Brown University Algebra Seminar

The Effect of Zeros of Elliptic Curve L-Functions at the Central Point on Nearby Zeros

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Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium ($200+$ protons / neutrons) even worse!

Info by shooting high-energy neutrons into nucleus.

Fundamental Equation:Quantum Mechanics

$$H\psi_n \;=\; E_n\psi_n$$

Similar to stat mech, consider eigenvalues of ensembles of matrices.

Real Symmetric (GOE), Complex Hermitian (GUE), Classical Compact Groups.

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, $B \subset \mathbb{R}^{n-1}$ a compact box. Define the n-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\dots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i \neq j_k \leq N\right\}}{N}$$

Results on Zeros (Assuming GRH):

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko)
2. Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)
3. n-level corr. for all automorphic cupsidal L-functions (Rudnick-Sarnak)
4. n-level corr. for the classical compact groups (Katz-Sarnak)
5. **insensitive to any finite set of zeros**

Measures of Spacings: n-Level Density and Families

Let $\phi(x) = \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions, $\widehat{\phi}$ compactly supported.

$$D_{n,f}(\phi) \;\; = \;\; \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(C_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(C_f \gamma_f^{(j_n)}\right)$$

$C_f =$ Conductor, Scale factor for low zeros.

- individual zeros contribute in limit
- most of contribution is from low zeros
- average over similar curves (family)

$$D_{n,\mathcal{F}}(\phi) \;\; = \;\; \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} D_{n,f}(\phi).$$

Limiting Behavior

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{1,f}(\phi) &= \lim_{N\to\infty}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_j \phi\left(\frac{\gamma_f^{(j)}\log C_f}{2\pi}\right)\\ &= \int \phi(x) W_{1,\mathcal{G}(\mathcal{F})}(x)dx\\ &= \int \widehat{\phi}(y)\widehat{W}_{1,\mathcal{G}(\mathcal{F})}(y)dy.\end{aligned}$$

Density Conjecture: Distribution of low zeros of L-functions agree with the distribution of eigenvalues near 1 of a classical compact group.

1-Level Densities

Fourier Transforms for 1-level densities:

$$
\begin{aligned}
\widehat{W}_{1,SO(\text{even})}(u) &= \delta(u) + \tfrac{1}{2}\eta(u) \\
\widehat{W}_{1,SO}(u) &= \delta(u) + \tfrac{1}{2} \\
\widehat{W}_{1,SO(\text{odd})}(u) &= \delta(u) - \tfrac{1}{2}\eta(u) + 1 \\
\widehat{W}_{1,Sp}(u) &= \delta(u) - \tfrac{1}{2}\eta(u) \\
\widehat{W}_{1,U}(u) &= \delta(u)
\end{aligned}
$$

where $\delta(u)$ is the Dirac Delta functional and

$$
\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1. \end{cases}
$$

Elliptic Curves:

$$E_t : y^2 \ = \ x^3 + A(T)x + B(T), \ \ A(T), B(T) \in \mathbb{Z}(T).$$

$$a_t(p) \ = \ - \sum_{x \bmod p} \left(\frac{x^3 + A(t)x + B(t)}{p} \right) \ = \ a_{t+mp}(p)$$

$$L(E, s) \ = \ \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \ = \ \prod_p L_p(E, s).$$

By GRH: All zeros on the critical line.

Rational solutions: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus T$.

Birch and Swinnerton-Dyer Conjecture:
Geometric rank r = analytic rank (order of vanishing at central point).

Tools to Study Low Zeros

- explicit formula relating zeros and Fourier coeffs;
- averaging formulas for the family;
- conductors easy to control (constant or monotone)

1-Level Expansion

$$
\begin{aligned}
D_{1,\mathcal{F}}(\phi) &= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{j}\phi\left(\frac{\log C_E}{2\pi}\gamma_E^{(j)}\right) \\
&= \frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\widehat{\phi}(0)+\phi_i(0) \\
&\quad -\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log C_E}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log C_E}\right)a_E(p) \\
&\quad -\frac{2}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}\sum_{p}\frac{\log p}{\log C_E}\frac{1}{p^2}\widehat{\phi}\left(2\frac{\log p}{\log C_E}\right)a_E^2(p) \\
&\quad +O\left(\frac{\log\log C_E}{\log C_E}\right)
\end{aligned}
$$

Want to move $\frac{1}{|\mathcal{F}|}\sum_{E\in\mathcal{F}}$, Leads us to study

$$A_{r,\mathcal{F}}(p) = \sum_{t \bmod p} a_t^r(p), \quad r = 1 \text{ or } 2.$$

Input

For many families

$$(1) : A_{1,\mathcal{F}}(p) \ = \ -rp \ + \ O(1)$$

$$(2) : A_{2,\mathcal{F}}(p) \ = \ p^2 \ + \ O(p^{3/2})$$

Rational Elliptic Surfaces (Rosen and Silverman): If rank r over $\mathbb{Q}(T)$:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p \le X} -\frac{A_{1,\mathcal{F}}(p) \log p}{p} \ = \ r$$

Surfaces with $j(T)$ non-constant (Michel):

$$A_{2,\mathcal{F}}(p) = p^2 + O\left(p^{3/2}\right).$$

One-Level Result

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E\in\mathcal{F}_N} \sum_j \phi\left(\frac{\log C_E}{2\pi}\gamma_E^{(j)}\right) \ = \ \int \phi(x) W_{\mathcal{G}}(x) dx + \ r\phi(0),$$

where

$$\mathcal{G} \ = \ \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Forced zeros seem independent.

