The Rectangle Game:

How to go from Rigging Games to Art and Theorems

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(Joint with PANTHers: Probability And Number Theory)

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Joint with many students and colleagues over the years.

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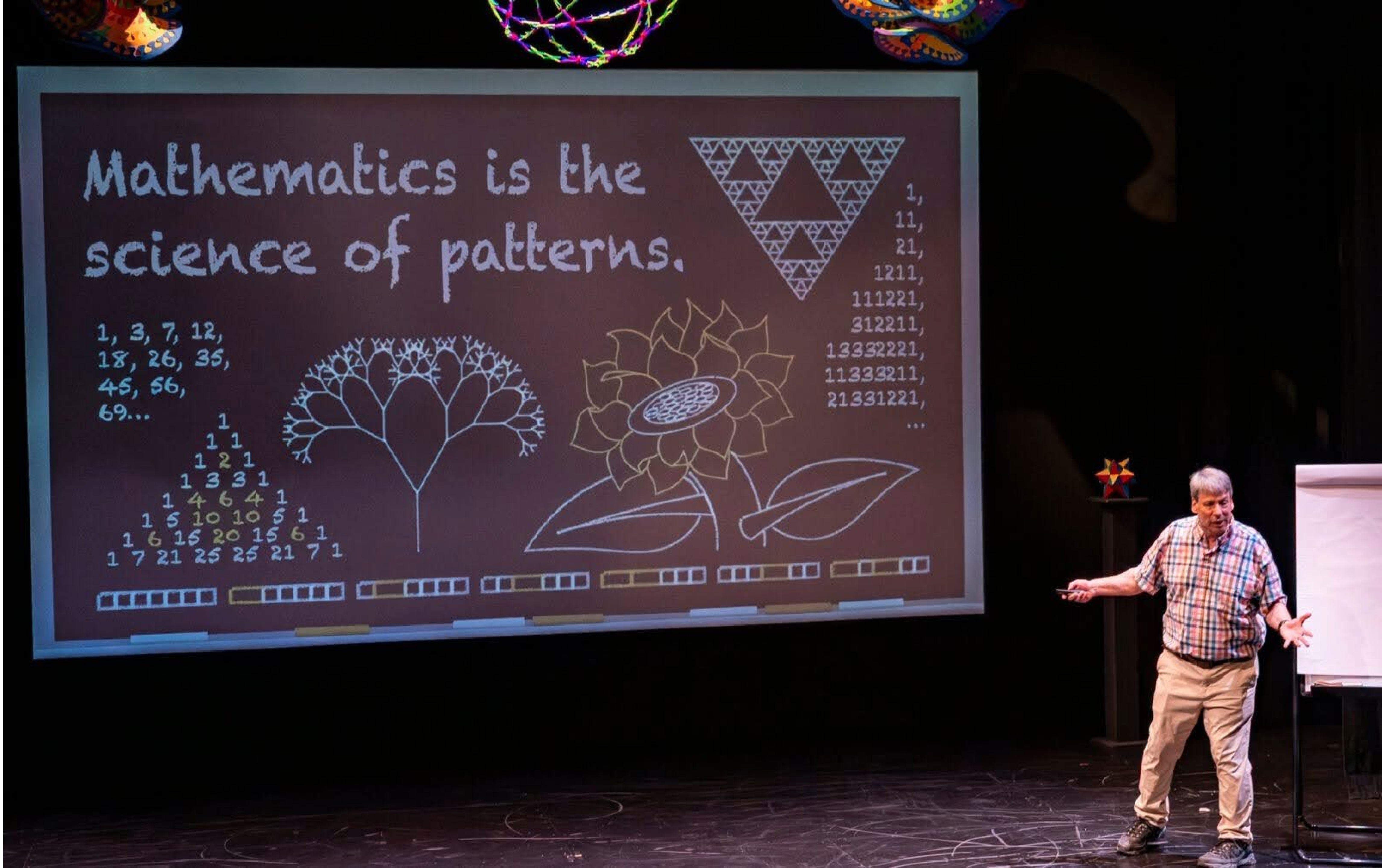
Art on off-Broadway

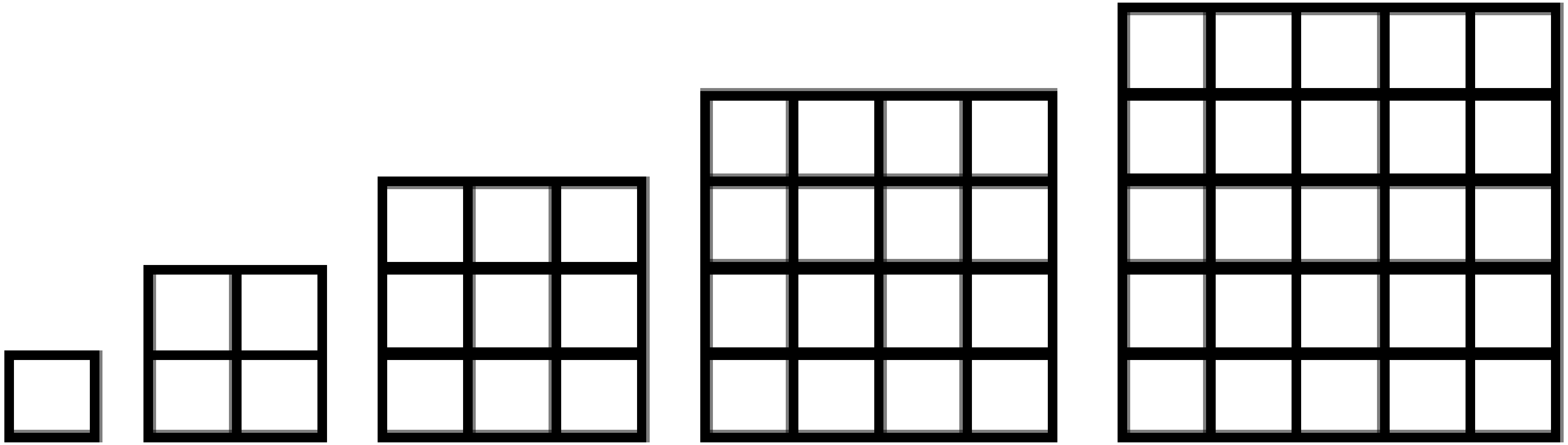
seahorse song: https://www.youtube.com/shorts/1giveD1wO6o



THE RECTANGLE GAME

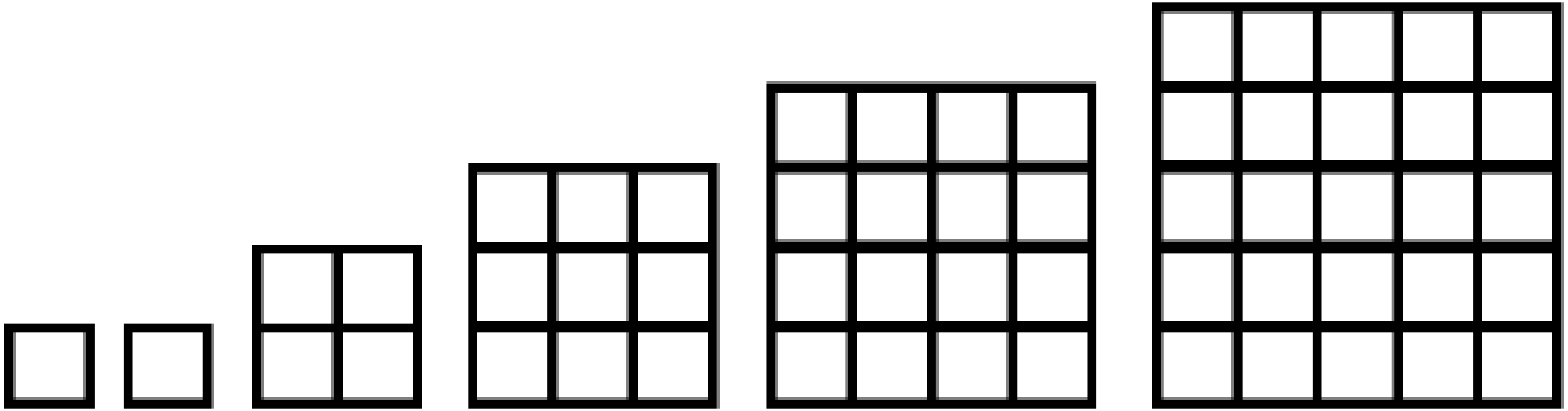
Tiling the Plane with Squares

Have $n \times n$ square for each n, place one at a time so that shape formed is always connected and a rectangle.

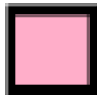


Tiling the Plane with Squares

Have $n \times n$ square for each n, extra 1×1 square, place one at a time so that shape formed is always connected and a rectangle.



Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$



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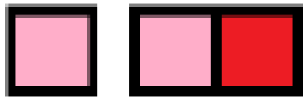

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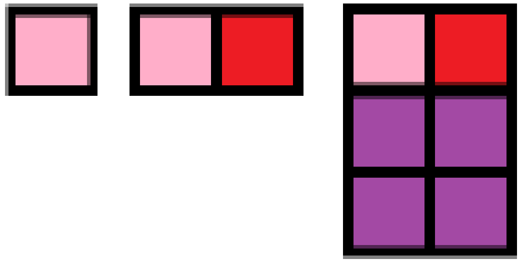
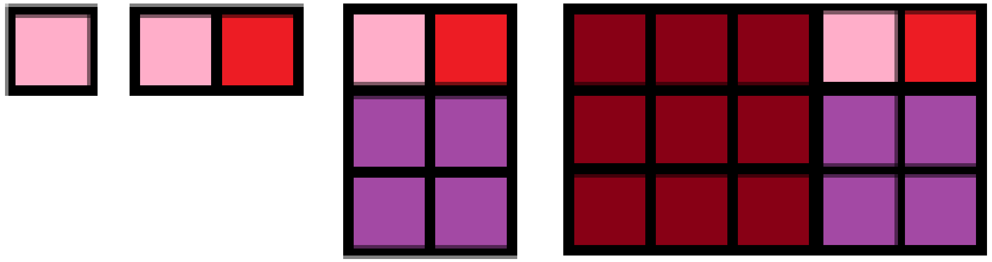
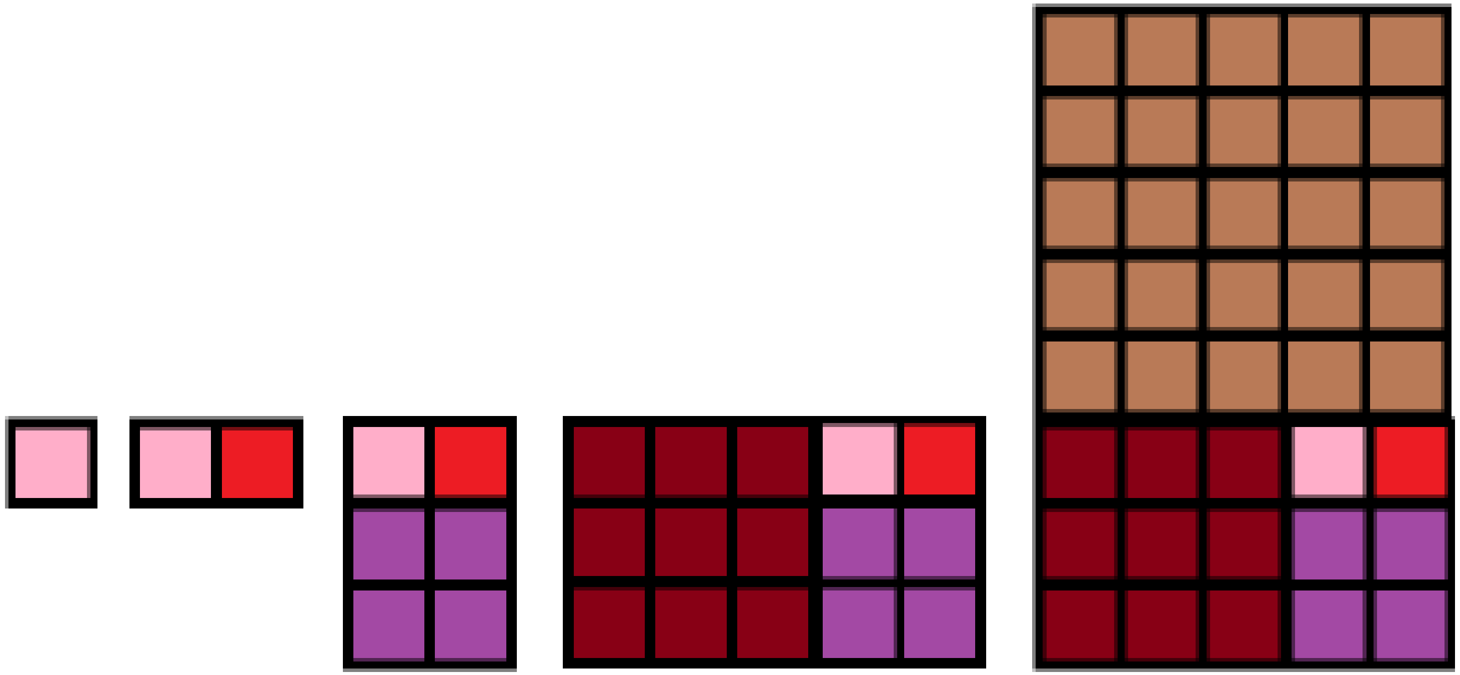


Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$

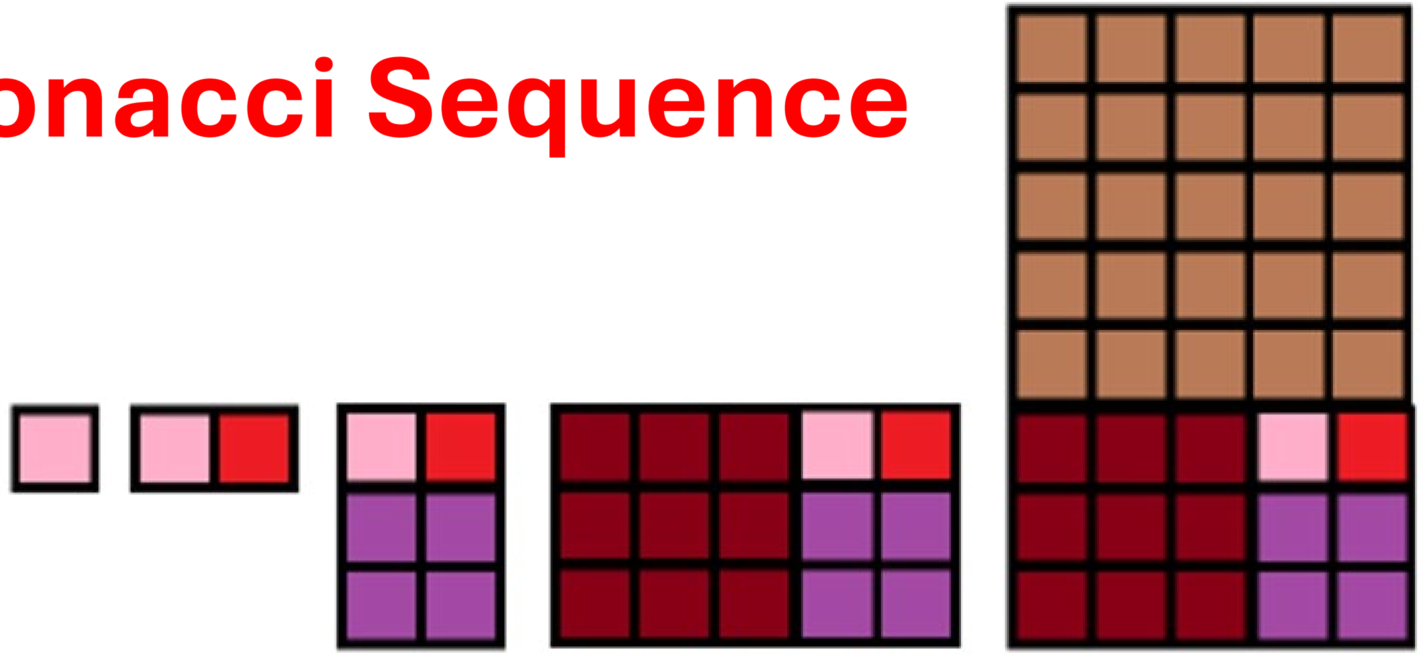


Tiling the Plane with Squares: $1 \times 1, 1 \times 1, 2 \times 2, 3 \times 3, \ldots$



1, 1, 2, 3, 5, ...

