From Random Matrix Theory to Number Theory

Steven J Miller
Dept of Math/Stats, Williams College

sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu
http://www.williams.edu/Mathematics/sjmiller

SMALL: Williams College, July 5, 2017
Introduction
Goals

- Determine correct scale and statistics to study eigenvalues and zeros of $L$-functions.

- See similar behavior in different systems.

- Discuss the tools and techniques needed to prove the results.
Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \ldots$.

Question: What rules govern the spacings between the $t_i$?
Fundamental Problem: Spacing Between Events

**General Formulation:** Studying system, observe values at $t_1, t_2, t_3, \ldots$

**Question:** What rules govern the spacings between the $t_i$?

**Examples:** Spacings between

- Energy Levels of Nuclei.
- Eigenvalues of Matrices.
- Zeros of $L$-functions.
- Summands in Zeckendorf Decompositions.
- Primes.
- $n^k \alpha \mod 1$. 
**Fundamental Problem: Spacing Between Events**

**General Formulation:** Studying system, observe values at 
$t_1, t_2, t_3, \ldots$.

**Question:** What rules govern the spacings between the $t_i$?

**Examples:** Spacings between
- Energy Levels of Nuclei.
- Eigenvalues of Matrices.
- Zeros of $L$-functions.
- Summands in Zeckendorf Decompositions.
- Primes.
- $n^k \alpha \mod 1$. 
Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.

2. Develop an explicit formula relating what we want to study to something we understand.

3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!
Classical Random Matrix Theory
Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.
Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.

Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

**Fundamental Equation:**

\[ H \psi_n = E_n \psi_n \]

- \( H \): matrix, entries depend on system
- \( E_n \): energy levels
- \( \psi_n \): energy eigenfunctions
Origins of Random Matrix Theory

- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\bar{A}^T = A$).
Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix $p$, define

$$\text{Prob}(A) = \prod_{1 \leq i \leq j \leq N} p(a_{ij}).$$

This means

$$\text{Prob}(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]) = \prod_{1 \leq i \leq j \leq N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) \, dx_{ij}.$$

Want to understand eigenvalues of $A$. 
Eigenvalue Distribution

\[ \delta(x - x_0) \text{ is a unit point mass at } x_0: \]
\[ \int f(x)\delta(x - x_0)\,dx = f(x_0). \]
Eigenvalue Distribution

\( \delta(x - x_0) \) is a unit point mass at \( x_0 \):
\[
\int f(x) \delta(x - x_0) \, dx = f(x_0).
\]

To each \( A \), attach a probability measure:

\[
\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)
\]
**Eigenvalue Distribution**

\[ \delta(x - x_0) \text{ is a unit point mass at } x_0: \]
\[ \int f(x)\delta(x - x_0)\,dx = f(x_0). \]

To each \( A \), attach a probability measure:

\[
\begin{align*}
\mu_{A,N}(x) &= \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right) \\
\int_{a}^{b} \mu_{A,N}(x)\,dx &= \frac{\# \left\{ \lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a, b] \right\}}{N}
\end{align*}
\]
Eigenvalue Distribution

\( \delta(x - x_0) \) is a unit point mass at \( x_0 \):
\[
\int f(x) \delta(x - x_0) \, dx = f(x_0).
\]

To each \( A \), attach a probability measure:

\[
\mu_{A,N}(x) = \frac{1}{N} \sum_{i=1}^{N} \delta \left( x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)
\]

\[
\int_{a}^{b} \mu_{A,N}(x) \, dx = \frac{\# \left\{ \lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a, b] \right\}}{N}
\]

The \( k \)th moment
\[
k^{th} \text{ moment} = \frac{\sum_{i=1}^{N} \lambda_i(A)^k}{2^k N^{k+1/2}} = \frac{\text{Trace}(A^k)}{2^k N^{k+1/2}}.
\]
Wigner’s Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d. r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all $A$, as $N \to \infty$

$$\mu_{A,N}(x) \to \begin{cases} \frac{2}{\pi} \sqrt{1 - x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Numerical examples

500 Matrices: Gaussian $400 \times 400$

\[ p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]
Numerical examples

The eigenvalues of the Cauchy distribution are NOT semicircular.

