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Random Fibonacci Tilings

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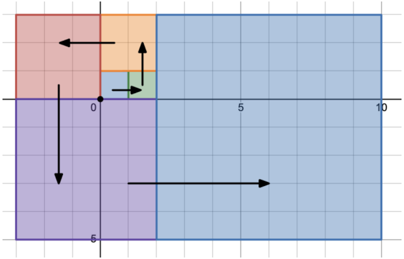
2026

Williams SMALL
Probability and Number Theory

Introduction

The construction of the standard Fibonacci spiral.



Introduction

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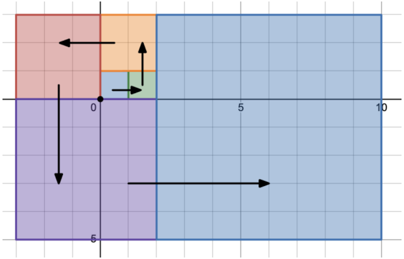

Idea

Odd indexed turns: randomly place a square left/right.
Even indexed turns: randomly place the square up/down.

Example

Consider the sequence:

$$\begin{bmatrix} \textbf{Left} & \textbf{Up} & \text{Left} & \textbf{Up} & \textbf{Left} \\ \text{Right} & \text{Down} & \textbf{Right} & \text{Down} & \text{Right} \end{bmatrix}$$

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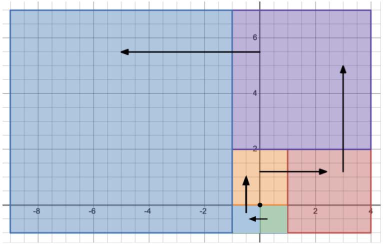
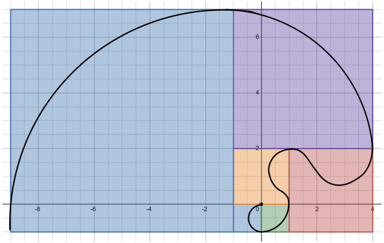
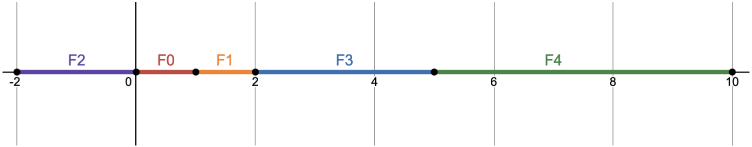
This yields the following tiling:





One-Dimensional General Case

Set $F_0 = F_1 = 1$...
We consider a 1-D version of our problem.



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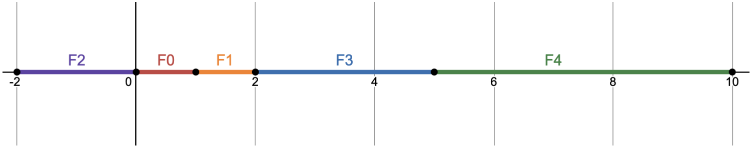

Let $b \in \mathbb{Z}^+$, and let X_i be a random variable

$$X_i := \begin{cases} F_i & \text{probability } p \\ 0 & \text{probability } 1-p. \end{cases}$$

Then,

$$\mathbb{E}(X_i) = pF_i$$

One-Dimensional Expectation

Let T_x be a variable representing the index of the first flip to cover b.

Theorem

$$\mathbb{E}(T_x) = \log_{\varphi}(b/p) + O(1)$$

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By linearity of expectation,

$$\mathbb{E}(X_i) = pF_i \implies \mathbb{E}(X_0 + \cdots + X_n) = p\sum_{i=0}^{n} F_i = p(F_{n+2} - 1).$$

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The expected number of intervals needed to cover b is on the order of the minimal n with $\mathbb{E}(X_0 + \cdots + X_n) \;\geq\; b$, so

$$p(F_{n+2} - 1) \;\geq\; b \iff F_{n+2} \geq \frac{b}{p} + 1 \iff n = \log_\varphi(b/p) + O(1)$$

Two-Dimensional Case

To cover (a, b), we must cover it horizontally and vertically. This breaks into two 1-D cases.

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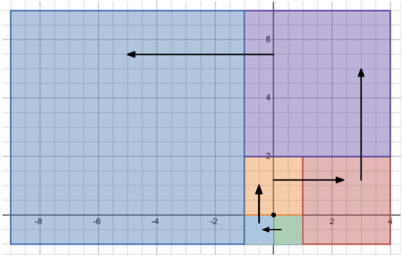

- Odd-indexed flips contribute only to the width
- Even-indexed flips contribute only to the height

Two-Dimensional Expectation

Theorem

$$\mathbb{E}\max\{T_x, T_y\} = \frac{1}{2}\cdot(\mathbb{E}T_x + \mathbb{E}T_y + \mathbb{E}|T_x - T_y|)$$

D-Dimensional Case

To cover $(a_1, a_2, \ldots, a_d)$, we must cover it on each the of d axes. This breaks into d 1-D cases.

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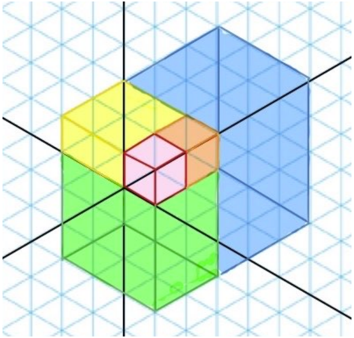


- k (mod d)-indexed flips contributes only to the k^{th} axis.

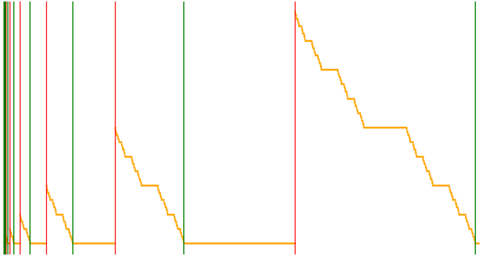
D-Dimension Expectation

Theorem

$$\mathbb{E}\max\{T_{x_1},\dots,T_{x_n}\} = \frac{1}{2^{n-1}}\left(\mathbb{E}[T_{x_1}] + \sum_{k=2}^{n} 2^{k-2}\left(\mathbb{E}[T_{x_k}] + \mathbb{E}\left|\max\{T_{x_1},\dots,T_{x_{k-1}}\} - T_{x_k}\right|\right)\right).$$

Open Questions and Possible Directions

- Find a closed form for $E_{n_b}(b)$ in 2 dimensions and higher.



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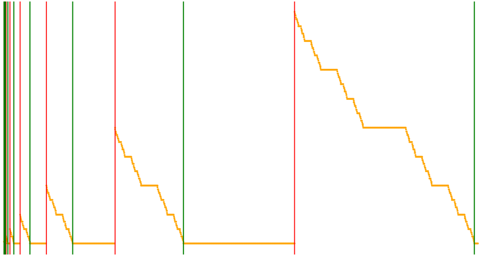

- Understand the general Fibonacci spirals (causality, etc.).

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Thanks ^–^

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