Fibonacci Sequence



1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597.

Each is the sum of the previous two: $F_{n+1} = F_n + F_{n-1}$, $F_1 = 1$, $F_2 = 2$.

Fibonacci and Art: I

Fibonacci and Art: II

Cake by Kayla Miller

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Fibonacci Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597.

Each is the sum of the previous two: $F_{n+1} = F_n + F_{n-1}$, $F_1 = 1$, $F_2 = 2$.

What patterns do you notice in the sequence?

Fourth graders at Lanesborough Elementary discovering the Fibonacci numbers.

Fibonacci Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597.

Odd, odd, even, odd, odd, even, odd, odd, even, ...

Fibonacci Sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597….

Odd, odd, even, odd, odd, even, odd, odd, even, ….

PROOF:

Clock arithmetic:

- a = b mod n means a-b is a multiple of n.
- a + b mod n equals (a mod n) + (b mod n) mod n.
- Periodic mod n – Pigeonhole/Box Principle.
    - Look at (F_i, F_{i+1}) mod n for i from 0 to n^2.

Related Questions:

What other related questions can we ask?

Related Questions:

What other related questions can we ask?

- What if we look modulo 3? Modulo n?
- How does the period depend on n?
- How many distinct residues are seen?
- How often is each residue seen?

Email me at sjm1@williams.edu if interested in pursuing....

Pascal's Triangle

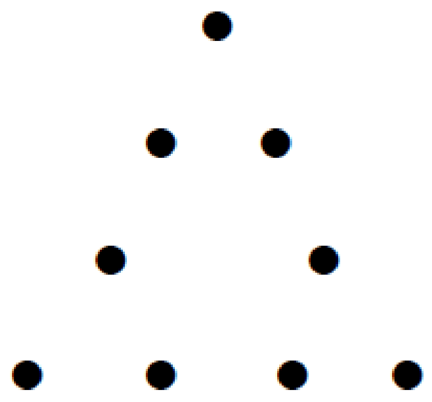
Pascal's triangle: k^{th} entry in the n^{th} row is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Pascal's Triangle Modulo 2

Modify Pascal's triangle: • if $\binom{n}{k}$ is odd, blank if even.

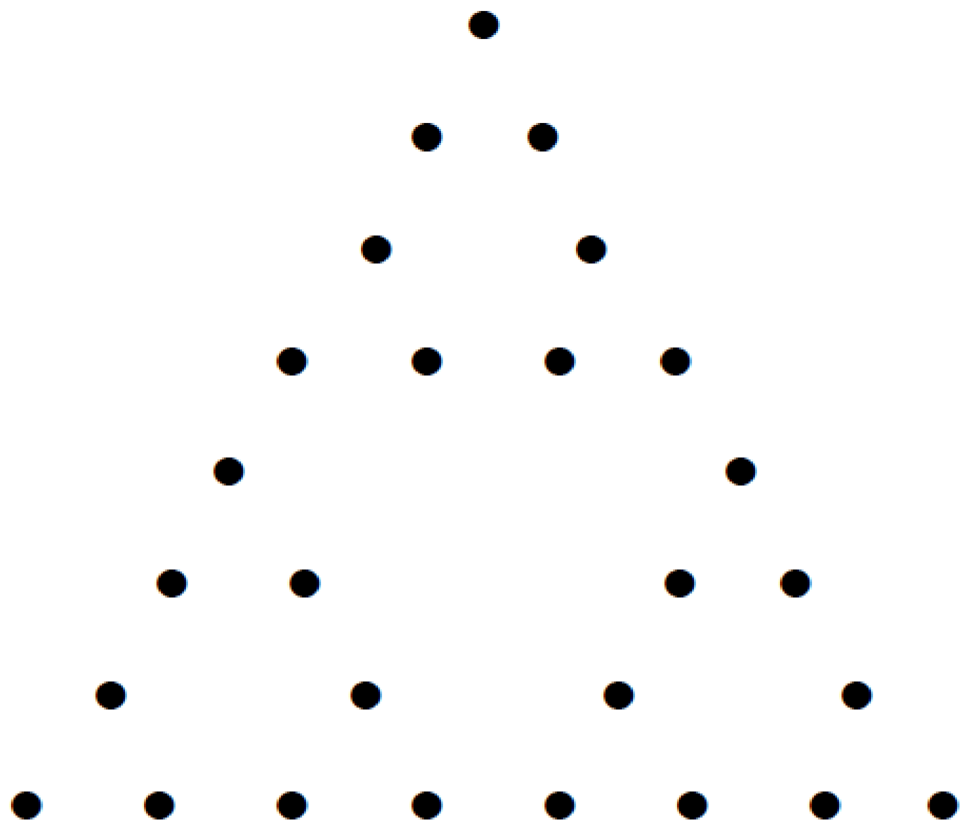
If we have just one row we would see •, if we have four rows we would see



Pascal's Triangle Modulo 2

Modify Pascal's triangle: $\bullet$ if $\binom{n}{k}$ is odd, blank if even.

For eight rows we find



Pascal mod 2 Plots

https://youtu.be/_vkGakVt1RA

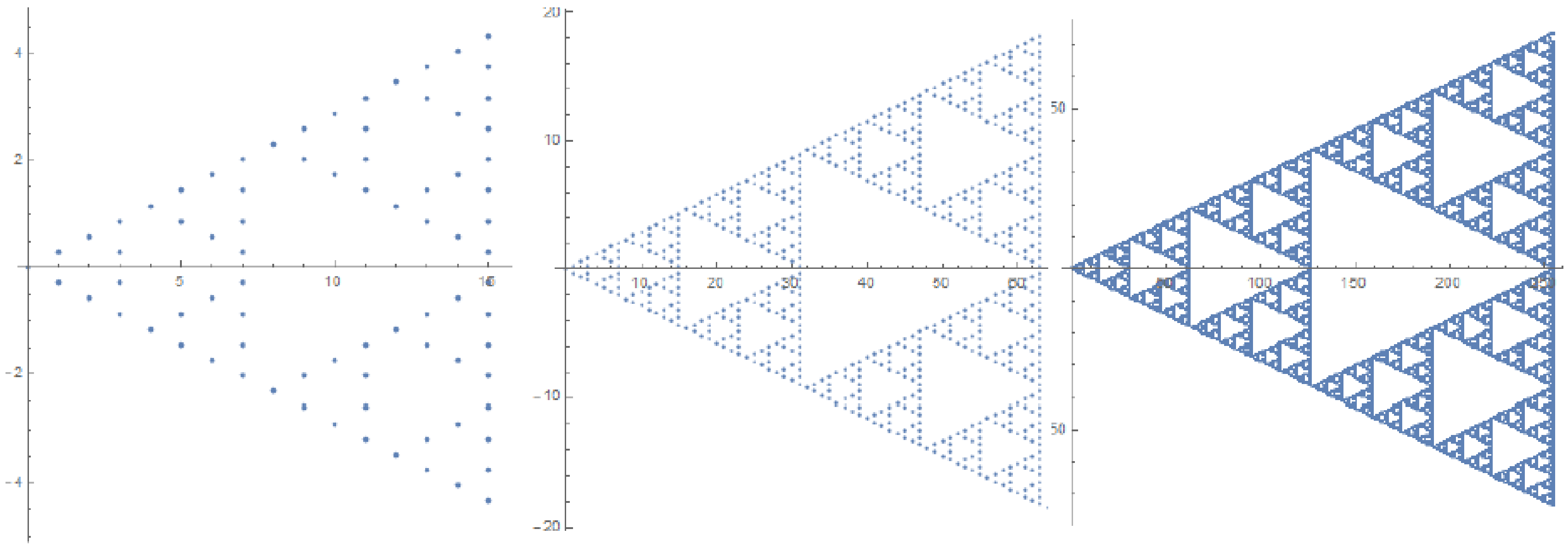
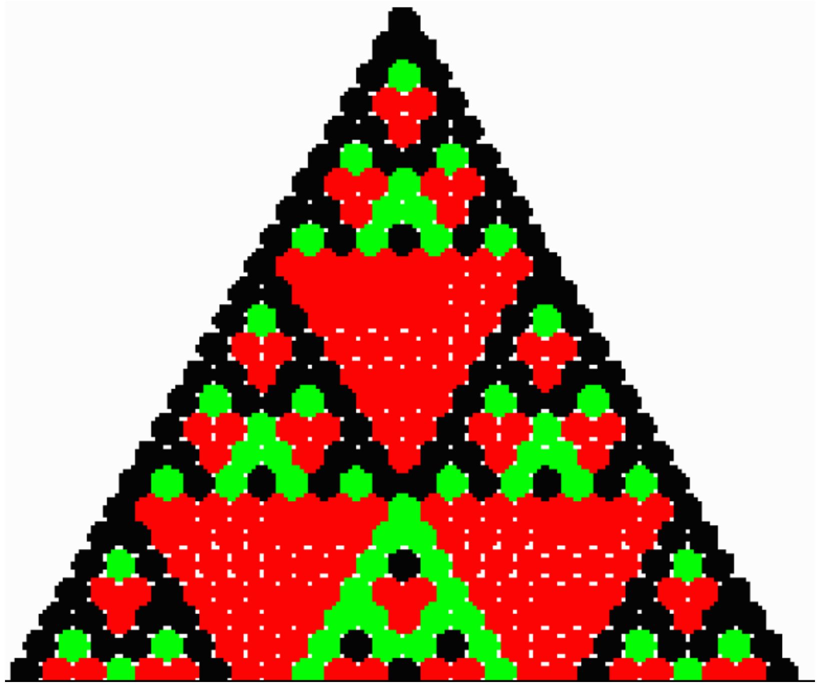


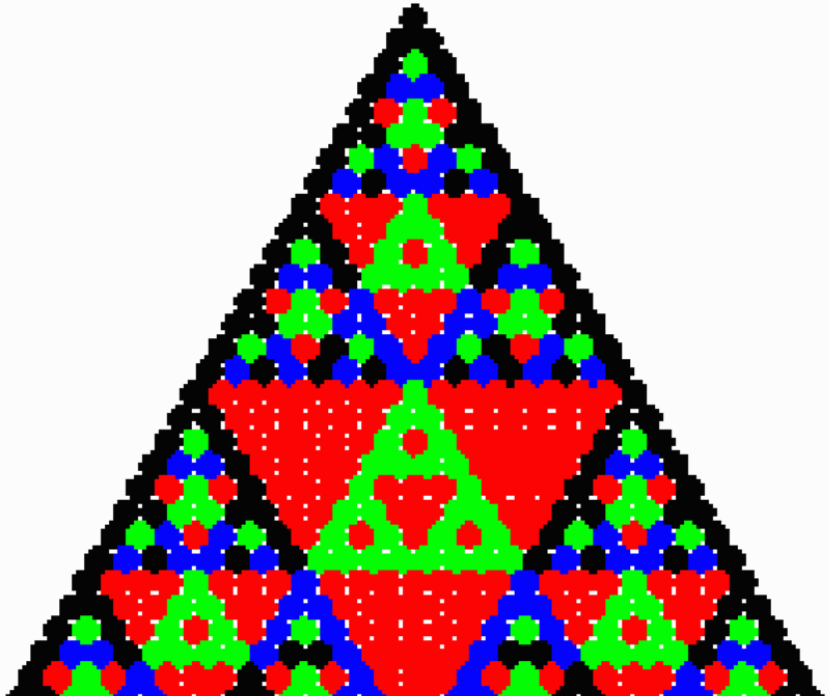
Figure: Plot of Pascal's triangle modulo 2 for 2^4, 2^8 and 2^{10} rows.