Interesting Families

Let $\mathcal{E} : y^2 = x^3 + A[t]x + B[T]$ be a one-parameter family of elliptic curves of rank r over $\mathbb{Q}$.

Natural sub-families

- Curves of rank r.
- Curves of rank $r + 2$.

Question: Does the sub-family of rank $r+2$ curves in a rank r family behave like the sub-family of rank $r+2$ curves in a rank $r + 2$ family?

Equivalently, does it matter how one conditions on a curve being rank $r + 2$?

Orthogonal Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

Independent Model:

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interaction Model:

Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$:

$$d\varepsilon_{2r}(\theta) \propto \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

with $1 \le j,k \le N-r$.

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Independent):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Comparing the RMT Models

Small support, 1-level densities for one-param families agree with $\rho_{r,\text{Indep}}$ and not $\rho_{r,\text{Inter}}$.

Curve E, conductor C_E, expect first zero $\frac{1}{2} + i\gamma_E^{(1)}$ with $\gamma_E^{(1)} \approx \frac{1}{\log C_E}$.

If r zeros at central point, if repulsion of zeros is of size $\frac{c_r}{\log C_E}$, might detect in 1-level density:

$$\frac{1}{|\mathcal{F}_N|} \sum_{E \in \mathcal{F}_N} \sum_j \phi\left(\frac{\gamma_E^{(j)} \log C_E}{2\pi}\right).$$

Corrections of size

$$\phi\left(x_0 + c_r\right) - \phi(x_0) \ \approx \ \phi'\left(x(x_0, c_r)\right) \cdot c_r.$$

Testing Random Matrix Theory Predictions

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: what percent have rank $\geq r + 2$?

2. **First (Normalized) Zero above Central Point:** Do extra zeros at the central point affect the distribution of zeros near the central point?

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(T)$.

RMT $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

Percent with rank r $\approx$ 32%
Percent with rank r+1 $\approx$ 48%
Percent with rank r+2 $\approx$ 18%
Percent with rank r+3 $\approx$ 2%

Problem: small data sets, sub-families, convergence rate $\log(\text{conductor})$.

Data on Excess Rank

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

Family: $a_1 : 0$ to 10, rest -10 to 10.

Percent with rank 0 = 28.60%
Percent with rank 1 = 47.56%
Percent with rank 2 = 20.97%
Percent with rank 3 = 2.79%
Percent with rank 4 = .08%

14 Hours, 2,139,291 curves (2,971 singular, 248,478 distinct).

Data on Excess Rank

$$y^2 + y = x^3 + Tx$$

Each data set 2000 curves from start.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{17}, but on logarithmic scale still small.

RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

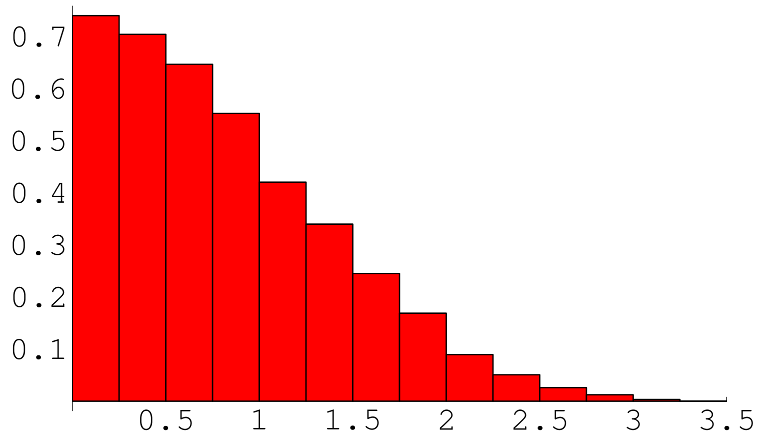

Figure 1a: 1st norm. evalue above 1: 23,040 SO(4) matrices
Mean = .709, Std Dev of the Mean = .601, Median = .709

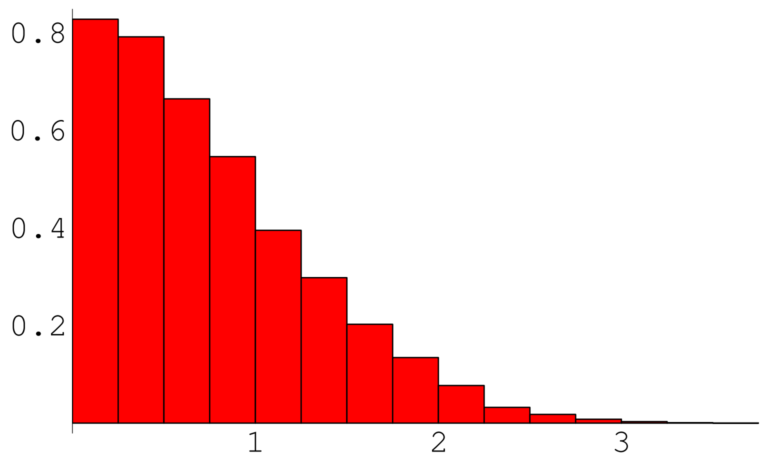

Figure 1b: 1st norm. evalue above 1: 23,040 SO(6) matrices
Mean = .635, Std Dev of the Mean = .574, Median = .635