Cauchy Distribution: \( p(x) = \frac{1}{\pi(1+x^2)} \)

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of $A$, but choose the matrix elements randomly and independently.

**Eigenvalue Trace Lemma**

Let $A$ be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) = \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1i_2}a_{i_2i_3} \cdots a_{i_Ni_1}.$$
SKETCH OF PROOF: Correct Scale

\[ \text{Trace}(A^2) = \sum_{i=1}^{N} \lambda_i(A)^2. \]

By the Central Limit Theorem:

\[ \text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}a_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \sim N^2 \]

\[ \sum_{i=1}^{N} \lambda_i(A)^2 \sim N^2 \]

Gives \( N\text{Ave}(\lambda_i(A)^2) \sim N^2 \) or \( \text{Ave}(\lambda_i(A)) \sim \sqrt{N} \).
SKETCH OF PROOF: Averaging Formula

Recall $k$-th moment of $\mu_{A,N}(x)$ is $\text{Trace}(A^k)/2^k N^{k/2+1}$.

Average $k$-th moment is

$$\int \cdots \int \frac{\text{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \leq j} p(a_{ij}) \, da_{ij}.$$ 

Proof by method of moments: Two steps

- Show average of $k$-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).
SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

\[
\frac{1}{2^2 N^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \cdot p(a_{11}) da_{11} \cdots p(a_{NN}) da_{NN}
\]

Integration factors as

\[
\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij}) da_{ij} \cdot \prod_{(k,l) \neq (i,j), k<l}^{(k,l) \neq (i,j)} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl}) da_{kl} = 1.
\]

Higher moments involve more advanced combinatorics (Catalan numbers).
SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

\[
\frac{1}{2^k N^{k/2 + 1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1 = 1}^{N} \cdots \sum_{i_k = 1}^{N} a_{i_1i_2} \cdots a_{i_ki_1} \cdot \prod_{i \leq j} p(a_{ij}) \, da_{ij}.
\]

Main contribution when the \( a_{i\ell i_{\ell+1}} \)'s matched in pairs, not all matchings contribute equally (if did would get a Gaussian and not a semi-circle; this is seen in Real Symmetric Palindromic Toeplitz matrices).

**GOE Conjecture:**

As $N \to \infty$, the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of $p$.

Until recently only known if $p$ is a Gaussian.

\[ \text{GOE}(x) \approx \frac{\pi}{2} xe^{-\pi x^2/4}. \]
Numerical Experiment: Uniform Distribution

Let \( p(x) = \frac{1}{2} \) for \( |x| \leq 1 \).

The local spacings of the central 3/5 of the eigenvalues of 5000 300x300 uniform matrices, normalized in batches of 20.

5000: 300 \times 300 uniform on \([-1, 1]\)
Cauchy Distribution

Let \( p(x) = \frac{1}{\pi (1 + x^2)} \).

The local spacings of the central 3/5 of the eigenvalues of 5000 300x300 Cauchy matrices, normalized in batches of 20.
Random Graphs

Degree of a vertex = number of edges leaving the vertex. Adjacency matrix: $a_{ij} =$ number edges b/w Vertex $i$ and Vertex $j$.

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.
McKay’s Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for $d$-regular graphs

\[ f(x) = \begin{cases} 
\frac{d}{2\pi(d^2-x^2)} \sqrt{4(d-1) - x^2} & |x| \leq 2\sqrt{d-1} \\
0 & \text{otherwise.}
\end{cases} \]
McKay’s Law (Kesten Measure) with $d = 6$

Fat Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \to \infty$ recover semi-circle).
3-Regular Graph with 2000 Vertices: Comparison with the GOE

Spacings between eigenvalues of 3-regular graphs and the GOE:
<table>
<thead>
<tr>
<th>Classical RMT</th>
<th>Checkerboard Ensemble</th>
<th>L-Functions</th>
<th>Katz-Sarnak Conj</th>
<th>Qs and Refs</th>
<th>Block Circulant</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Checkerboard Ensemble**
Definition