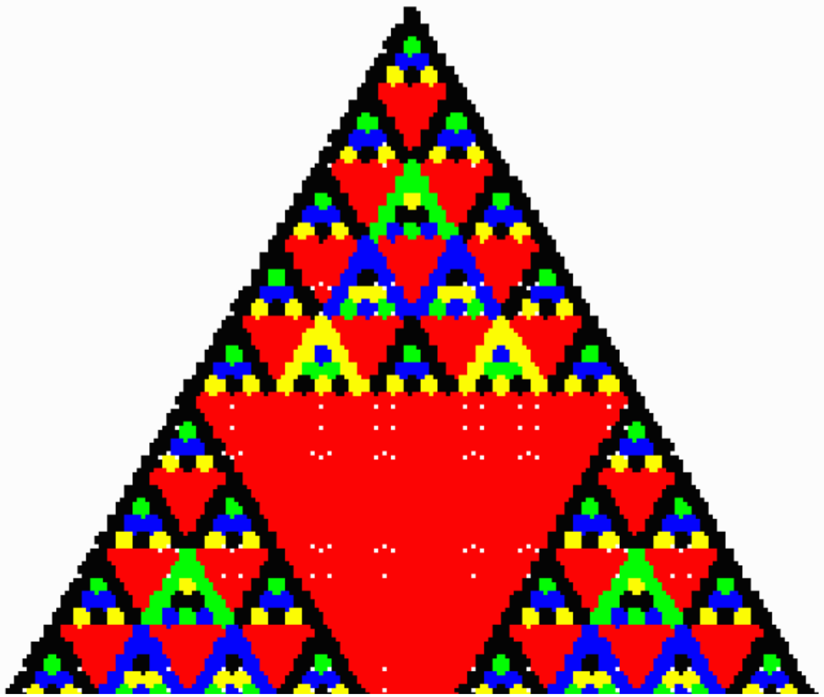
https://www.youtube.com/watch?v=tt4_4YajqRM
(start 1:35)
Fixed: https://youtu.be/_vkGakVt1RA?t=264 (start 4:24)

https://youtu.be/_vkGakVt1RA?t=265

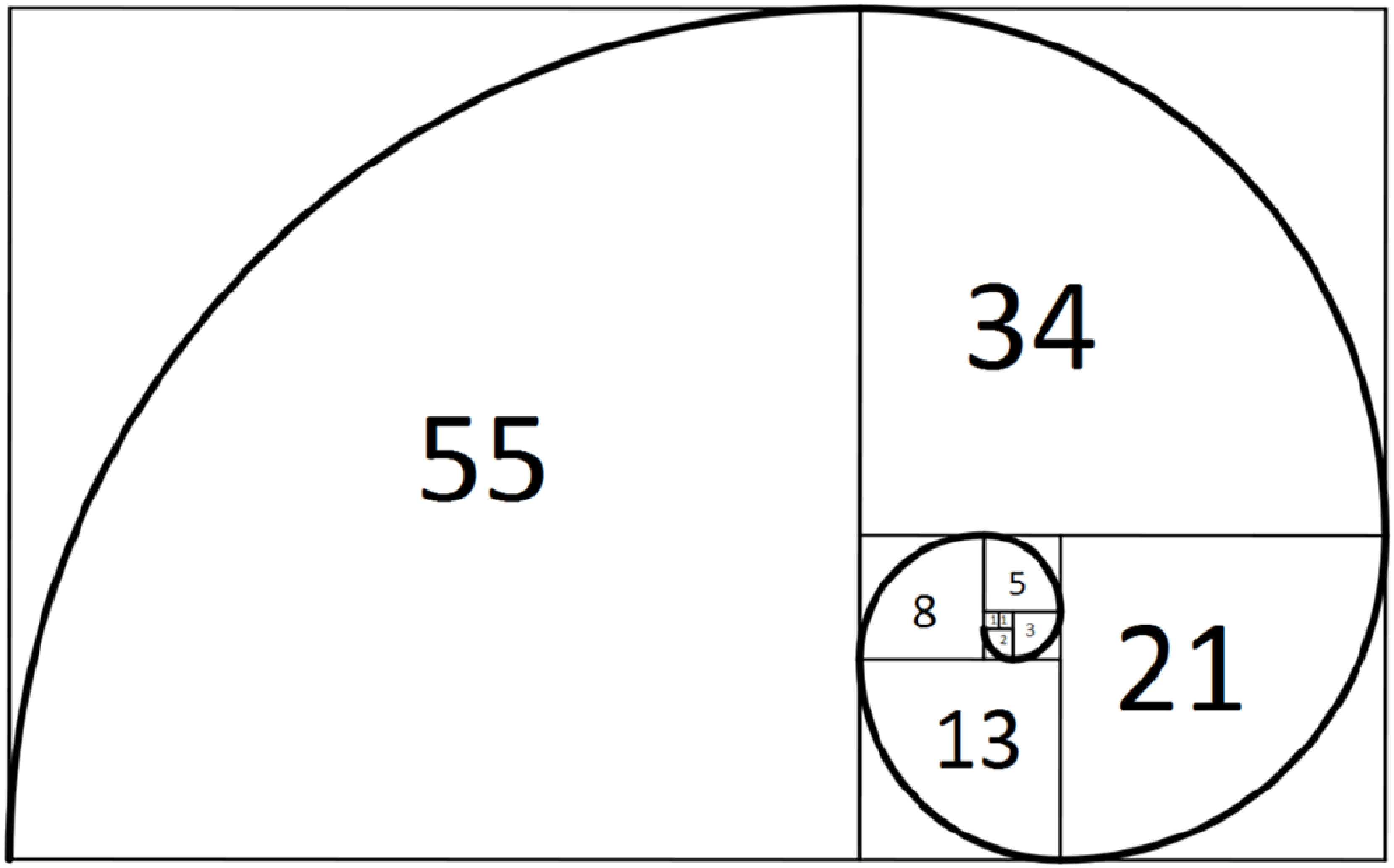
Pascal mod 3 mod 4 mod 5







Sums of Squares of Fibonacci Numbers



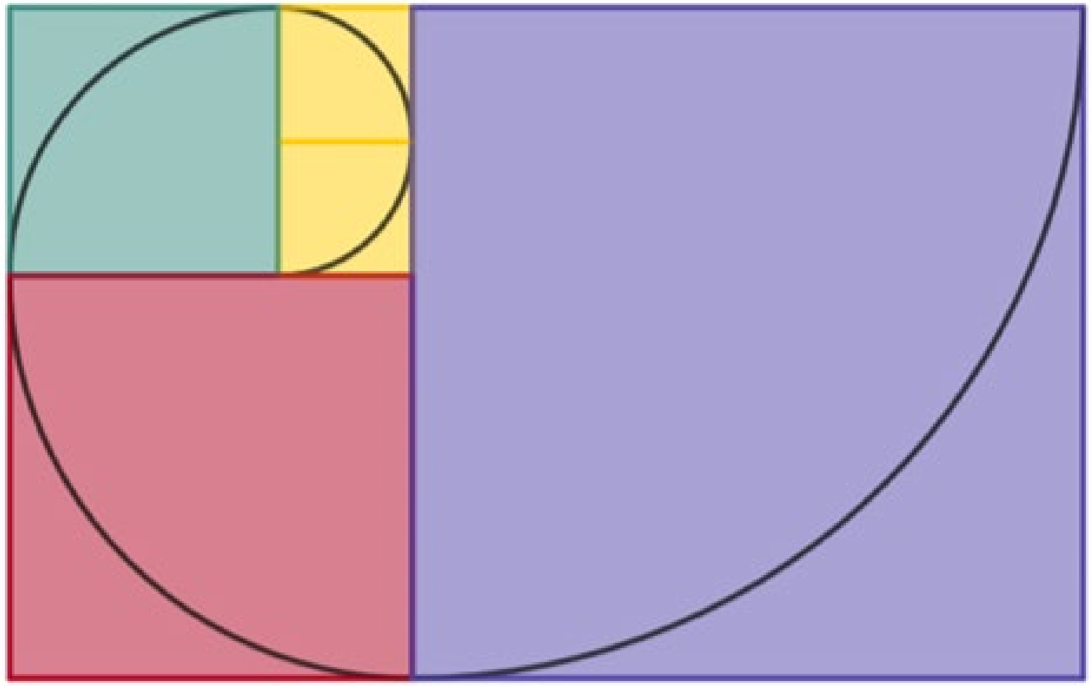
$$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 + 21^2 + 34^2 + 55^2 = 55 \cdot 89.$$

Geometric Interpretation (Tile Plane): $1^2 + 1^2 + 2^2 + \cdots + F_n^2 = F_n \cdot F_{n+1}.$

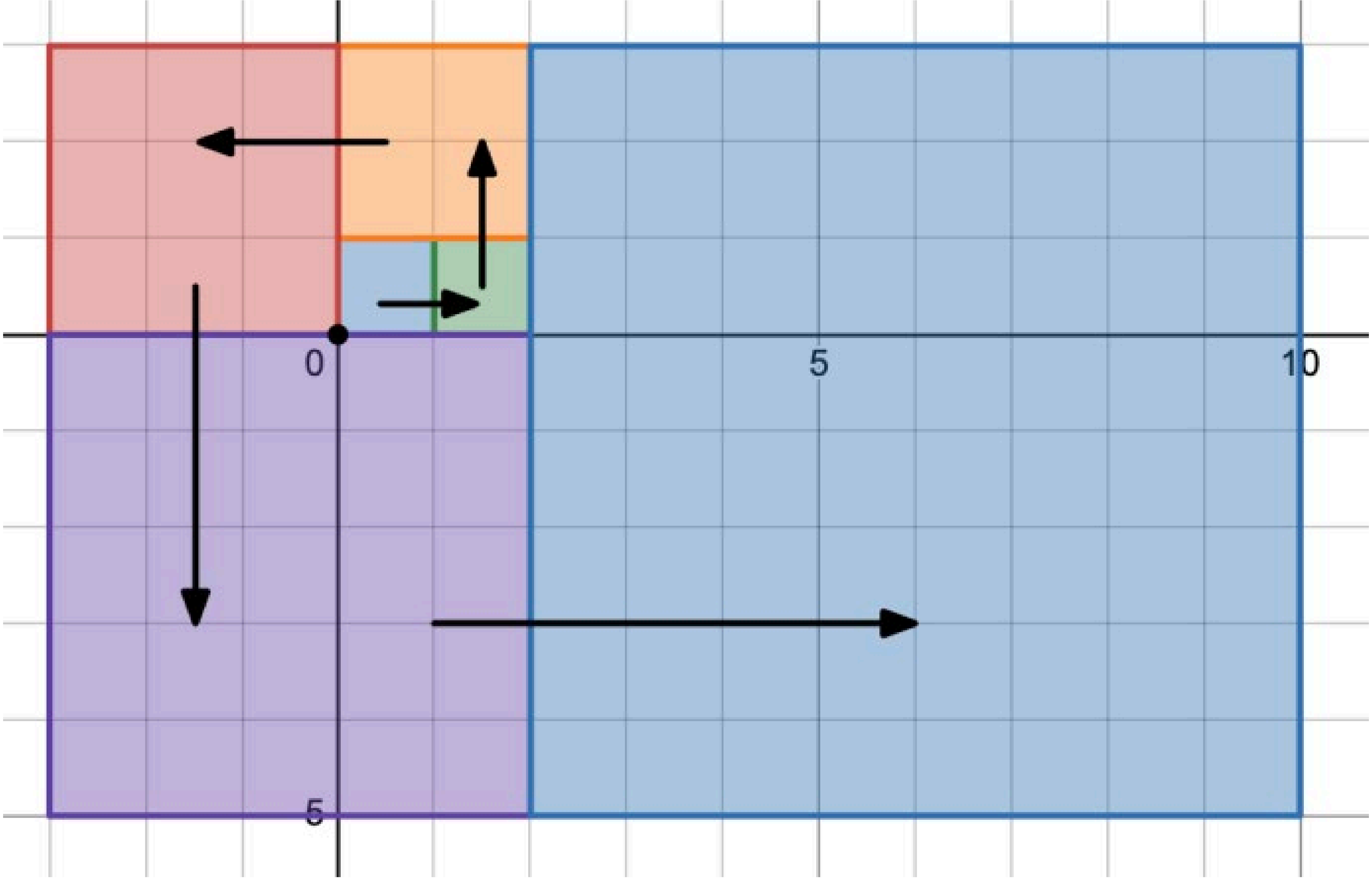
Random “Fibonacci” Spirals

The Fibonacci Rectangle Game: Two First-Move Classes and a Triangle-Induced Choice (Douglass Larsson Engholm and Steven J. Miller), to appear in the Fibonacci Quarterly.

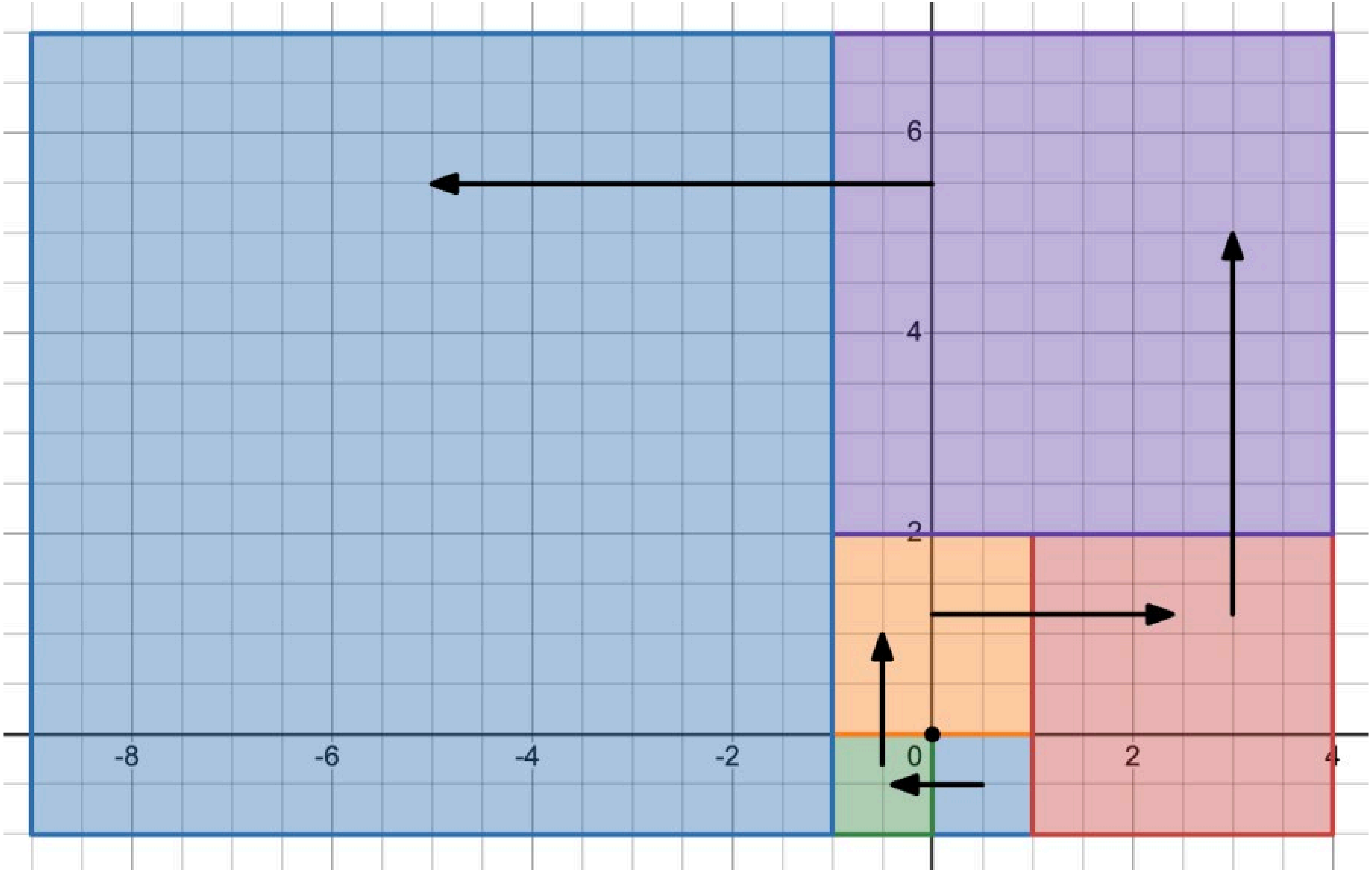
https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/Fibonacci_Rectangle_Game_Engholm_Miller.pdf