Rank 0 Curves: 1st Normalized Zero above Central Point

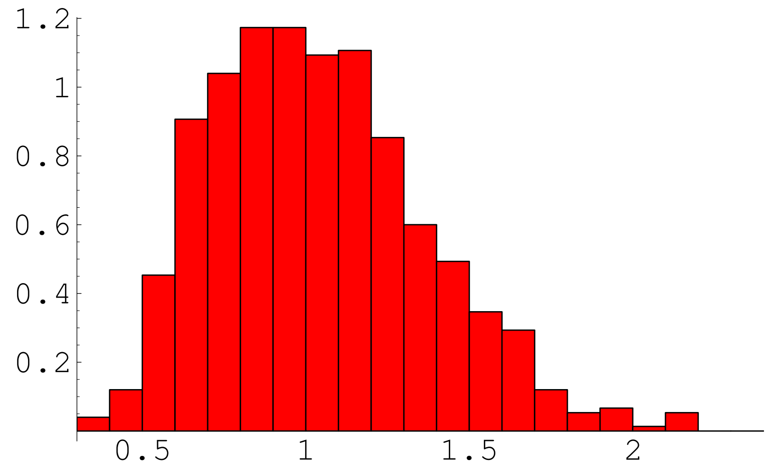

Figure 2a: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$, $\sigma_\mu = .32$

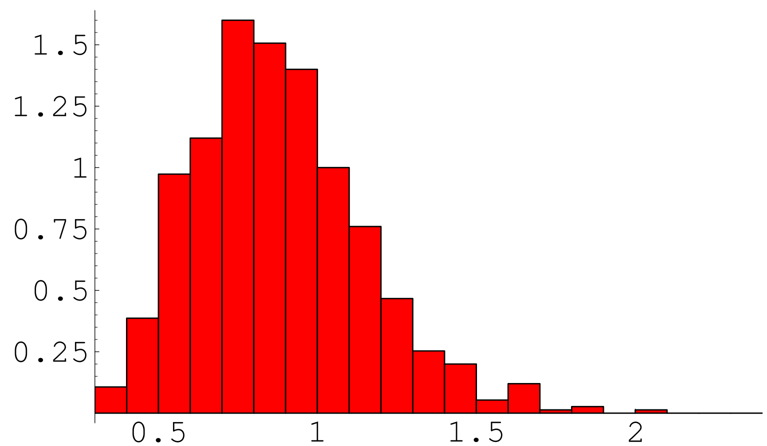

Figure 2b: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, $\sigma_\mu = .27$