The $N \times N (k, w)$-checkerboard ensemble is the ensemble of matrices $M = (m_{ij})$ given by

$$m_{ij} = \begin{cases} a_{ij} & \text{if } i \not\equiv j \mod k \\ w & \text{if } i \equiv j \mod k \end{cases}$$

where the $a_{ij} = a_{ji}$ are iid with mean 0, variance 1, and finite higher moments, and $w$ is constant.
A \((3, w)\)-checkerboard matrix is of the form

\[
\begin{pmatrix}
  w & a_{0,1} & a_{0,2} & w & a_{0,4} & \cdots & a_{0,N-1} \\
  a_{1,0} & w & a_{1,2} & a_{1,3} & w & \cdots & a_{1,N-1} \\
  a_{2,0} & a_{2,1} & w & a_{2,3} & a_{2,4} & \cdots & w \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{0,N-1} & a_{1,N-1} & w & a_{3,N-1} & a_{4,N-1} & \cdots & w
\end{pmatrix}
\]

SMALL ’15, ’16: Checkerboard: Split Behavior

**Figure:** Histogram of normalized eigenvalues for 500 $100 \times 100$ 2-checkerboard matrices.

SMALL ’17: Generalized Checkerboard: Work in Progress
Figure: Histogram of normalized eigenvalues for complex symmetric matrix.
Introduction to $L$-Functions
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]

Unique Factorization: \( n = p_1^{r_1} \cdots p_m^{r_m}. \)
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]

Unique Factorization: \( n = p_1^{r_1} \cdots p_m^{r_m}. \)

\[ \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right] \left[1 + \frac{1}{3s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right] \cdots \]

\[ = \sum_{n} \frac{1}{n^s}. \]
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \#\{p: p \text{ is prime}, p \leq x\} \]

Properties of \( \zeta(s) \) and Primes:
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \#\{p : p \text{ is prime}, p \leq x\} \]

Properties of \( \zeta(s) \) and Primes:

- \( \lim_{s \to 1^+} \zeta(s) = \infty, \pi(x) \to \infty \).
Riemann Zeta Function (cont)

\[ \zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \]

\[ \pi(x) = \#\{p : p \text{ is prime}, p \leq x\} \]

Properties of \( \zeta(s) \) and Primes:

- \( \lim_{s \to 1^+} \zeta(s) = \infty, \pi(x) \to \infty. \)
- \( \zeta(2) = \frac{\pi^2}{6}, \pi(x) \to \infty. \)
Riemann Zeta Function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\rho \text{ prime}} \left(1 - \frac{1}{\rho^s}\right)^{-1}, \quad \text{Re}(s) > 1. \]

**Functional Equation:**

\[ \xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1 - s). \]

**Riemann Hypothesis (RH):**

All non-trivial zeros have \( \text{Re}(s) = \frac{1}{2} \); can write zeros as \( \frac{1}{2} + i\gamma \).

**Observation:** Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices \( \overline{A^T} = A \).
General $L$-functions

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s, f)^{-1}, \quad \text{Re}(s) > 1.$$ 

Functional Equation:

$$\Lambda(s, f) = \Lambda_{\infty}(s, f)L(s, f) = \Lambda(1 - s, f).$$

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$. 
Elliptic Curves: Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.

Addition of distinct points $P$ and $Q$  

Adding a point $P$ to itself

\[ E(\mathbb{Q}) \approx E(\mathbb{Q})_{tors} \oplus \mathbb{Z}^r \]
Elliptic curve $L$-function

$E : y^2 = x^3 + ax + b$, associate $L$-function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \mod p\}.$$
Elliptic curve $L$-function

$E : y^2 = x^3 + ax + b$, associate $L$-function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \mod p\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s, E)$ at $s = 1/2$. 
Properties of zeros of $L$-functions

- infinitude of primes, primes in arithmetic progression.
- Chebyshev’s bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ ‘most’ of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from $L$-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.
Distribution of zeros