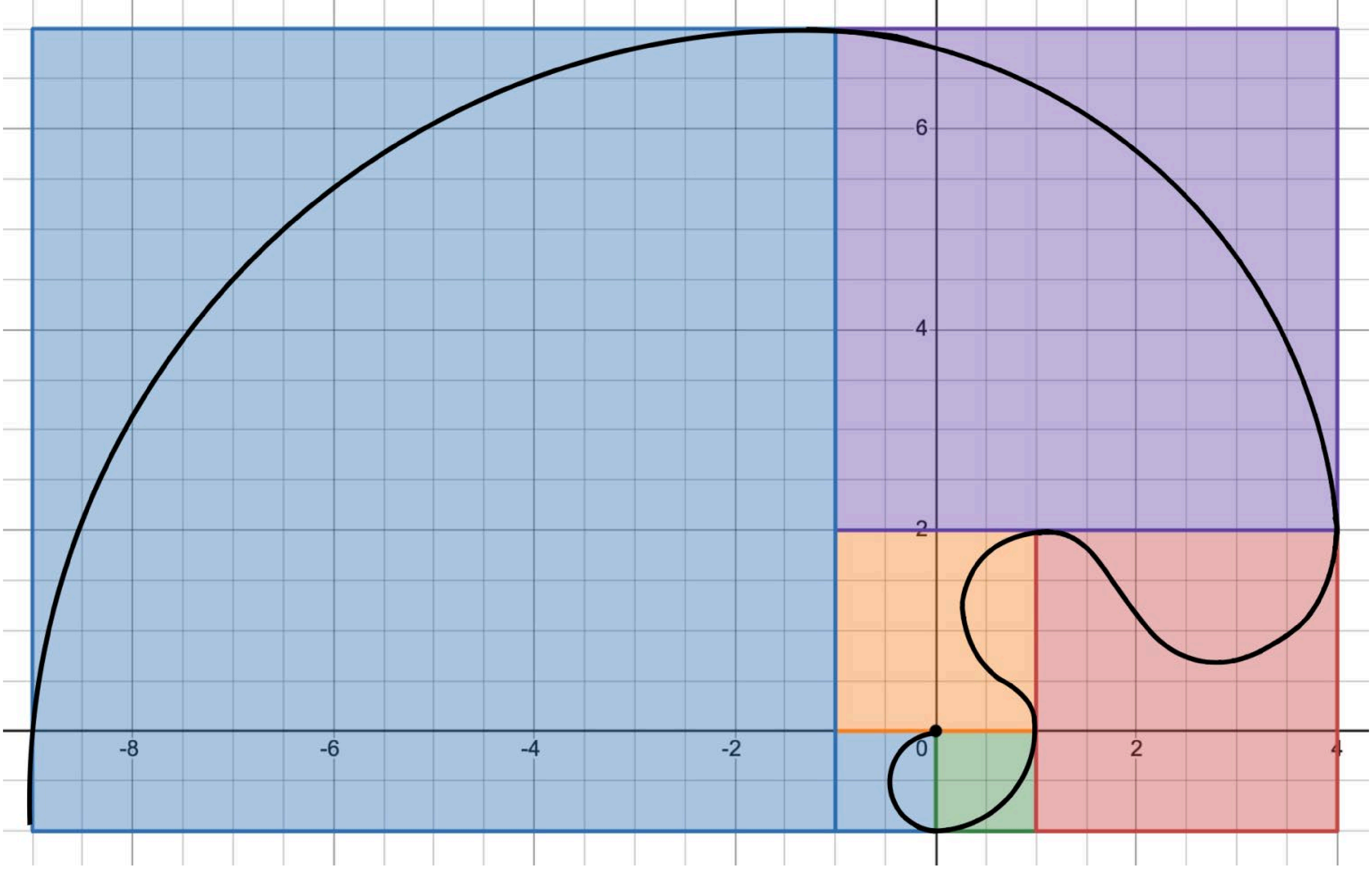


Up – Left – Down – Right – Up – Left – Down – Right – Up – Left - …


0
5
10
5

HOW DO WE CONNECT? TRY IT ON THE GRAPH PAPER!




6
4
2
-8
-6
-4
-2
0
2
4

Theorem 1. *After $2n$ turns, the vertical edge distance is $\sum_{F\in B} F$ for a uniformly chosen $B \subseteq \{F_0, F_2, \ldots, F_{2n}\}$. After $2n+1$ turns, the horizontal edge distance is $\sum_{F\in A} F$ for a uniformly chosen $A \subseteq \{F_1, F_3, \ldots, F_{2n+1}\}$.*

Theorem 2. *Let $E_n(b) = \{B \subseteq \{F_0, F_2, \ldots, F_{2n}\} : \sum_{F\in B} F \geq b\}$ and choose n_b such that $F_{2n_b} \leq b < F_{2(n_b+1)}$. Then, $E_n(b) = \varnothing$ for $n < n_b$, and*

$$|E_n(b)| = (2^{n+1} - 2^{n_b+1}) + |E_{n_b}(b)| \qquad (n \geq n_b).$$

Theorem 3. *Let T_y be the first time the top edge reaches height b. Then*

$$\mathbb{P}(T_y > 2n) = \begin{cases} 1, & n < n_b, \\ 2^{-(n-n_b)} - \dfrac{|E_{n_b}(b)|}{2^{n+1}}, & n \geq n_b, \end{cases}$$

$$\mathbb{E}[T_y] = 2\left(n_b + 2 - \frac{|E_{n_b}(b)|}{2^{n_b}}\right).$$

Theorem 4. *If $T(a,b)$ is the first time the tiling covers (a,b), then*

$$T(a,b) = \max\{T_x, T_y\}, \qquad \mathbb{E}[T(a,b)] = \frac{1}{2}\left(\mathbb{E}[T_x] + \mathbb{E}[T_y] + \mathbb{E}|T_x - T_y|\right).$$

By symmetry this applies to all $(a,b) \in \mathbb{Z}^2$ after replacing a, b by $|a|, |b|$.

FIBONACCI DIGITS

Digits of Fibonacci Numbers

An Invitation to Fibonacci Digits (Justin Cheigh, Guilherme Zeus Dantas e Moura, Jacob Lehmann Duke, Annika Mauro, Zoe McDonald, Anna Mello, Kayla Miller, Steven J. Miller, Santiago Velazquez Iannuzzelli), Fibonacci Quarterly **63** (2025), no. 4, 702--710.
https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/FibOutreachPaper_Digits40.pdf

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597….

- For a given base B, for what fixed subsets of $\{0, \ldots, B-1\}$ must all Fibonacci numbers have at least one member as a digit?

- Reconsider the previous problem, but now allow finitely many exceptions. In other words, for what subsets will all sufficiently large Fibonacci numbers have at least one of these as a digit?

Digits of Fibonacci Numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597….

F_{300} = 222232244629420445529739893461909967206666939096499764990979600
Note last 19 digits do not contain a 1, 2, 3, 5 or 8! If 50% chance odds 1 in 2 million (remember 3x+1 and 27…).

Theorem (Jarden). The last $n \geq 3$ digits of the Fibonacci numbers repeat every $15 \cdot 10^{n-1}$ times.

If we look at the Fibonacci numbers modulo 10, we get {1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, 1, 7, 8, 5, 3, 8, 1, 9, 0, 9, 9, 8, 7, 5, 2, 7, 9, 6, 5, 1, 6, 7, 3, 0, 3, 3, 6, 9, 5, 4, 9, 3, 2, 5, 7, 2, 9, 1, 0, 1, 1, 2, 3, 5, . . . }, and we see that the last digit is periodic with period 60; we leave to the audience the fun exercise of finding the period of the last two digits!

S	Do first $15 \cdot 10^{n-1}$ terms contain a digit in S?	n checked
$\{0,\ldots,9\} \setminus \{1\}$	no (disregarding $F_1, F_2 = 1$)	8
$\{0,\ldots,9\} \setminus \{2\}$	yes (disregarding $F_3 = 2$)	3
$\{0,\ldots,9\} \setminus \{3\}$	no (disregarding $F_4 = 3$)	8
$\{0,\ldots,9\} \setminus \{4\}$	yes	3
$\{0,\ldots,9\} \setminus \{5\}$	no (disregarding $F_5 = 5$, $F_{10} = 55$)	8
$\{0,\ldots,9\} \setminus \{6\}$	yes	5
$\{0,\ldots,9\} \setminus \{7\}$	no	10
$\{0,\ldots,9\} \setminus \{8\}$	no (disregarding $F_6 = 8$)	8
$\{0,\ldots,9\} \setminus \{2,4\}$	no (disregarding $F_3 = 2$)	10
$\{0,\ldots,9\} \setminus \{2,6\}$	no (disregarding $F_3 = 2$)	10
$\{0,\ldots,9\} \setminus \{4,6\}$	no	10

TABLE 1. Result from computing $15 \cdot 10^{n-1}$ terms of Fibonacci sequence mod 10^{n-1} for various S. We say "disregarding" to mean that our result holds for all Fibonacci numbers except for the choice of Fibonacci number labeled disregarded. Notice that the only disregarded Fibonacci numbers are all single-digit Fibonacci numbers save for 55, which is the only multi-digit Fibonacci number whose decimal digits are all the same (see [L]).

As a result of the computations in Table 1, we can conclude that all Fibonacci numbers contain:

- a digit in $\{0, \ldots, 9\} \setminus \{6\}$,
- a digit in $\{0, \ldots, 9\} \setminus \{4\}$,
- a digit in $\{0, \ldots, 9\} \setminus \{2\}$ (disregarding $F_2 = 2$).

Proposition 3.3. *No Fibonacci number can be written in base 10 using only the digit* 6.

PANTHers Results at the Conference

- **Starting:** For any arbitrary finite sequence of digits S there exists infinitely many Fibonacci numbers starting with S.

- **Ending:** The Fibonacci sequence $\{F_n\}$ fails to produce every possible n-digit ending combination if and only if the sequence is incomplete modulo 10^n. With Burr's criterion, the sequence is complete modulo 10^n if and only if $n < 3$.

- **Proportion:** For any d (or finite string of digits), the proportion of Fibonacci numbers that contain d (or the string) in their decimal expansion approaches 100% as n tends to infinity.

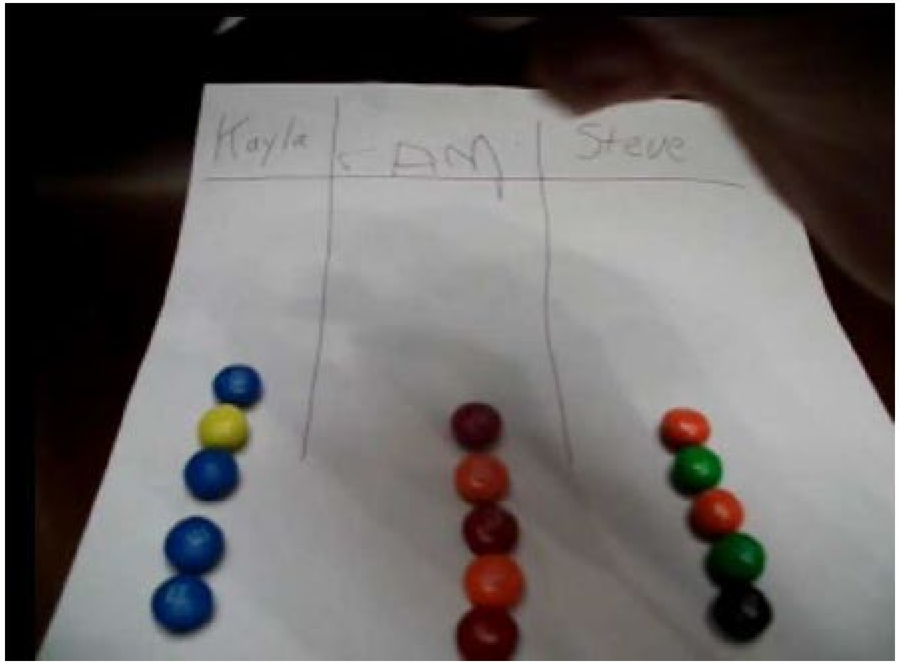
From M&Ms to Mathematics, or, How I learned to answer questions and help my kids love math.

The M&M Game: From Morsels to Modern Mathematics (with Ivan Badinski, Nathan McCue, Cameron Miller, Kayla Miller and Michael Stone). Mathematics Magazine **(90** (2017), no. 3, 197--207**)** pdf (pruned published version: pdf).

Generalizations of the M&M Game (Snehesh Das, Steven J. Miller, Geremias Polanco, Yilong Wu, April Yang, Chris Yao), to appear in The PUMP Journal of Undergraduate Research pdf

M&M Game Rules

Cam (4 years): `If you're born on the same day, do you die on the same day?`



(1) Everyone starts off with k M&Ms (we did 5).
(2) All toss fair coins, eat an M&M if and only if head.

Be active – ask questions!

What are natural questions to ask?

Be active – ask questions!

What are natural questions to ask?

Question 1: How likely is a tie (as a function of k)?

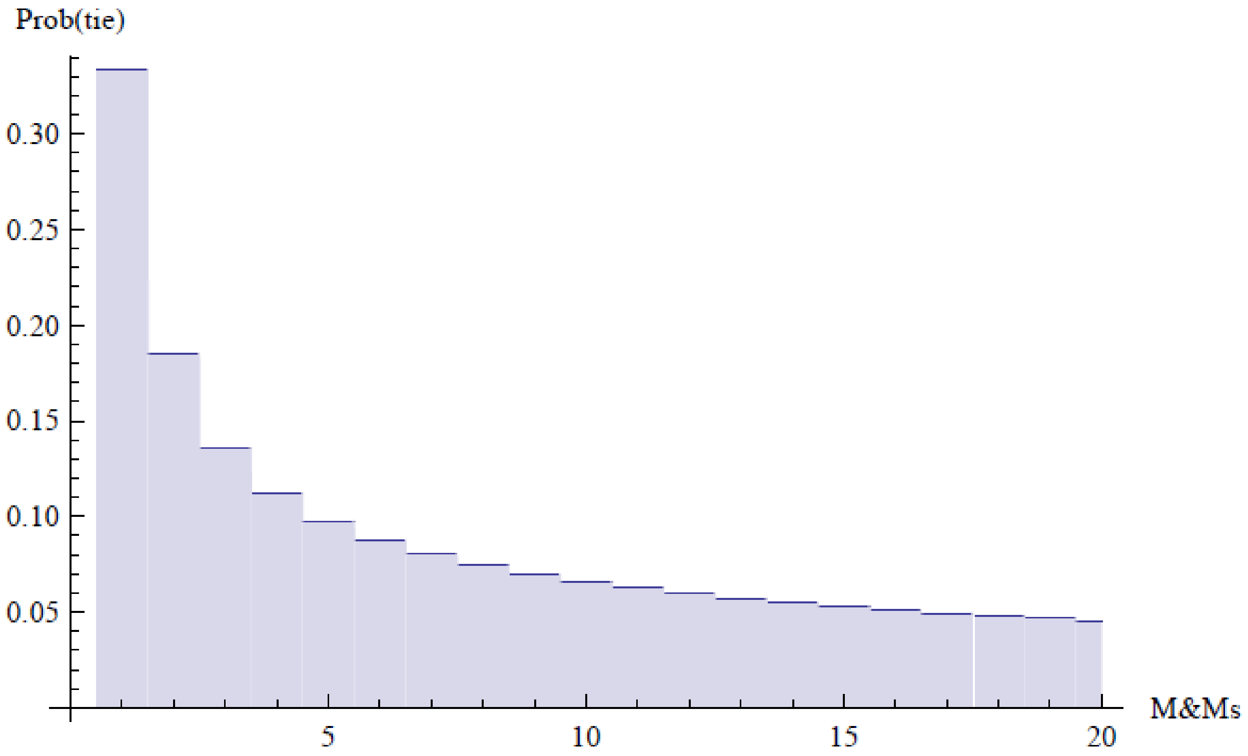
Question 2: How long until one dies?