Rank 2 Curves: 1st Norm. Zero above the Central Point

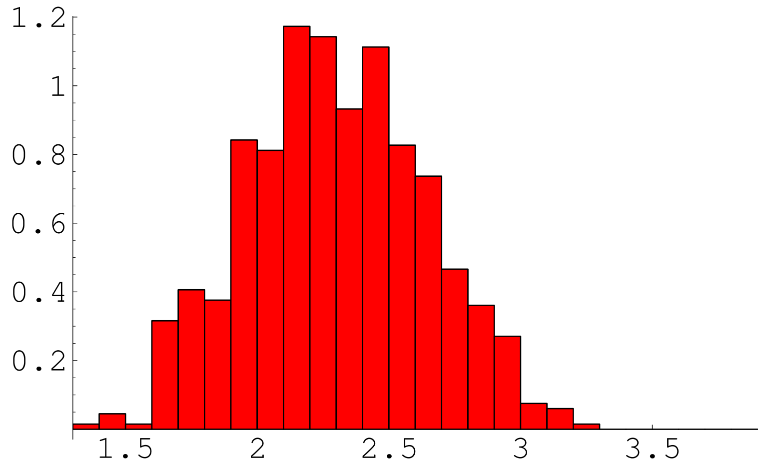

Figure 3a: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

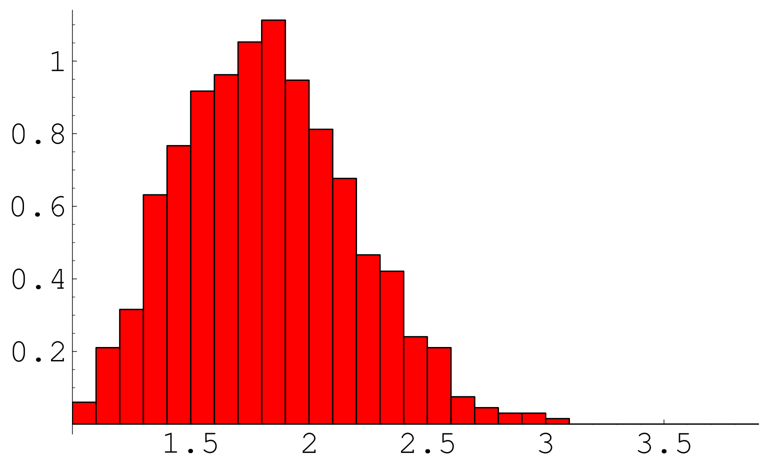

Figure 3b: 665 rank 2 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

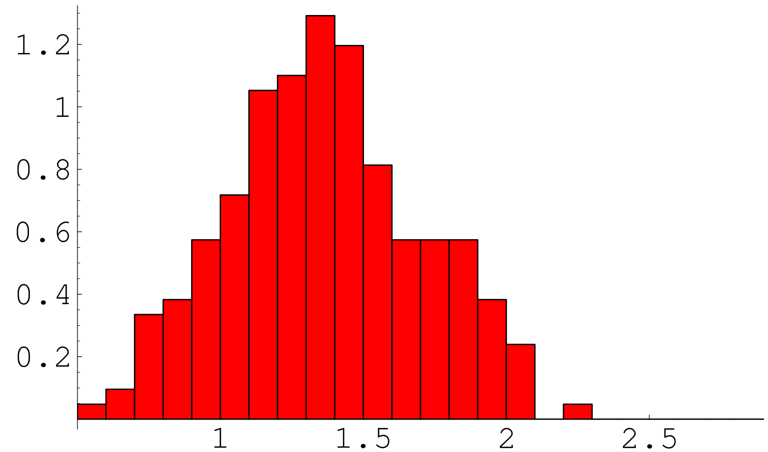

Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

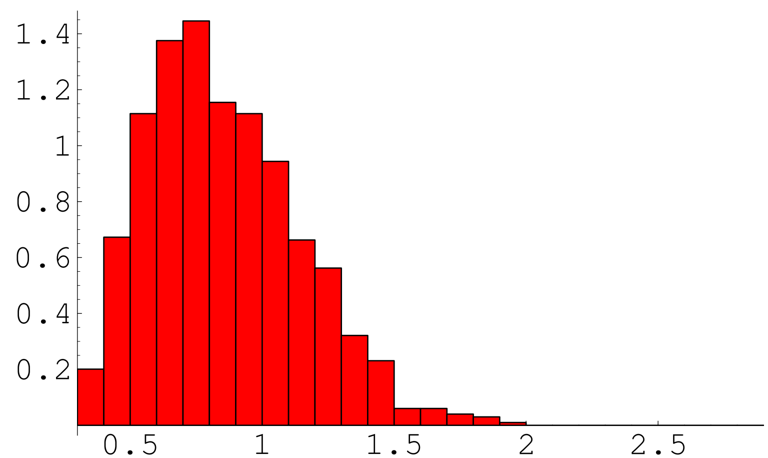

Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	Median $\widetilde{\mu}$	Mean μ	StDev σ_μ	log(conductor)	Number
1: [0,1,1,1,T]	1.28	1.33	0.26	[3.93, 9.66]	7
2: [1,0,0,1,T]	1.39	1.40	0.29	[4.66, 9.94]	11
3: [1,0,0,2,T]	1.40	1.41	0.33	[5.37, 9.97]	11
4: [1,0,0,-1,T]	1.50	1.42	0.37	[4.70, 9.98]	20
5: [1,0,0,-2,T]	1.40	1.48	0.32	[4.95, 9.85]	11
6: [1,0,0,T,0]	1.35	1.37	0.30	[4.74, 9.97]	44
7: [1,0,1,-2,T]	1.25	1.34	0.42	[4.04, 9.46]	10
8: [1,0,2,1,T]	1.40	1.41	0.33	[5.37, 9.97]	11
9: [1,0,-1,1,T]	1.39	1.32	0.25	[7.45, 9.96]	9
10: [1,0,-2,1,T]	1.34	1.34	0.42	[3.26, 9.56]	9
11: [1,1,-2,1,T]	1.21	1.19	0.41	[5.73, 9.92]	6
12: [1,1,-3,1,T]	1.32	1.32	0.32	[5.04, 9.98]	11
13: [1,-2,0,T,0]	1.31	1.29	0.37	[4.73, 9.91]	39
14: [-1,1,-3,1,T]	1.45	1.45	0.31	[5.76, 9.92]	10
All Curves	1.35	1.36	0.33	[3.26, 9.98]	209
Distinct Curves	1.35	1.36	0.33	[3.26, 9.98]	196

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	**Median** $\widetilde{\mu}$	**Mean** μ	**StDev** σ_μ	**log(conductor)**	**Number**
1: [0,1,1,1,T]	0.80	0.86	0.23	[15.02, 15.97]	49
2: [1,0,0,1,T]	0.91	0.93	0.29	[15.00, 15.99]	58
3: [1,0,0,2,T]	0.90	0.94	0.30	[15.00, 16.00]	55
4: [1,0,0,-1,T]	0.80	0.90	0.29	[15.02, 16.00]	59
5: [1,0,0,-2,T]	0.75	0.77	0.25	[15.04, 15.98]	53
6: [1,0,0,T,0]	0.75	0.82	0.27	[15.00, 16.00]	130
7: [1,0,1,-2,T]	0.84	0.84	0.25	[15.04, 15.99]	63
8: [1,0,2,1,T]	0.90	0.94	0.30	[15.00, 16.00]	55
9: [1,0,-1,1,T]	0.86	0.89	0.27	[15.02, 15.98]	57
10: [1,0,-2,1,T]	0.86	0.91	0.30	[15.03, 15.97]	59
11: [1,1,-2,1,T]	0.73	0.79	0.27	[15.00, 16.00]	124
12: [1,1,-3,1,T]	0.98	0.99	0.36	[15.01, 16.00]	66
13: [1,-2,0,T,0]	0.72	0.76	0.27	[15.00, 16.00]	120
14: [-1,1,-3,1,T]	0.90	0.91	0.24	[15.00, 15.99]	48
All Curves	0.81	0.86	0.29	[15.00,16.00]	996
Distinct Curves	0.81	0.86	0.28	[15.00,16.00]	863

