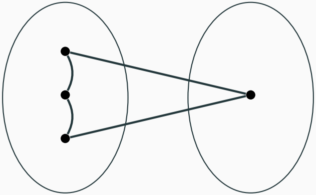
Fourth Moment: non-simple 4-cycles



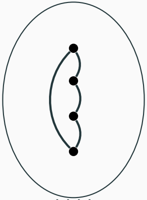
We compute the contributions manually in this case

- $2(b(b-1)+b^2)N$
- $2(a(a-1)+a^2)N$
- $4abN$

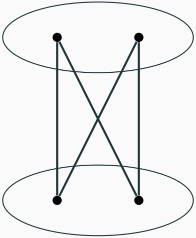
Fourth moment



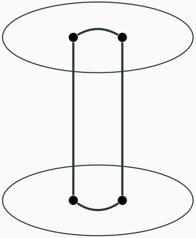
BAAB



AAAA



BBBB



ABAB

Third and Fourth Moments

Corollary
Let G be a random (a, b)-regular two-level graph on $N = 2n$ vertices, and let A be its adjacency matrix. For fixed a and b, as $N \to \infty$,

$$\mathbb{E}[m_3(A)] = \frac{2(a-1)^3 + 6ab(b-1)}{N} + O\left(\frac{1}{N^2}\right).$$

Moreover,

$$\begin{aligned}\mathbb{E}[m_4(A)] = 2(a+b)^2 - (a+b) \\ + \frac{1}{N}\Big[2(a-1)^4 + 2(b-1)^4 \\ + 8a(a-1)b(b-1) + 4a^2b^2\Big] \\ + O\left(\frac{1}{N^2}\right).\end{aligned}$$

Considerations for Higher Order Moments

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- In general, we can count these cycles by breaking it up into a simple cycle and the number of excursion edges.
- We already know how to calculate the number of cycles, so we must count the number of symmetries that these tree-like excursions create.

Core Cycle Counts $B_s(a, b)$ via Parity Linear Recurrences

- Let $C_l(a, b)$ be the number of l-cycles.
- Windings around the cycle are accounted for via divisibility:

$$B_s(a, b) = \sum_{\substack{3 \le l \le s, \\ l | s}} 2l\, C_l(a, b)$$

- *Example ($s = 6$):* $B_6 = 12C_6 + 6C_3$ (includes twice-wound triangles!).

Excursion Counting

Definition: $h_{k,s}(d)$ counts length-k walks spending s steps on the core cycle and $k - s$ steps executing tree-like excursions.

Base Conditions ($k = 0$):

$$h_{0,0}(d) = 1, \qquad h_{0,s}(d) = 0 \quad \text{for } s > 0$$

Step-by-Step Recurrence ($k \to k + 1$):

$$\begin{aligned}
h_{k+1,0}(d) &= d \cdot h_{k,1}(d) && \text{(Return to core root)} \\
h_{k+1,1}(d) &= h_{k,0}(d) + (d-1)h_{k,2}(d) && \text{(Adjacent to core)} \\
h_{k+1,s}(d) &= h_{k,s-1}(d) + (d-1)h_{k,s+1}(d) && \text{for } s \geq 2 \text{ (Deep tree excursions)}
\end{aligned}$$

Parity Rule

$h_{k,s}(d) = 0$ whenever $k \not\equiv s \pmod 2$.

Excursion Coefficients

Generating coefficients row-by-row via the 2D recurrence:

$k \setminus s$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	d	0	1	0	0	0
3	0	$2d-1$	0	1	0	0
4	$2d^2-d$	0	$3d-2$	0	1	0
5	0	$5d^2-6d+2$	0	$4d-3$	0	1

- **Tree Column ($s=0$):** Gives the Kesten–McKay bulk moment $m_k(d)$.
- **Unicyclic Term ($s \geq 3$):** $h_{5,3}(d) = 4d-3$, which directly multiplies $B_3(a,b) = 6C_3$ to form $6(4d-3)C_3$ in $R_5(a,b)$.

The total count for the number of cycles then becomes:

$$R_k(a,b) = \sum_{s=3}^{k} h_{k,s}(d) B_s(a,b).$$

$$R_3(a,b) = 6C_3$$
$$R_4(a,b) = 8C_4$$
$$R_5(a,b) = 10C_5 + 6(4d-3)C_3$$
$$R_6(a,b) = 12C_6 + 6C_3 + 8(5d-4)C_4$$

- The lower order terms do depend on a and b, while the leading order term does not
- Closed walks with $\beta = k$ impact terms of order

$$\frac{1}{N^{k+i}}$$

where $i \geq 0$

Further Directions

- Generalize the work to higher moments and possibly to lower orders of N
- Further explore combinatorial interpretations of lower order terms
- Extend the analysis to other classes of graphs to gain further insight

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Thank You

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