Question 3: Generalize the game: More people? Biased coin?

Important to ask questions – curiousity is good and to be encouraged! Value to the journey and not knowing the answer.

Let's gather some data!

Probability of a tie in the M&M game (2 players)

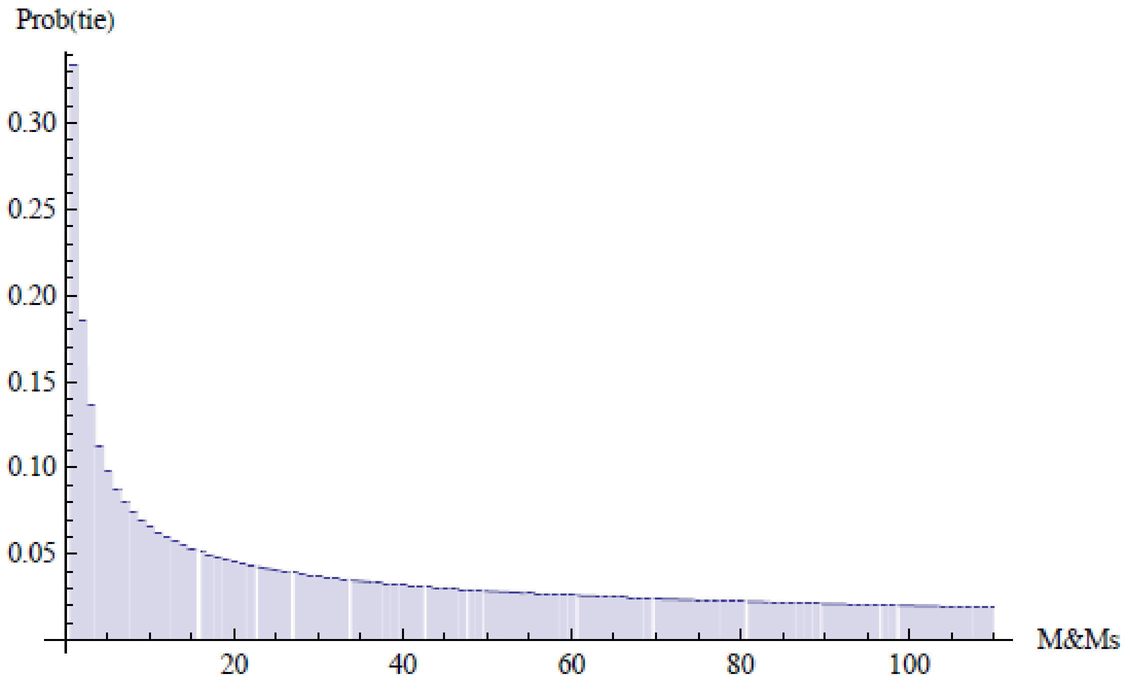


Prob(tie) $\approx$ 33% (1 M&M), 19% (2 M&Ms), 14% (3 M&Ms), 10% (4 M&Ms).

Welcome to Statistics and Inference!

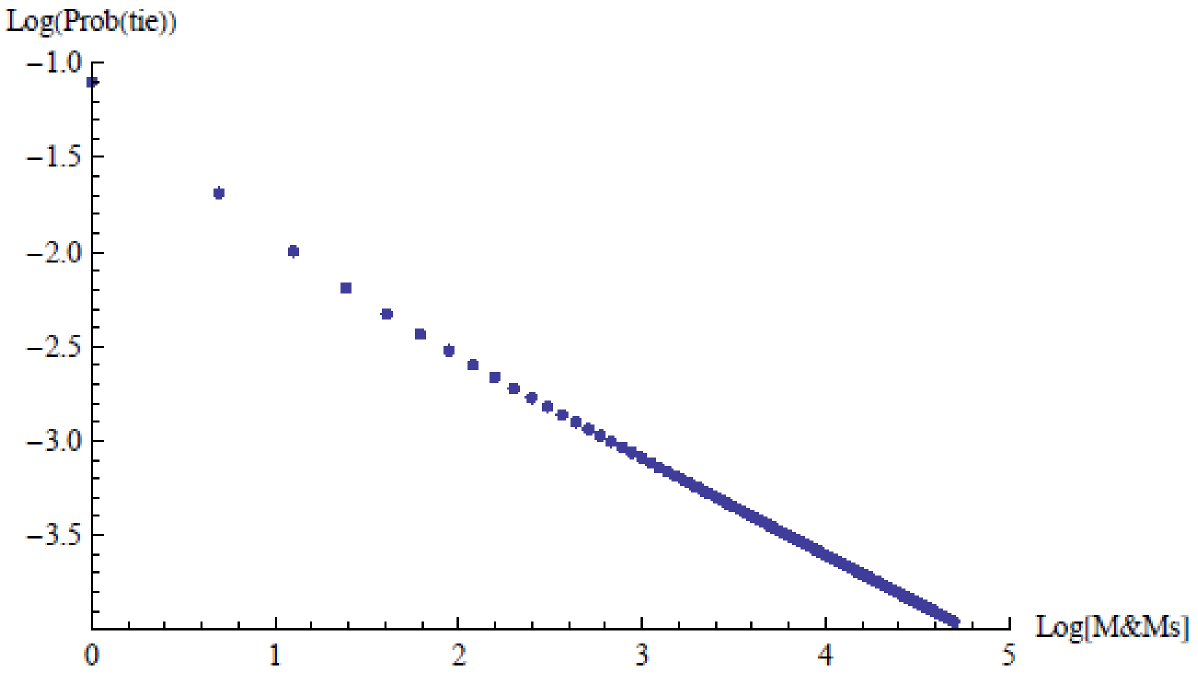
- Goal: Gather data, see pattern, extrapolate.
- Methods: Simulation, analysis of special cases.
- Presentation: It matters how we show data, and which data we show.

Viewing M&M Plots



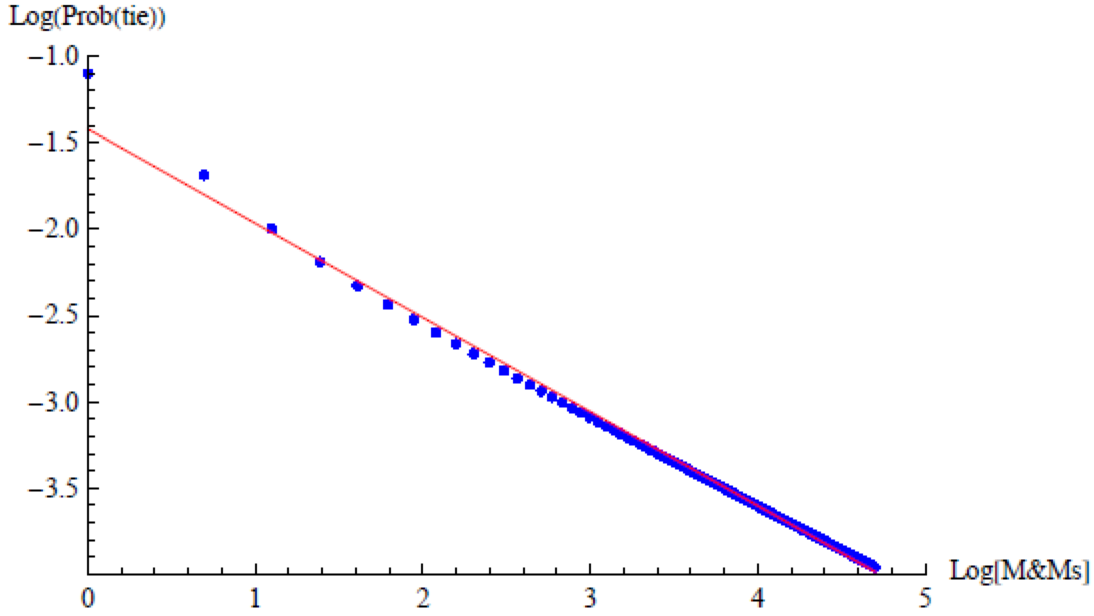
Hard to predict what comes next.

Viewing M&M Plots: Log-Log Plot



Not *just* sadistic teachers: logarithms useful!

Viewing M&M Plots: Log-Log Plot



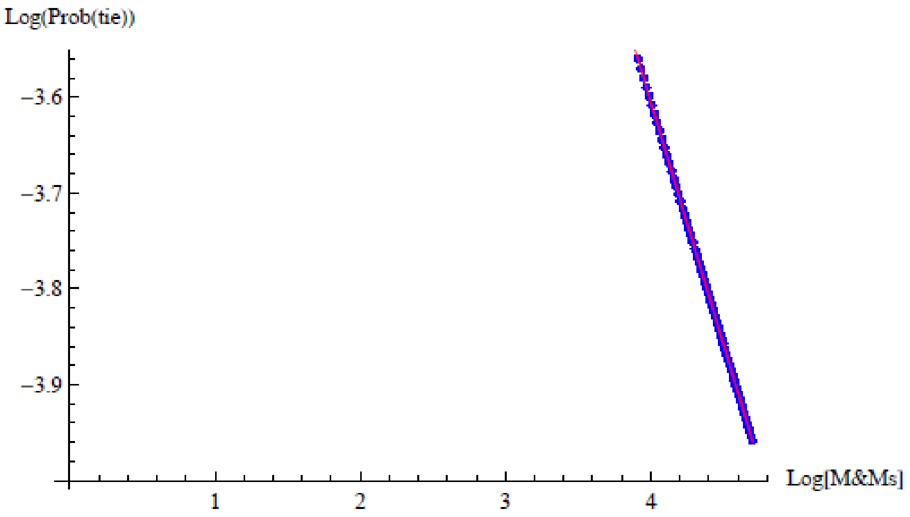
Best fit line:

$\log(\text{Prob}(\text{tie})) = -1.42022 - 0.545568 \log(\#\text{M\&Ms})$ or $\text{Prob}(k) \approx 0.2412/k^{.5456}$.

Predicts probability of a tie when $k = 220$ is 0.01274, but answer is 0.0137. **What gives?**

Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from $k = 50$ to 110.



Best fit line:

$\log(\mathrm{Prob(tie)}) = -1.58261 - 0.50553 \log(\#\mathrm{M\&Ms})$ or

$\mathrm{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).

Get 0.01344 for $k = 220$ (answer 0.01347); **much better!**

Solving the M&M Game

Overpower with algebra: Assume k M&Ms, two people, fair coins:

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2} \cdot \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is a binomial coefficient.

"Simplifies" to $4^{-k}\ {}_2F_1(k, k, 1, 1/4)$, a special value of a hypergeometric function! (Look up / write report.)

Obviously way beyond the classroom – is there a better way?

Solving the M&M Game (cont)

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.
Each person has exactly $k-1$ heads in first $n-1$ tosses, then ends with a head.

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}.$$

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Solving the M&M Game (cont)

Use the lesson from the Hoops Game: Memoryless process!

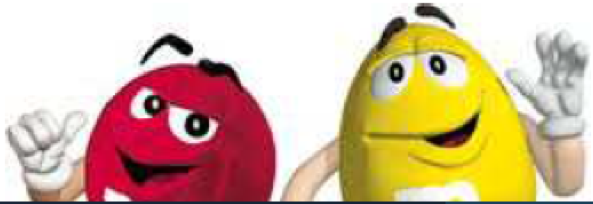
If neither eat, as if toss didn't happen. Now game is finite.

Much better perspective: each "turn" one of three equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is 1/3 or about 33%

$$\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^{n} \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \binom{1}{1} \frac{1}{3}.$$



Solving the M&M Game (cont)

Interpretation: Let Cam have c M&Ms and Kayla have k; write as (c, k).

Then each of the following happens 1/3 of the time after a 'turn':

- $(c, k) \longrightarrow (c-1, k-1)$.
- $(c, k) \longrightarrow (c-1, k)$.
- $(c, k) \longrightarrow (c, k-1)$.

Solving the M&M Game (cont): Assume $k = 4$

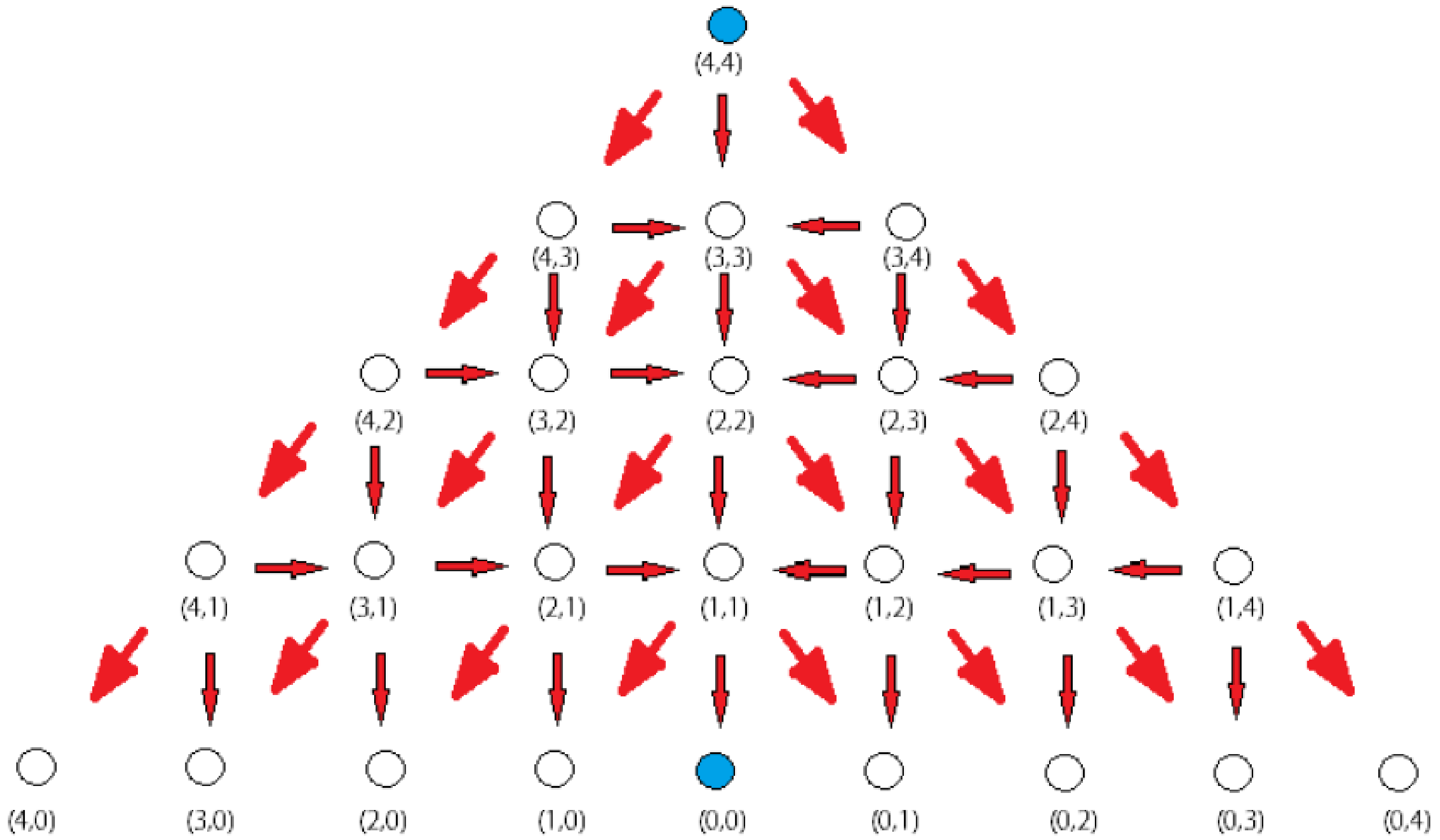


Figure: The M&M game when $k = 4$. Count the paths! Answer 1/3 of probability hit (1,1).