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 0

first set $\log(\text{cond}) \in [15, 15.5)$; second set $\log(\text{cond}) \in [15.5, 16]$. Median $\widetilde{\mu}$, Mean μ, Std Dev (of Mean) σ_μ.

Family	$\widetilde{\mu}$	μ	σ_μ	Number	$\widetilde{\mu}$	μ	σ_μ	Number
1: [0,1,3,1,T]	1.59	1.83	0.49	8	1.71	1.81	0.40	19
2: [1,0,0,1,T]	1.84	1.99	0.44	11	1.81	1.83	0.43	14
3: [1,0,0,2,T]	2.05	2.03	0.26	16	2.08	1.94	0.48	19
4: [1,0,0,-1,T]	2.02	1.98	0.47	13	1.87	1.94	0.32	10
5: [1,0,0,T,0]	2.05	2.02	0.31	23	1.85	1.99	0.46	23
6: [1,0,1,1,T]	1.74	1.85	0.37	15	1.69	1.77	0.38	23
7: [1,0,1,2,T]	1.92	1.95	0.37	16	1.82	1.81	0.33	14
8: [1,0,1,-1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
9: [1,0,1,-2,T]	1.74	1.74	0.43	14	1.82	1.90	0.40	14
10: [1,0,-1,1,T]	2.00	2.00	0.32	22	1.81	1.94	0.42	18
11: [1,0,-2,1,T]	1.97	1.99	0.39	14	2.17	2.14	0.40	18
12: [1,0,-3,1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
13: [1,1,0,T,0]	1.89	1.88	0.31	20	1.82	1.88	0.39	26
14: [1,1,1,1,T]	2.31	2.21	0.41	16	1.75	1.86	0.44	15
15: [1,1,-1,1,T]	2.02	2.01	0.30	11	1.87	1.91	0.32	19
16: [1,1,-2,1,T]	1.95	1.91	0.33	26	1.98	1.97	0.26	18
17: [1,1,-3,1,T]	1.79	1.78	0.25	13	2.00	2.06	0.44	16
18: [1,-2,0,T,0]	1.97	2.05	0.33	24	1.91	1.92	0.44	24
19: [-1,1,0,1,T]	2.11	2.12	0.40	21	1.71	1.88	0.43	17
20: [-1,1,-2,1,T]	1.86	1.92	0.28	23	1.95	1.90	0.36	18
21: [-1,1,-3,1,T]	2.07	2.12	0.57	14	1.81	1.81	0.41	19
All Curves	1.95	1.97	0.37	350	1.85	1.90	0.40	388
Distinct Curves	1.95	1.97	0.37	335	1.85	1.91	0.40	366

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 0 over $\mathbb{Q}(T)$

- Observe the medians and means of the small conductor set to be larger than those from the large conductor set.

- For all curves the Pooled and Unpooled Two-Sample t-Procedure give t-statistics of 2.5 with over 600 degrees of freedom.

- For distinct curves the t-statistics is 2.16 (respectively 2.17) with over 600 degrees of freedom (about a 3% chance).

- Provides evidence against the null hypothesis (that the means are equal) at the $.05$ confidence level (though not at the $.01$ confidence level).

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 0 over $\mathbb{Q}(T)$

Apply non-parametric tests to further support our claim that the repulsion decreases as the conductors increase.

- Write a plus sign if the small conductor set has a larger mean and a minus sign if not.

- Observe four minus signs and seventeen plus signs.

- The null hypothesis is that each is equally likely to be larger; thus the number of minus signs is a random variable from a binomial distribution with $N = 21$ and $\theta = \frac{1}{2}$.

- The probability of observing four or fewer minus signs is about 3.6%, supporting the claim of decreasing repulsion with increasing conductor.

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$
(Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point