- $\zeta(s) \neq 0$ for $\Re(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- **GRH:** error terms.
- **GSH:** Chebyshev’s bias.
- **Analytic rank, adjacent spacings:** $h(D)$. 
Explicit Formula (Contour Integration)

\[-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p (1 - p^{-s})^{-1}\]
Explicit Formula (Contour Integration)

\[-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} = \frac{d}{ds} \sum_p \log (1 - p^{-s}) = \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).\]
Explicit Formula (Contour Integration)

\[-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1}\]

\[= \frac{d}{ds} \sum_p \log (1 - p^{-s})\]

\[= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{ps} + \text{Good}(s).\]

Contour Integration:

\[\int - \frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} \, ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \, \frac{ds}{s}.\]
Explicit Formula (Contour Integration)

\[-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \]

\[= \frac{d}{ds} \sum_p \log (1 - p^{-s}) \]

\[= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s). \]  

Contour Integration:

\[\int -\frac{\zeta'(s)}{\zeta(s)} \phi(s) \, ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) p^{-s} \, ds. \]
Explicit Formula (Contour Integration)

\[-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1}\]

= \frac{d}{ds} \sum_p \log (1 - p^{-s})

= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).

Contour Integration (see Fourier Transform arising):

\[\int -\frac{\zeta'(s)}{\zeta(s)} \phi(s) ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma \log p} e^{-it \log p} ds.\]

Knowledge of zeros gives info on coefficients.
Explicit Formula: Example

**Dirichlet L-functions:** Let $\phi$ be an even Schwartz function and $L(s, \chi) = \sum_n \chi(n)/n^s$ a Dirichlet $L$-function from a non-trivial character $\chi$ with conductor $m$ and zeros $\rho = \frac{1}{2} + i\gamma_{\chi}$. Then

$$\sum_{\rho} \phi \left( \gamma_{\chi} \frac{\log(m/\pi)}{2\pi} \right) = \int_{-\infty}^{\infty} \phi(y) dy$$

$$-2 \sum_{p} \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( \frac{\log p}{\log(m/\pi)} \right) \frac{\chi(p)}{p^{1/2}}$$

$$-2 \sum_{p} \frac{\log p}{\log(m/\pi)} \hat{\phi} \left( 2 \frac{\log p}{\log(m/\pi)} \right) \frac{\chi^2(p)}{p} + O\left( \frac{1}{\log m} \right).$$
Katz-SarnakDensity Conjectures
70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20}$th zero (from Odlyzko).
Measures of Spacings: $n$-Level Correlations

\[ \{ \alpha_j \} \text{ increasing sequence, box } B \subset \mathbb{R}^{n-1}. \]

**$n$-level correlation**

\[
\lim_{N \to \infty} \frac{\# \left\{ \left( \alpha_{j_1} - \alpha_{j_2}, \ldots, \alpha_{j_{n-1}} - \alpha_{j_n} \right) \in B, j_i \neq j_k \right\}}{N}
\]

(Instead of using a box, can use a smooth test function.)
Measures of Spacings: $n$-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbb{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at $10^{20}$ (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. $n$-level correlations for all automorphic cuspidal $L$-functions (Rudnick-Sarnak).
4. $n$-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.
Let \( g_i \) be even Schwartz functions whose Fourier Transform is compactly supported, \( L(s, f) \) an \( L \)-function with zeros \( \frac{1}{2} + i \gamma_f \) and conductor \( Q_f \):

\[
D_{n,f}(g) = \sum_{j_1, \ldots, j_n \atop j_i \neq \pm j_k} g_1 \left( \frac{\log Q_f}{2\pi} \gamma_{f,j_1} \right) \cdots g_n \left( \frac{\log Q_f}{2\pi} \gamma_{f,j_n} \right)
\]

- Properties of \( n \)-level density:
  - Individual zeros contribute in limit
  - Most of contribution is from low zeros
  - Average over similar \( L \)-functions (family)
**n-Level Density**

*n-level density*: \( \mathcal{F} = \bigcup \mathcal{F}_N \) a family of \( L \)-functions ordered by conductors, \( g_k \) an even Schwartz function: 

\[
D_{n,\mathcal{F}}(g) = \lim_{N \to \infty} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{j_1, \ldots, j_n} g_1 \left( \frac{\log Q_f}{2\pi} \gamma_{j_1;f} \right) \cdots g_n \left( \frac{\log Q_f}{2\pi} \gamma_{j_n;f} \right)
\]