Solving the M&M Game (cont): Assume $k = 4$

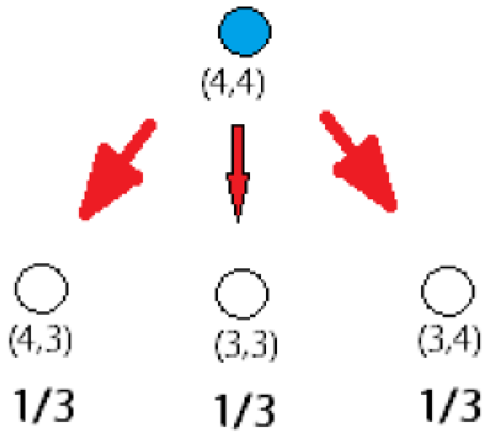


Figure: The M&M game when $k = 4$, going down one level.

Solving the M&M Game (cont): Assume $k = 4$

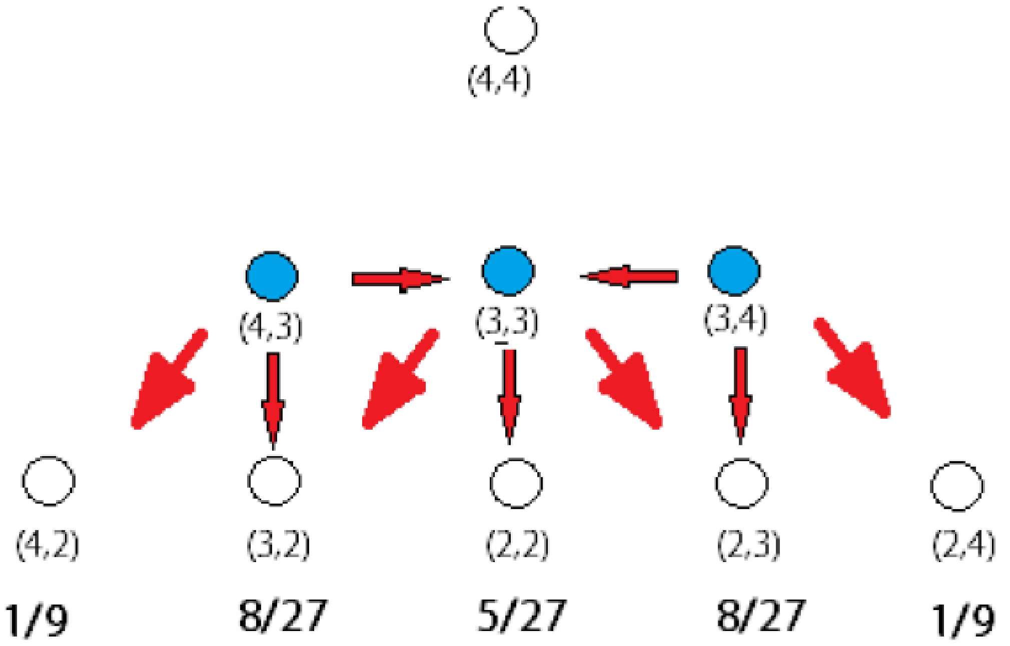


Figure: The M&M game when $k = 4$, removing probability from the second level.

Solving the M&M Game (cont): Assume $k = 4$

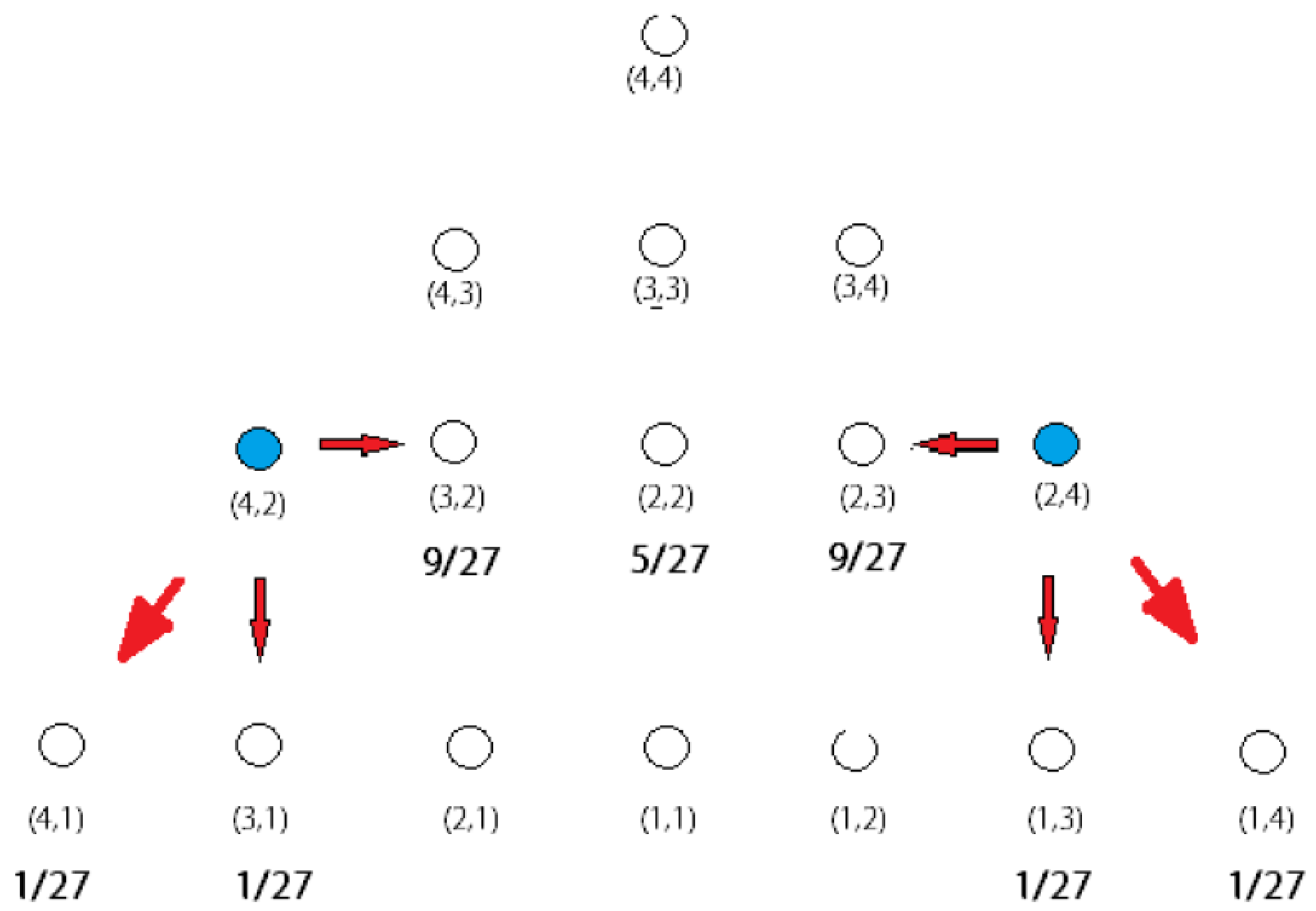


Figure: Removing probability from two outer on third level.

Solving the M&M Game (cont): Assume $k = 4$

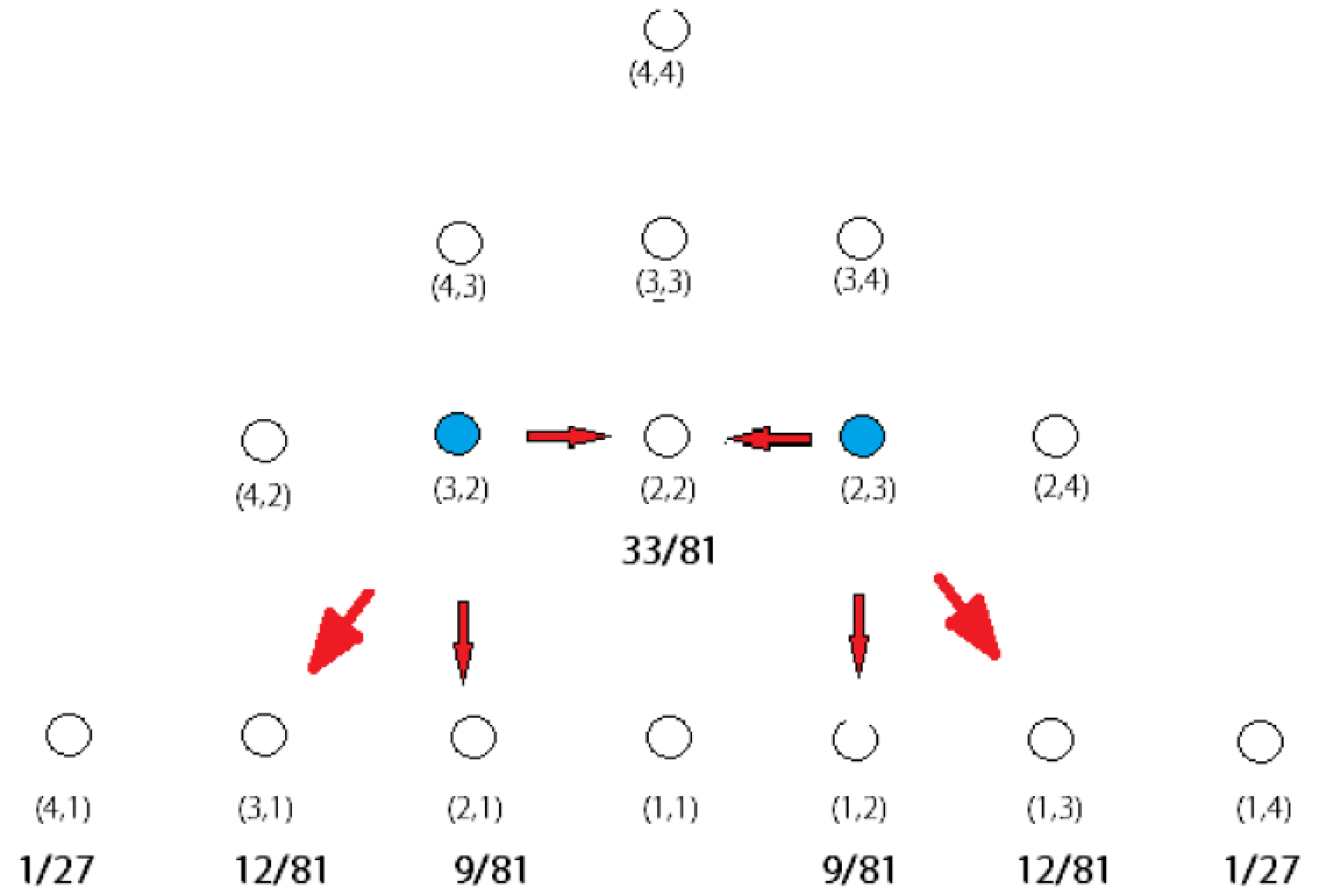


Figure: Removing probability from the (3,2) and (2,3) vertices.

Solving the M&M Game (cont): Assume $k = 4$

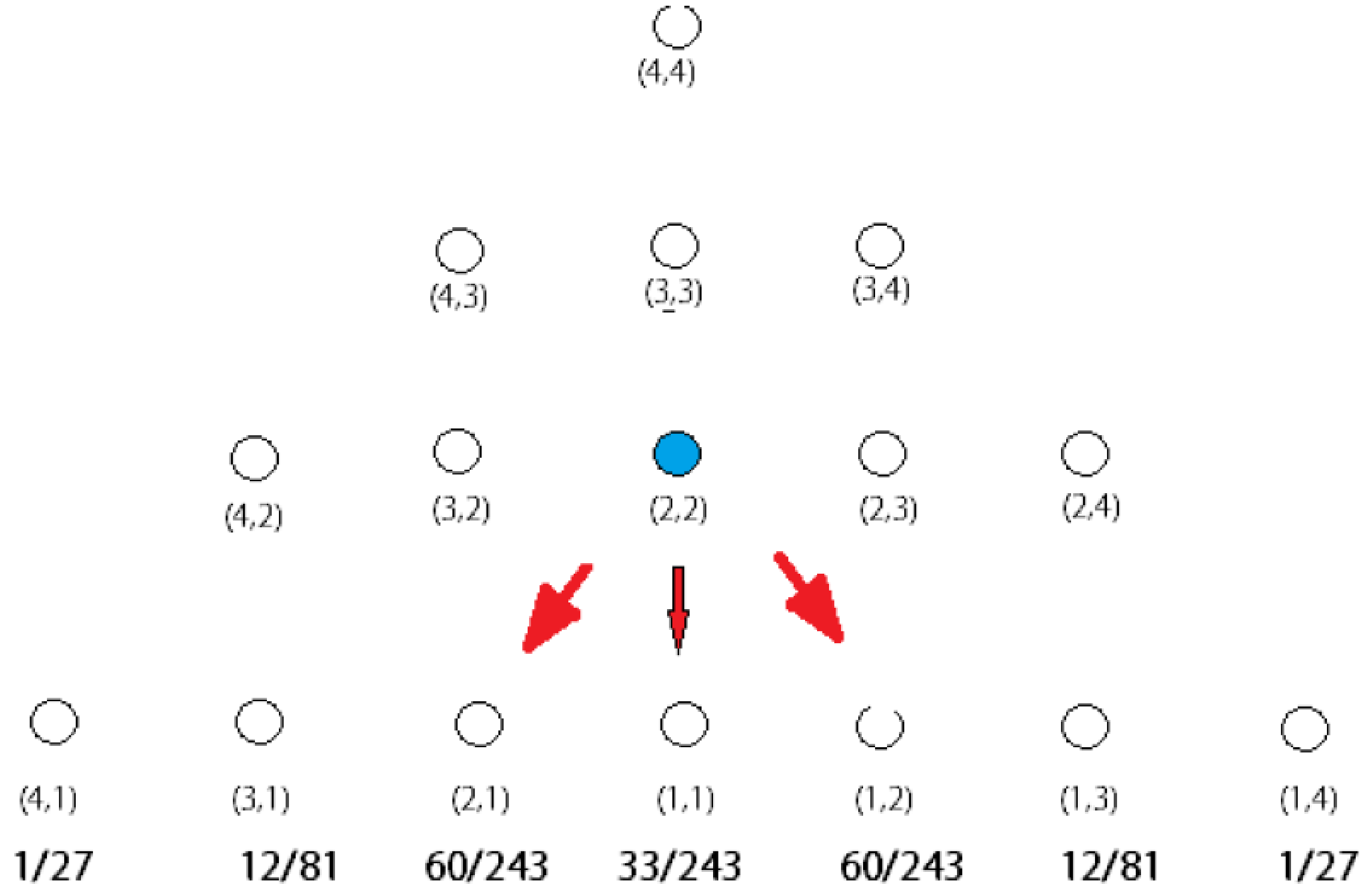


Figure: Removing probability from the (2,2) vertex.