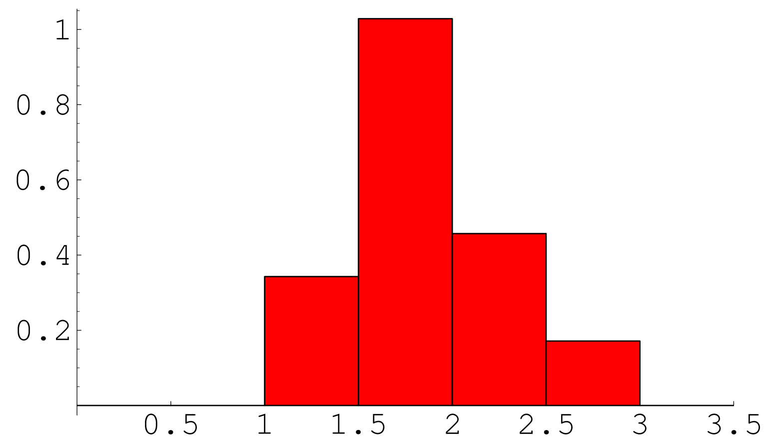

Figure $5a$:35 **curves**, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

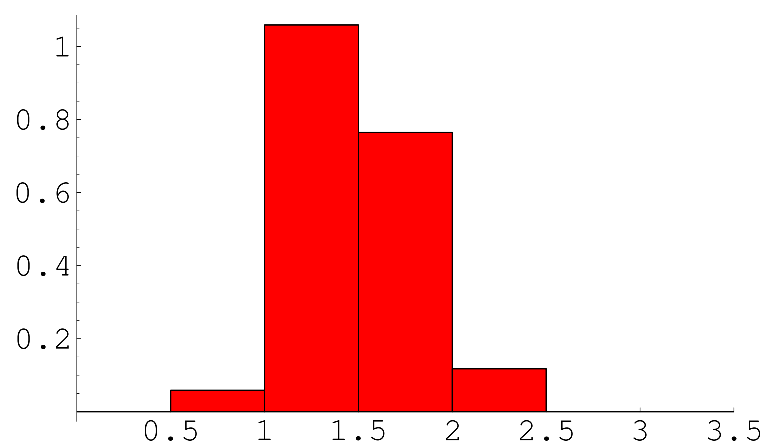

Figure $5b$:34 **curves**, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 2 over $\mathbb{Q}(T)$

$\log(\text{cond}) \in [15, 16]$, $t \in [0, 120]$, median is 1.64.

Family	Mean	Standard Deviation	log(conductor)	Number
1: [1,T,0,-3-2T,1]	1.91	0.25	[15.74,16.00]	2
2: [1,T,-19,-T-1,0]	1.57	0.36	[15.17,15.63]	4
3: [1,T,2,-T-1,0]	1.29		[15.47, 15.47]	1
4: [1,T,-16,-T-1,0]	1.75	0.19	[15.07,15.86]	4
5: [1,T,13,-T-1,0]	1.53	0.25	[15.08,15.91]	3
6: [1,T,-14,-T-1,0]	1.69	0.32	[15.06,15.22]	3
7: [1,T,10,-T-1,0]	1.62	0.28	[15.70,15.89]	3
8: [0,T,11,-T-1,0]	1.98		[15.87,15.87]	1
9: [1,T,-11,-T-1,0]				
10: [0,T,7,-T-1,0]	1.54	0.17	[15.08,15.90]	7
11: [1,T,-8,-T-1,0]	1.58	0.18	[15.23,25.95]	6
12: [1,T,19,-T-1,0]				
13: [0,T,3,-T-1,0]	1.96	0.25	[15.23, 15.66]	3
14: [0,T,19,-T-1,0]				
15: [1,T,17,-T-1,0]	1.64	0.23	[15.09, 15.98]	4
16: [0,T,9,-T-1,0]	1.59	0.29	[15.01, 15.85]	5
17: [0,T,1,-T-1,0]	1.51		[15.99, 15.99]	1
18: [1,T,-7,-T-1,0]	1.45	0.23	[15.14, 15.43]	4
19: [1,T,8,-T-1,0]	1.53	0.24	[15.02, 15.89]	10
20: [1,T,-2,-T-1,0]	1.60		[15.98, 15.98]	1
21: [0,T,13,-T-1,0]	1.67	0.01	[15.01, 15.92]	2
All Curves	1.61	0.25	[15.01, 16.00]	64

Repulsion or Attraction?

Conductors in $[15, 16]$; first set is rank 0 curves from 14 one-parameter families of rank 0 over $\mathbb{Q}$; second set rank 2 curves from 21 one-parameter families of rank 0 over $\mathbb{Q}$. The t-statistics exceed 6.

Family	**2nd vs 1st Zero**	**3rd vs 2nd Zero**	**Number**
Rank 0 Curves	2.16	3.41	863
Rank 2 Curves	1.93	3.27	701

The additional repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Comparison b/w One-Param Families of Different Rank

First normalized zero above the central point.

- The first family is the 701 rank 2 curves from the 21 one-parameter families of rank 0 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$;
- the second family is the 64 rank 2 curves from the 21 one-parameter families of rank 2 over $\mathbb{Q}(T)$ with $\log(\text{cond}) \in [15, 16]$.

Family	**Median**	**Mean**	**Std. Dev.**	**Number**
Rank 2 Curves from Rank 0 Families	1.926	1.936	0.388	701
Rank 2 Curves from Rank 2 Families	1.642	1.610	0.247	64

- t-statistic is 6.60, indicating the means differ.
- The mean of the first normalized zero of rank 2 curves in a family above the central point (for conductors in this range) depends on *how* we choose the curves.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- While the normalized zeros are repelled from the central point (and by different amounts for the two sets), the *differences* between the normalized zeros are statistically independent of this repulsion (t-statistics < 2).