As \( N \to \infty \), \( n \)-level density converges to

\[
\int g(\mathbf{x}) \rho_{n,\mathcal{G}(\mathcal{F})}(\mathbf{x}) \, d\mathbf{x} = \int \hat{g}(\mathbf{u}) \hat{\rho}_{n,\mathcal{G}(\mathcal{F})}(\mathbf{u}) \, d\mathbf{u}.
\]

**Conjecture (Katz-Sarnak)**

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.
1-Level Densities

Let $\mathcal{G}$ be one of the classical compact groups: Unitary, Symplectic, Orthogonal (or $\text{SO}(\text{even}), \text{SO}(\text{odd})$). If $\text{supp}(\hat{g}) \subset (-1, 1)$, 1-level density of $\mathcal{G}$ is

$$\hat{g}(0) - c_\mathcal{G} \frac{g(0)}{2},$$

where

$$c_\mathcal{G} = \begin{cases} 
0 & \mathcal{G} \text{ is Unitary} \\
1 & \mathcal{G} \text{ is Symplectic} \\
-1 & \mathcal{G} \text{ is Orthogonal.}
\end{cases}$$
Identifying the Symmetry Groups

• Often suggested by monodromy group in the function field.

• Tools: Explicit Formula, Summation Formula.

• How to identify symmetry group in general? One possibility is by the signs of the functional equation: Folklore Conjecture: If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise SO(even). (False!)

Explicit Formula

- $\pi$: cuspidal automorphic representation on $GL_n$.
- $Q_\pi > 0$: analytic conductor of $L(s, \pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake params: $\{\alpha_{\pi,i}(p)\}_{i=1}^n$; $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s, \pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n \left(1 - \alpha_{\pi,i}(p)p^{-s}\right)^{-1}$.
- $$\sum_j g \left(\gamma_{\pi,j} \frac{\log Q_\pi}{2\pi}\right) = \hat{g}(0) - 2 \sum_{p, \nu} \hat{g} \left(\frac{\nu \log p}{\log Q_\pi}\right) \frac{\lambda_\pi(p^\nu) \log p}{p^\nu/2 \log Q_\pi}$$
Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_1,p \times \pi_2,p$ are $\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} = \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{1 \leq i \leq n, 1 \leq j \leq m}$.

Theorem (Dueñez-Miller)

If $\mathcal{F}$ and $\mathcal{G}$ are nice families of $L$-functions, then $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

Breaks analysis of compound families into simple ones.

1-Level Density

Assuming conductors constant in family \( \mathcal{F} \), have to study

\[ \text{\( \nu \)th moment}: \lambda_f(p^\nu) = \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu \]

\[
S_1(\mathcal{F}) = -2 \sum_p \hat{g} \left( \frac{\log p}{\log R} \right) \frac{\log p}{\sqrt{p \log R}} \left[ \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p) \right] 
\]

\[
S_2(\mathcal{F}) = -2 \sum_p \hat{g} \left( 2 \frac{\log p}{\log R} \right) \frac{\log p}{p \log R} \left[ \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p^2) \right] 
\]

The corresponding classical compact group determined by

\[
\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 
0 & \text{Unitary} \\
1 & \text{Symplectic} \\
-1 & \text{Orthogonal.} 
\end{cases}
\]
Takeaways

Very similar to Central Limit Theorem.

- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.

- First moment zero save for families of elliptic curves.

- Higher moments control convergence and can depend on arithmetic of family.
Correspondences

Similarities between $L$-Functions and Nuclei:

Zeros $\leftrightarrow$ Energy Levels

Schwartz test function $\longrightarrow$ Neutron

Support of test function $\leftrightarrow$ Neutron Energy.
Open Questions
and References
Open Questions: Low-lying zeros of $L$-functions

- Generalize excised ensembles for higher weight $GL_2$ families where expect different discretizations.

- Obtain better estimates on vanishing at the central point by finding optimal test functions for the second and higher moment expansions.