Solving the M&M Game (cont): Assume $k = 4$

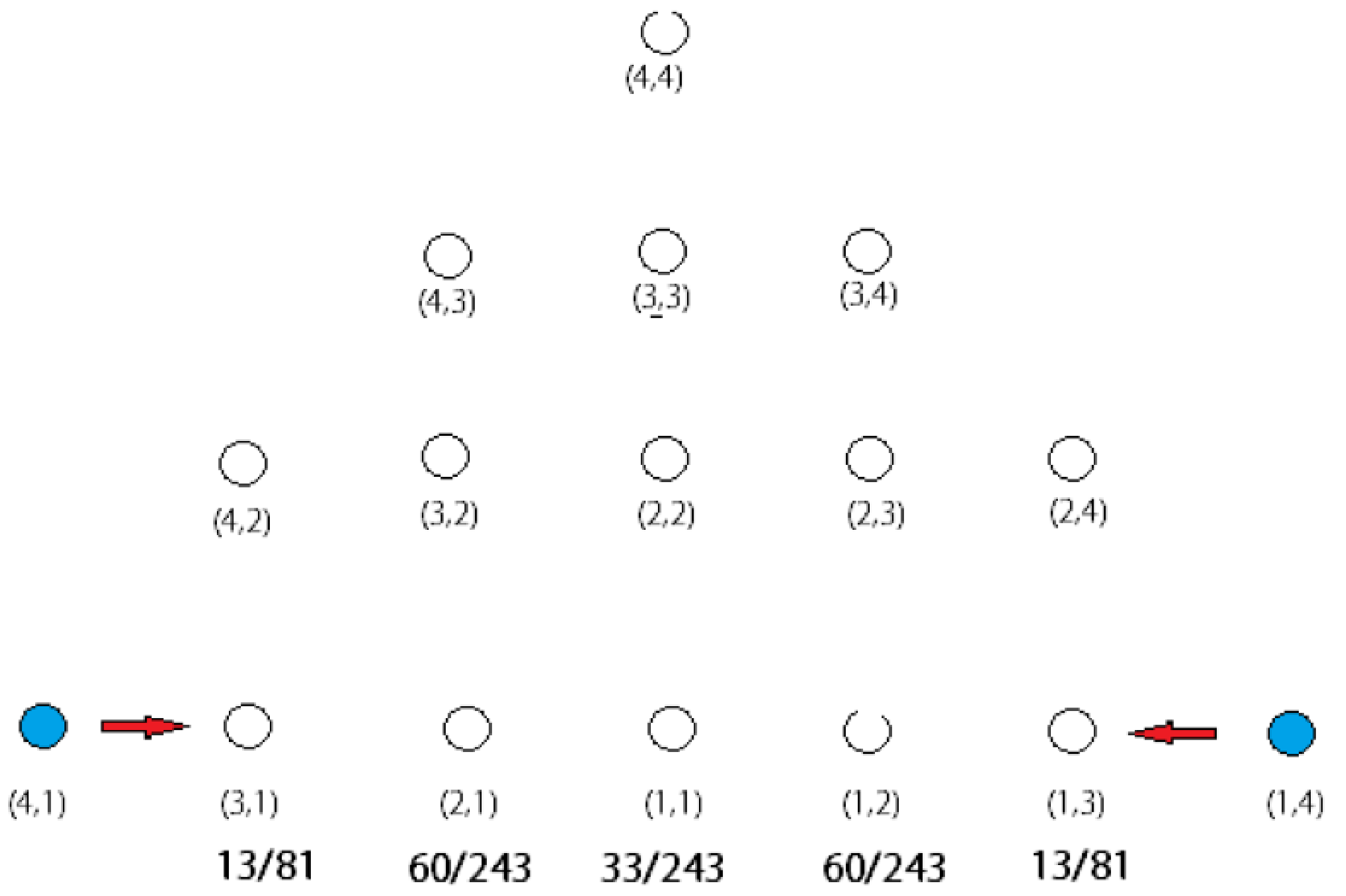


Figure: Removing probability from the (4,1) and (1,4) vertices.

Solving the M&M Game (cont): Assume $k = 4$

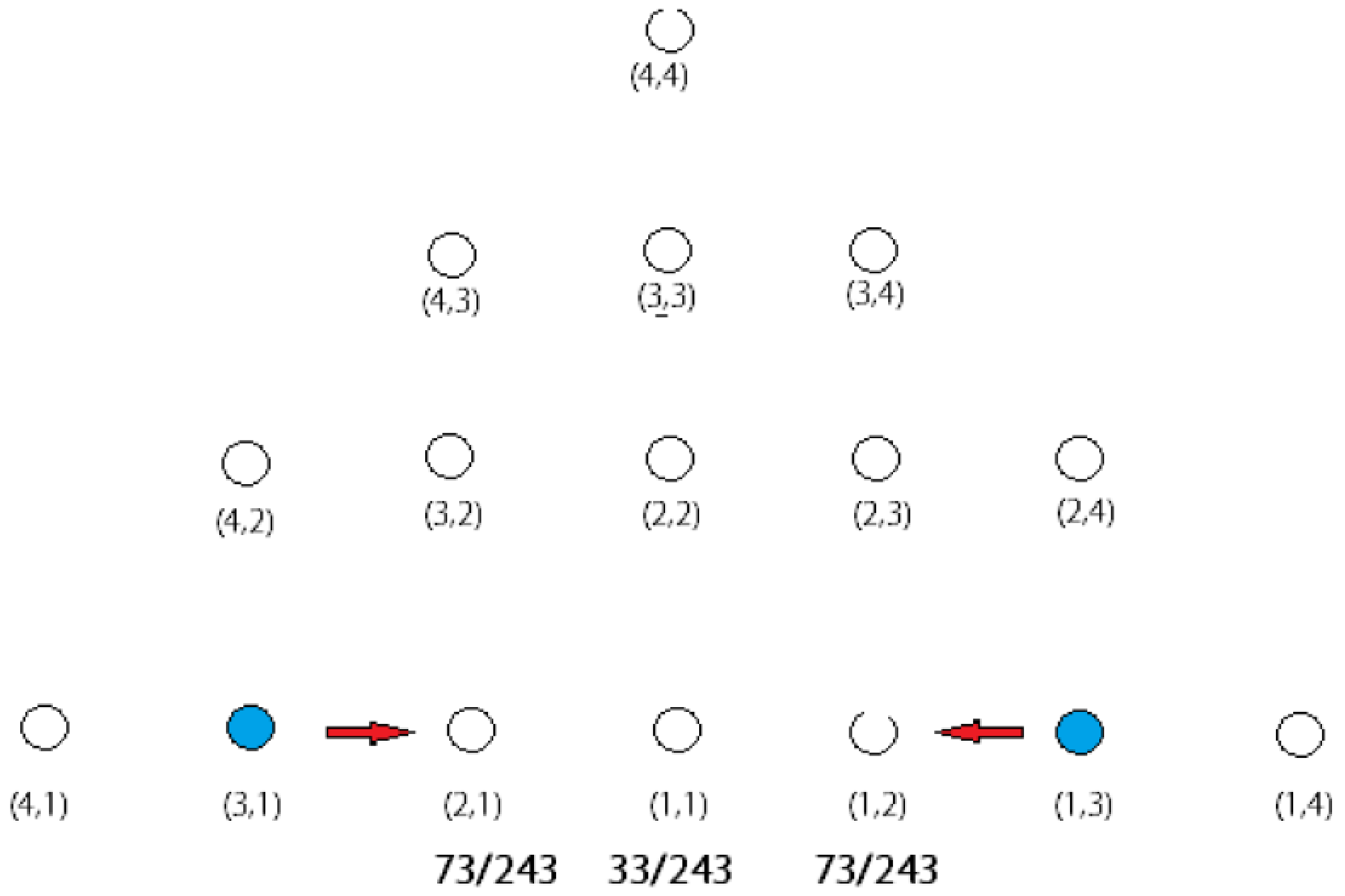


Figure: Removing probability from the (3,1) and (1,3) vertices.

Solving the M&M Game (cont): Assume $k = 4$

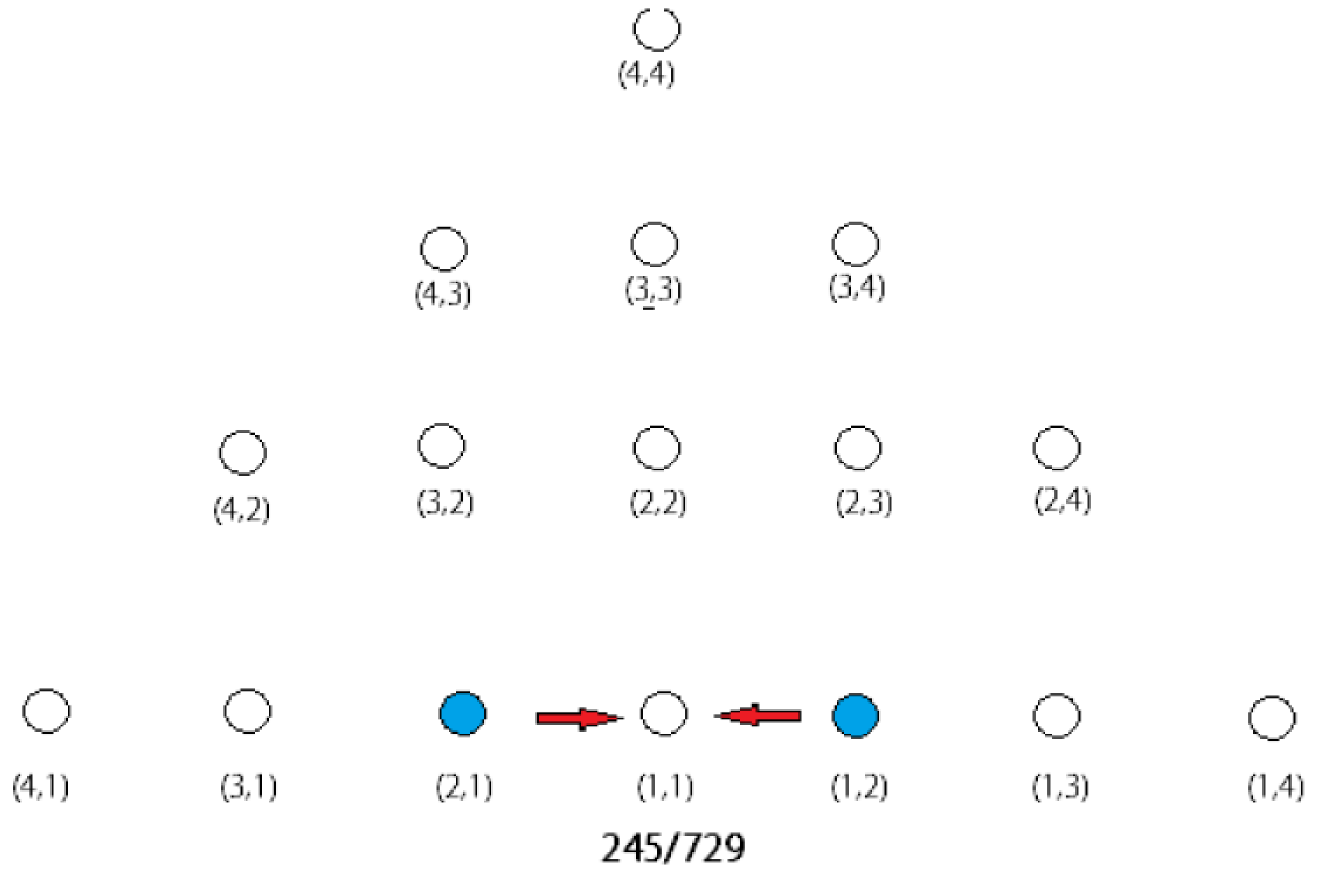


Figure: Removing probability from (2,1) and (1,2) vertices. Answer is 1/3 of (1,1) vertex, or 245/2187 (about 11%).

Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonaccis: $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0, F_1 = 1$.

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, ...
http://www.youtube.com/watch?v=kkGeOWYOFoA.

Binet's Formula (can prove via 'generating functions'):

$$F_n \ = \ \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

M&Ms: For $c, k \geq 1$: $x_{c,0} = x_{0,k} = 0$; $x_{0,0} = 1$, and if $c, k \geq 1$:

$$x_{c,k} \ = \ \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Reproduces the tree but a lot 'cleaner'.