- While for a given range of log-conductors the average second normalized zero of a rank 0 curve is close to the average first normalized zero of a rank 2 curve, they are not the same and the additional repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Spacings b/w Norm Zeros: Rank 2 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	**23 Rank 4 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

Conclusions and Future Work

- Theoretical supports the Independent Model and Birch and Swinnerton-Dyer Conjecture for one-parameter families over $\mathbb{Q}(T)$ as the conductors tend to infinity.

- Experimental suggests a different answer for finite conductors:
  - ◇ First normalized zero is repelled by zeros at the central point.
  - ◇ The more central point zeros the greater the repulsion.
  - ◇ Repulsion decreases as the conductor increases.
  - ◇ Difference b/w adjacent normalized zeros stat. indep. of the repulsion.

- What is the right model for rank $r+2$ curves from rank r one-parameter families over $\mathbb{Q}(T)$: Independent, Interaction or other?

- Unlike the excess rank investigations, noticeable convergence to the limiting theoretical results as we increase the conductors.

Appendices

The first appendix list various standard conjectures. The second appendix gives the formula to numerically approximate the analytic rank of an elliptic curve. For a curve of conductor C_E, one needs about $\sqrt{C_E} \log C_E$ Fourier coefficients. The third is the statement (with assumptions) of the main theoretical result for the one-level density of one-parameter families of Elliptic curves over $\mathbb{Q}(T)$.

Appendix I: Standard Conjectures

Generalized Riemann Hypothesis (for Elliptic Curves) *Let $L(s, E)$ be the (normalized) L-function of the elliptic curve E. Then the non-trivial zeros of $L(s, E)$ satisfy $Re(s) = \frac{1}{2}$.*

Birch and Swinnerton-Dyer Conjecture [BSD1], [BSD2] *Let E be an elliptic curve of geometric rank r over $\mathbb{Q}$ (the Mordell-Weil group is $\mathbb{Z}^r \oplus T$, T is the subset of torsion points). Then the analytic rank (the order of vanishing of the L-function at the central point) is also r.*

Tate's Conjecture for Elliptic Surfaces [Ta] *Let $\mathcal{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathcal{E}, s)$ be the L-series attached to $H^2_{ét}(\mathcal{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathcal{E}, s)$ has a meromorphic continuation to $\mathbf{C}$ and satisfies $-ord_{s=2} L_2(\mathcal{E}, s) = rank\ NS(\mathcal{E}/\mathbb{Q})$, where $NS(\mathcal{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathcal{E}$. Further, $L_2(\mathcal{E}, s)$ does not vanish on the line $Re(s) = 2$.*

Most of the 1-param families we investigate are rational surfaces, where Tate's conjecture is known. See [RSi].

Appendix II: Numerically Approximating Ranks: Preliminaries

Cusp form f, level N, weight 2:

$$\begin{aligned} f(-1/Nz) &= -\epsilon N z^2 f(z) \\ f(i/y\sqrt{N}) &= \epsilon y^2 f(iy/\sqrt{N}). \end{aligned}$$

Define

$$\begin{aligned} L(f,s) &= (2\pi)^s \Gamma(s)^{-1} \int_0^{i\infty} (-iz)^s f(z) \frac{dz}{z} \\ \Lambda(f,s) &= (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s) = \int_0^{\infty} f(iy/\sqrt{N}) y^{s-1} dy. \end{aligned}$$

Get

$$\Lambda(f,s) = \epsilon \Lambda(f, 2-s), \quad \epsilon = \pm 1.$$

To each E corresponds an f, write $\int_0^{\infty} = \int_0^1 + \int_1^{\infty}$ and use transformations.

Algorithm for $L^r(s, E)$: I

$$\begin{aligned}\Lambda(E,s) &= \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_0^1 f(iy/\sqrt{N})y^{s-1}dy + \int_1^\infty f(iy/\sqrt{N})y^{s-1}dy \\ &= \int_1^\infty f(iy/\sqrt{N})(y^{s-1} + \epsilon y^{1-s})dy.\end{aligned}$$

Differentiate k times with respect to s:

$$\Lambda^{(k)}(E,s) = \int_1^\infty f(iy/\sqrt{N})(\log y)^k(y^{s-1} + \epsilon(-1)^k y^{1-s})dy.$$

At $s = 1$,

$$\Lambda^{(k)}(E,1) = (1 + \epsilon(-1)^k) \int_1^\infty f(iy/\sqrt{N})(\log\ y)^k dy.$$

Trivially zero for half of k; let r be analytic rank.

Algorithm for $L^r(s, E)$: II

$$\begin{aligned}\Lambda^{(r)}(E,1) &= 2\int_1^\infty f(iy/\sqrt{N})(\log\ y)^r dy \\ &= 2\sum_{n=1}^\infty a_n \int_1^\infty e^{-2\pi ny/\sqrt{N}}(\log\ y)^r dy.\end{aligned}$$

Integrating by parts

$$\Lambda^{(r)}(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^\infty \frac{a_n}{n}\int_1^\infty e^{-2\pi ny/\sqrt{N}}(\log\ y)^{r-1}\frac{dy}{y}.$$

We obtain

$$L^{(r)}(E,1) = 2r!\sum_{n=1}^\infty \frac{a_n}{n} G_r\left(\frac{2\pi n}{\sqrt{N}}\right),$$

where

$$G_r(x) = \frac{1}{(r-1)!}\int_1^\infty e^{-xy}(\log y)^{r-1}\frac{dy}{y}.$$

Expansion of $G_r(x)$

$$G_r(x) = P_r\left(\log\frac{1}{x}\right) + \sum_{n=1}^{\infty}\frac{(-1)^{n-r}}{n^r\cdot n!}x^n$$