- Further explore $L$-function Ratios Conjecture to predict lower order terms in families, compute these terms on number theory side.

See Dueñez-Huynh-Keating-Miller-Snaith, Miller, and the Ratios papers.
Publications: Random Matrix Theory


Publications: L-Functions


Publications: Elliptic Curves


Publications: $L$-Function Ratio Conjecture


Block Circulant Ensemble

With Murat Koloğlu, Gene Kopp, Fred Strauch and Wentao Xiong.
The Ensemble of $m$-Block Circulant Matrices

Symmetric matrices periodic with period $m$ on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

\[
\begin{pmatrix}
  c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
  c_1 & d_0 & d_1 & d_2 & c_4 & d_3 & c_3 & d_2 \\
  c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
  c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
  c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
  d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
  c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
  d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0 
\end{pmatrix}
\]

Choose distinct entries i.i.d.r.v.
Oriented Matchings and Dualization

Compute moments of eigenvalue distribution (as $m$ stays fixed and $N \to \infty$) using the combinatorics of pairings. Rewrite:

$$M_n(N) = \frac{1}{N^{n^2+1}} \sum_{1 \leq i_1, \ldots, i_n \leq N} E(a_{i_1} a_{i_2} a_{i_3} \cdots a_{i_n})$$

$$= \frac{1}{N^{n^2+1}} \sum_{\sim} \eta(\sim) m_{d_1}(\sim) \cdots m_{d_l}(\sim).$$

where the sum is over oriented matchings on the edges $\{(1, 2), (2, 3), \ldots, (n, 1)\}$ of a regular $n$-gon.
Oriented Matchings and Dualization

**Figure:** An oriented matching in the expansion for $M_n(N) = M_6(8)$. 
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.
Only Topology Matters

Think of pairings as topological identifications; the contributing ones give rise to orientable surfaces.

Contribution from such a pairing is $m^{-2g}$, where $g$ is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.
Computing the Even Moments

**Theorem: Even Moment Formula**

\[ M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k \left( \frac{1}{N} \right), \]

with \( \varepsilon_g(k) \) the number of pairings of the edges of a \((2k)\)-gon giving rise to a genus \( g \) surface.

J. Harer and D. Zagier (1986) gave generating functions for the \( \varepsilon_g(k) \).
Harer and Zagier

\[ \sum_{g=0}^{\lceil k/2 \rceil} \varepsilon_g(k) r^{k+1-2g} = (2k - 1)!! \ c(k, r) \]

where

\[ 1 + 2 \sum_{k=0}^{\infty} c(k, r) x^{k+1} = \left( \frac{1 + x}{1 - x} \right)^r. \]

Thus, we write

\[ M_{2k} = m^{-(k+1)}(2k - 1)!! \ c(k, m). \]
A multiplicative convolution and Cauchy’s residue formula yield

**Theorem: Koloğlu, Kopp and Miller**

Limiting spectral density $f_m(x)$ of the real symmetric $m$-block circulant ensemble is

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \left( \frac{m}{r+s+1} \right) \left( \frac{2r+2s}{r+s} \right)! \left( \frac{1}{2} \right)^s (mx^2)^r.$$  

As $m \to \infty$, $f_m(x)$ approaches the semicircle distribution.
Results (continued)

Figure: Plot for $f_1$ and histogram of eigenvalues of 100 circulant matrices of size $400 \times 400$. 
Results (continued)

Figure: Plot for $f_2$ and histogram of eigenvalues of 100 2-block circulant matrices of size $400 \times 400$. 
Results (continued)

**Figure:** Plot for $f_3$ and histogram of eigenvalues of 100 3-block circulant matrices of size $402 \times 402$. 
Results (continued)

**Figure:** Plot for $f_4$ and histogram of eigenvalues of 100 4-block circulant matrices of size $400 \times 400$. 
Results (continued)

**Figure:** Plot for $f_8$ and histogram of eigenvalues of 100 8-block circulant matrices of size $400 \times 400$. 
Results (continued)

Figure: Plot for $f_{20}$ and histogram of eigenvalues of 100 20-block circulant matrices of size $400 \times 400$. 