Solving the Recurrence

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1.$

- $x_{1,0} = x_{0,1} = 0.$
- $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%.$

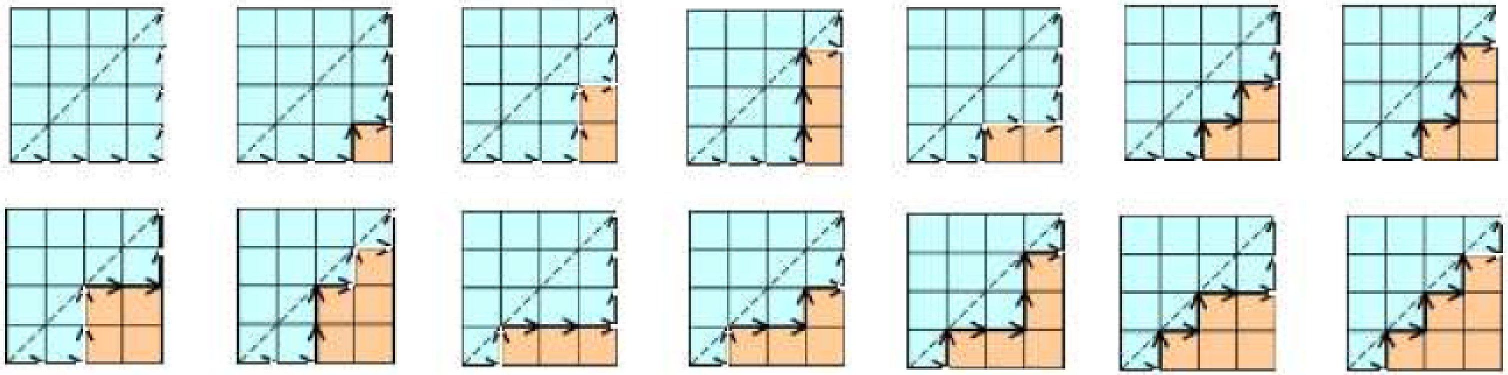
- $x_{2,0} = x_{0,2} = 0.$
- $x_{2,1} = \frac{1}{3}x_{1,0} + \frac{1}{3}x_{1,1} + \frac{1}{3}x_{2,0} = \frac{1}{9} = x_{1,2}.$
- $x_{2,2} = \frac{1}{3}x_{1,1} + \frac{1}{3}x_{1,2} + \frac{1}{3}x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} = \frac{5}{27} \approx 18.5\%.$

Try Simpler Cases!!!

Try and find an easier problem and build intuition.

Walking from (0,0) to (k, k) with allowable steps (1,0), (0,1) and (1,1), hit (k, k) before hit top or right sides.

Generalization of the Catalan problem. There don't have (1,1) and stay on or below the main diagonal.



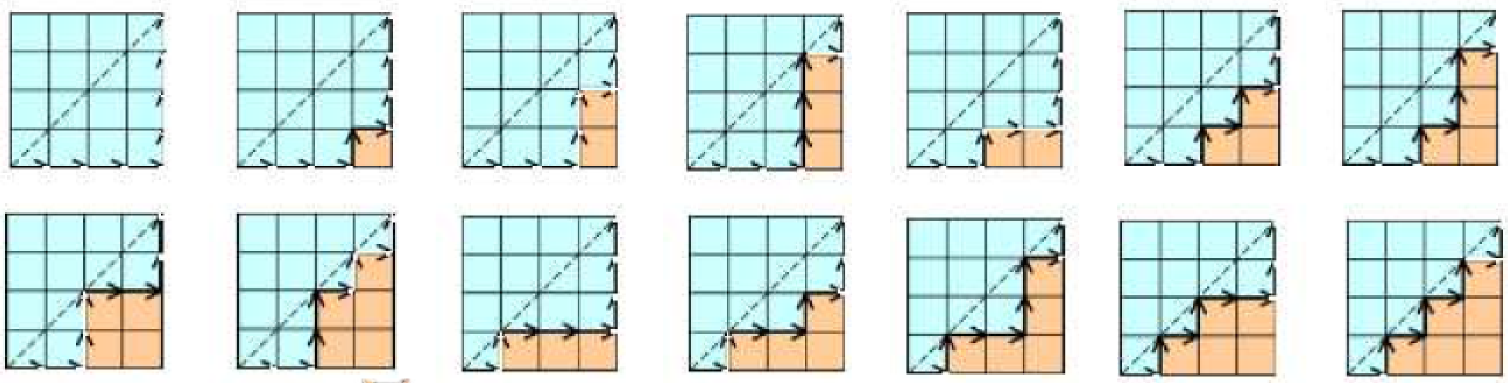
Interpretation: Catalan numbers are valid placings of (and).

Try Simpler Cases!!!

Try and find an easier problem and build intuition.

Walking from (0,0) to (k, k) with allowable steps (1,0), (0,1) and (1,1), hit (k, k) before hit top or right sides.

Generalization of the alan problem. There don't have (1,1) and stay on or below the main diagonal.



Interpretation: alan numbers are valid placings of (and).

Examining Probabilities of a Tie

When $k = 1$, Prob(tie) = 1/3.

When $k = 2$, Prob(tie) = 5/27.

When $k = 3$, Prob(tie) = 11/81.

When $k = 4$, Prob(tie) = 245/2187.

When $k = 5$, Prob(tie) = 1921/19683.

When $k = 6$, Prob(tie) = 575/6561.

When $k = 7$, Prob(tie) = 42635/531441.

When $k = 8$, Prob(tie) = 355975/4782969.

Examining Ties: Multiply by 3^{2k-1} to clear denominators.

When $k = 1$, get 1.

When $k = 2$, get 5.

When $k = 3$, get 33.

When $k = 4$, get 245.

When $k = 5$, get 1921.

When $k = 6$, get 15525.

When $k = 7$, get 127905.

When $k = 8$, get 1067925.

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ...

OEIS: `http://oeis.org/`.

Our sequence: `http://oeis.org/A084771`.

The web exists! Use it to build conjectures, suggest proofs....

OEIS (continued)

A084771 Coefficients of 1/sqrt(1-10*x+9*x^2); also, a(n) is the central coefficient of (1+5*x+4*x^2)^n. 5

1, 5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808673, 5614995765, 48416454529, 418895174885, 3634723102113, 31616937184725, 275621102802945, 2407331941640325, 21061836725455905, 184550106298084725 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. - N-E. Fahssi, Mar 30 2008

Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and three kinds of steps (1,1). [Joerg Arndt, Jul 01 2011]

Sums of squares of coefficients of (1+2*x)^n. [Joerg Arndt, Jul 06 2011]

The Hankel transform of this sequence gives A103488 . - Philippe DELEHAM, Dec 02 2007

REFERENCES Paul Barry and Aoife Hennessy, Generalized Narayana Polynomials, Riordan Arrays, and Lattice Paths, Journal of Integer Sequences, Vol. 15, 2012, #12.4.8.- From N. J. A. Sloane, Oct 08 2012

Michael Z. Spivey and Laura L. Steil, The k-Binomial Transforms and the Hankel Transform, Journal of Integer Sequences, Vol. 9 (2006), Article 06.1.1.

LINKS Table of n, a(n) for n=0..19.

Tony D. Noe, On the Divisibility of Generalized Central Trinomial Coefficients, Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7.

FORMULA G.f.: 1/sqrt(1-10*x+9*x^2).

Binomial transform of A059304. G.f.: Sum_{k>=0} binomial(2*k, k)*(2*x)^k/(1-x)^(k+1). E.g.f.: exp(5*x)*BesselI(0, 4*x). - Vladeta Jovovic (vladeta(AT)eunet.rs), Aug 20 2003

a(n) = sum(k=0..n, sum(j=0..n-k, C(n,j)*C(n-j,k)*C(2*n-2*j,n-j))). - Paul Barry, May 19 2006

a(n) = sum(k=0..n, 4^k*(C(n,k))^2) [From heruneedollar (heruneedollar(AT)gmail.com), Mar 20 2010]

Asymptotic: a(n) ~ 3^(2*n+1)/(2*sqrt(2*Pi*n)). [Vaclav Kotesovec, Sep 11 2012]

Conjecture: n*a(n) +5*(-2*n+1)*a(n-1) +9*(n-1)*a(n-2)=0. - R. J. Mathar,

Lessons

- Always ask questions.
- Many ways to solve a problem.
- Experience is useful and a great guide.
- Need to look at the data the right way.
- Often don't know where the math will take you.
- Value of continuing education: more math is better.
- Connections: My favorite quote: `If all you have is a hammer, pretty soon every problem looks like a nail.`

Teşekkürler!

BONUS MATERIAL

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = ?$

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;

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Zeckendorf's Theorem

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Example: $51 = 34 + 17 = F_8 + 17$.

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Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 4 = F_8 + F_6 + 4$.

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + 1$.

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Example: $83 = 55 + 21 + 5 + 2 = F_9 + F_7 + F_4 + F_2$.
Observe: 51 miles $\approx$ 82.1 kilometers.

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Observe: 51 miles $\approx$ 82.1 kilometers.

Decomposition Interpretation: Fibonaccis unique sequence such that all integers sums of non-adjacent terms!

Central Limit Type Theorem

As $n \to \infty$ distribution of number of summands in Zeckendorf decomposition for $m \in [F_n, F_{n+1})$ is Gaussian (normal).

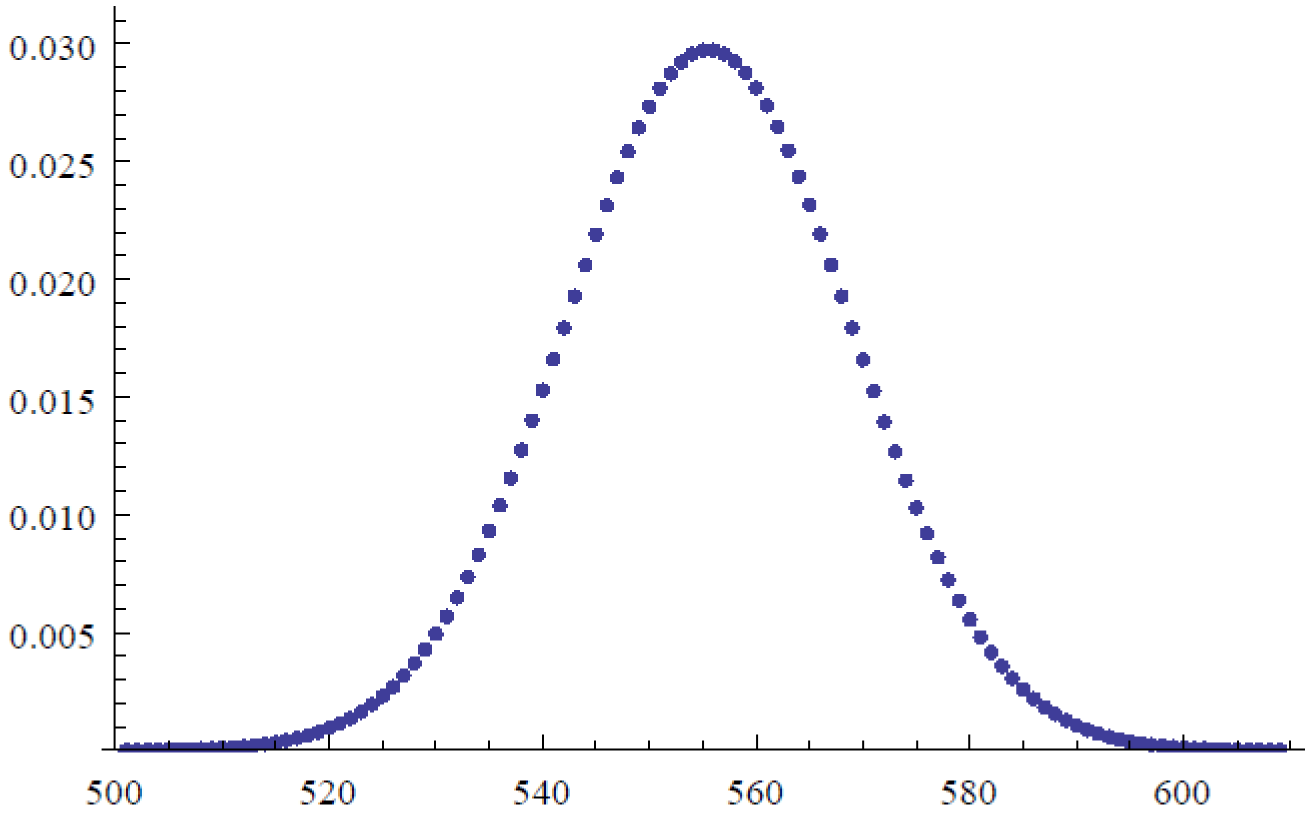

Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

$$m = \sum_{j=1}^{k(m)=n} F_{i_j}, \quad \nu_{m;n}(x) = \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Theorem (Zeckendorf Gap Distribution)

Gap measures $\nu_{m;n}$ converge almost surely to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.

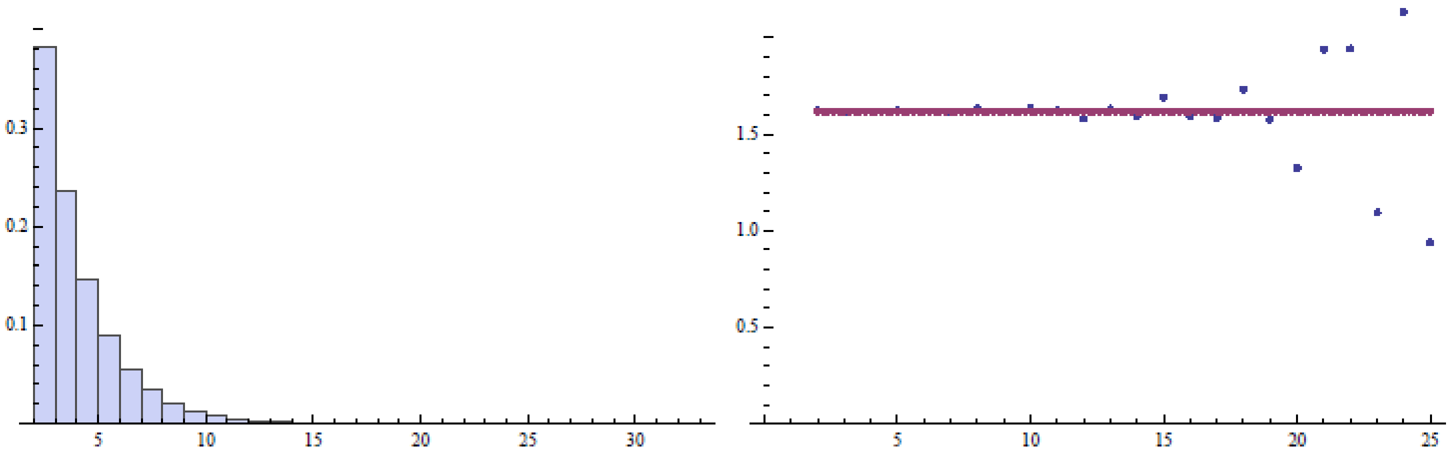

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{2010} \approx 10^{208}$.

Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

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Proof: Consider $C + P - 1$ cookies in a line.
Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.
Divides the cookies into P sets.

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Example: 8 cookies and 5 people ($C = 8$, $P = 5$):

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Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

Let $p_{n,k} = \#\left\{N \in [F_n, F_{n+1})\text{: the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\right\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n,\ i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1,\ d_j := i_j - i_{j-1} - 2\ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1,\ d_j \geq 0.$$

Cookie counting $\Rightarrow$ $p_{n,k} = \binom{n-2k+1\ +\ k-1}{k-1} = \binom{n-k}{k-1}$.