$P_r(t)$ is a polynomial of degree r, $P_r(t) = Q_r(t-\gamma)$.

$$\begin{aligned}
Q_1(t) &= t;\\
Q_2(t) &= \frac{1}{2}t^2 + \frac{\pi^2}{12};\\
Q_3(t) &= \frac{1}{6}t^3 + \frac{\pi^2}{12}t - \frac{\zeta(3)}{3};\\
Q_4(t) &= \frac{1}{24}t^4 + \frac{\pi^2}{24}t^2 - \frac{\zeta(3)}{3}t + \frac{\pi^4}{160};\\
Q_5(t) &= \frac{1}{120}t^5 + \frac{\pi^2}{72}t^3 - \frac{\zeta(3)}{6}t^2 + \frac{\pi^4}{160}t - \frac{\zeta(5)}{5} - \frac{\zeta(3)\pi^2}{36}.
\end{aligned}$$

For $r = 0$,

$$\Lambda(E,1) = \frac{\sqrt{N}}{\pi}\sum_{n=1}^{\infty}\frac{a_n}{n}\,e^{-2\pi n y/\sqrt{N}}.$$

Need about $\sqrt{N}$ or $\sqrt{N}\log N$ terms.

Appendix III: 1-Level Density

Definitions:

$$D_{n,\mathcal{F}}(\phi) = \frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i\left(\frac{\log C_E}{2\pi}\gamma_E^{(j_i)}\right)$$

$D_{n,\mathcal{F}}^{(r)}(\phi)$: n-level density with contribution of r zeros at central point removed.

$\mathcal{F}_N$: Rational one-parameter family, $t \in [N, 2N]$, conductors monotone.

ASSUMPTIONS

1-parameter family of Ell Curves, rank r over $\mathbb{Q}(T)$, rational surface. Assume

- GRH;
- $j(t)$ non-constant;
- Sq-Free Sieve if $\Delta(t)$ has irr poly factor of deg ≥ 4.

Pass to positive percent sub-seq where conductors polynomial of degree m.

ϕ_i even Schwartz, support σ_i:

- $\sigma_1 < \min\left(\frac{1}{2}, \frac{2}{3m}\right)$ for 1-level
- $\sigma_1 + \sigma_2 < \frac{1}{3m}$ for 2-level.

MAIN RESULT

Theorem (Miller 2004): Under previous conditions, as $N \to \infty$, $n = 1, 2$:

$$D^{(r)}_{n,\mathcal{F}_N}(\phi) \longrightarrow \int \phi(x) W_{\mathcal{G}}(x)\,dx,$$

where

$$\mathcal{G} \;=\; \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

1 and 2-level densities confirm Katz-Sarnak, B-SD predictions for small support.

Examples

Constant-Sign Families:

1. $y^2 = x^3 + 2^4(-3)^3(9t+1)^2$,
   $9t+1$ Square-Free: all even.

2. $y^2 = x^3 \pm 4(4t+2)x$,
   $4t+2$ Square-Free:
   $+$ all odd, $-$ all even.

3. $y^2 = x^3 + tx^2 - (t+3)x + 1$,
   t^2+3t+9 Square-Free: all odd.

First two rank 0 over $\mathbb{Q}(T)$, third is rank 1.

Without 2-Level Density, couldn't say *which* orthogonal group.

Examples (cont)

Rational Surface of Rank 6 over $\mathbf{Q}(t)$:

$$\begin{aligned} y^2 &= x^3 + (2at - B)x^2 + (2bt - C)(t^2 + 2t - A + 1)x \\ &\quad +(2ct - D)(t^2 + 2t - A + 1)^2 \end{aligned}$$

$$\begin{aligned} A &= 8,916,100,448,256,000,000 \\ B &= -811,365,140,824,616,222,208 \\ C &= 26,497,490,347,321,493,520,384 \\ D &= -343,107,594,345,448,813,363,200 \\ a &= 16,660,111,104 \\ b &= -1,603,174,809,600 \\ c &= 2,149,908,480,000 \end{aligned}$$

Need GRH, Sq-Free Sieve to handle sieving.

Appendix IV: t-Statistics

The Pooled Two-Sample t-Procedure is

$$t \;=\; \left(\overline{X_1} - \overline{X_2}\right) \Big/ s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},$$

where $\overline{X_i}$ is the sample mean of n_i observations of population i, s_i is the sample standard deviation and

$$s_p \;=\; \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$$

is the pooled variance; t has a t-distribution with $n_1 + n_2 - 2$ degrees of freedom.

The Unpooled Two-Sample t-Procedure is

$$t \;=\; \left(\overline{X_1} - \overline{X_2}\right) \Big/ \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}};$$

this is approximately a t distribution with

$$\frac{(n_1 - 1)\,(n_2 - 1)\,(n_2 s_1^2 + n_1 s_2^2)^2}{(n_2 - 1)n_2^2 s_1^4 + (n_1 - 1)n_1^2 s_2^4}$$

degrees of freedom

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