The Reversed Zeckendorf Game

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(with Zoë X. Batterman, Aditya Jambhale, Kishan Sharma, and Andrew K. Yang)

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Outline

- Zeckendorf decompositions, the Zeckendorf Game, and known results
- The Reversed Zeckendorf Game
- Varying the starting position
- The Build-Up 1-2-3 Game
- Concluding remarks and future directions

The *Zeckendorf decomposition* of a positive integer n is its expression as a sum of distinct, non-adjacent Fibonacci numbers.

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Theorem (Zeckendorf)

Any positive integer n admits a unique Zeckendorf decomposition up to the ordering of the summands.

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Each F_i in a decomposition is a game *chip*. We call the collection of F_i's the i^{th} *bin*, and the *height* h_i of the i^{th} bin is $\#F_i$.

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Players alternate turns, and the last player to move wins (i.e., the first player to run out of moves loses).

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Combine: If $h_i > 0$ and $h_{i-1} > 0$, then the move is

$$F_{i-1} \wedge F_i \longmapsto F_{i+1}.$$

The move $F_1 \wedge F_1 \longmapsto F_2$ is also a combine.

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Split: If $h_i > 1$ with $i > 2$, then the move is

$$2F_i \longmapsto F_{i-2} \wedge F_{i+1}.$$

The move $2F_2 \longmapsto F_3 \wedge F_1$ is also a split.

Earlier Results

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For $n \geq 3$, Player 2 wins with optimal play on both sides.

The proof of the second statement is non-constructive and uses a *strategy-stealing argument*. Unfortunately, this means we do not explicitly know the winning strategy!

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Place a directed edge between two vertices if it is possible to travel from one decomposition to another in one game turn.

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Graph-theoretic approach: We create a directed graph of the forwards game by associating a vertex to each possible decomposition of n into Fibonacci numbers.

Place a directed edge between two vertices if it is possible to travel from one decomposition to another in one game turn.

The game starts with n ones and always terminates at a Zeckendorf decomposition, so the starting and ending nodes are unique.

We obtain the reversed game by switching the starting and ending nodes and reversing the arrows in the directed graph.

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We use the same terminology for chips, bins, and heights as in the forwards game. The game terminates at n copies of $F_1 = 1$, and the last player to move wins.

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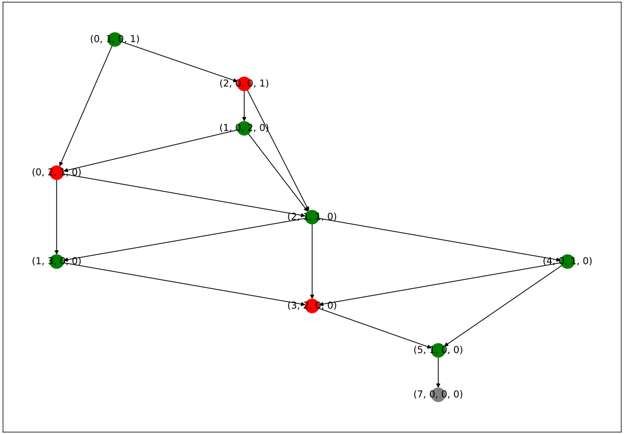
Combine: If $h_{i-2} > 0$ and $h_{i+1} > 0$ with $i > 2$, then the move is

$$F_{i-2} \wedge F_{i+1} \longmapsto 2F_i.$$

The move $F_3 \wedge F_1 \longmapsto 2F_2$ is also a combine.

Game Tree for $n = 7$

Figure: Green/red node means the player to move at that state is winning/losing. Player 1 wins with optimal play.



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Player 1 has a winning strategy in the Reversed Zeckendorf Game whenever

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We have found both a constructive and a non-constructive proof.

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Suppose that Player 2 has a forced win for some $n = F_{i+1} \wedge F_{i-2}$.

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If Player 1's first move is the combine $F_{i+1} \wedge F_{i-2} \longmapsto 2F_i$, then Player 2 has a forced win starting at the state $2F_i$ with their turn to play.

Player 2's only move leads to the game state $F_i \wedge F_{i-1} \wedge F_{i-2}$ with Player 1 on move. Thus, Player 2 has a forced win starting at the game state $F_i \wedge F_{i-1} \wedge F_{i-2}$ with Player 1 on move.

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However, Player 1 can steal Player 2's winning strategy by instead performing a split on their first move.

This forces the game state $F_i \wedge F_{i-1} \wedge F_{i-2}$ with Player 2 on move. Thus, Player 1 wins by stealing Player 2's winning strategy from the third statement.

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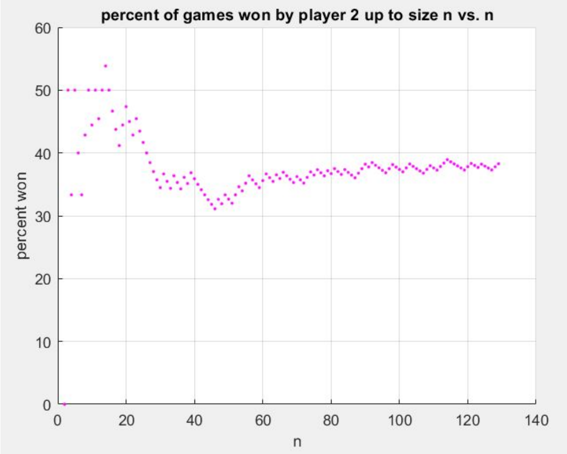

Source: SMALL 2023.

Conjecture (SMALL 2023)

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In the limit as $n \to \infty$, the proportion of Player 1 wins is $\varphi^{-1} \approx 0.618$, where

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

is the golden ratio.

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The proof uses a strategy-stealing argument.

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a	b	c		*Player having forced win*
Even	*Even*	*Even*		*Player 2*
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Odd	*Even*	*Odd*		*Player 1*
Odd	*Even*	*Even*	$a > c$	*Player 2*
Odd	*Even*	*Even*	$a < c$	*Player 1*
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Even	*Even*	*Odd*	$a < c$	*Player 2*
Even	*Odd*	*Odd*		*Player 1*
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Odd	*Even*	*Even*	$a < c$	*Player 1*
Even	*Even*	*Odd*	$a > c$	*Player 1*
Even	*Even*	*Odd*	$a < c$	*Player 2*
Even	*Odd*	*Odd*		*Player 1*
Odd	*Odd*	*Even*		*Player 1*

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The proofs are constructive and give explicit winning strategies.

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Choose an integer n. Two players begin by taking turns placing down a one, two, or three until their sum equals n. This generates an ordered triple (a, b, c).

The players then play the Reversed Zeckendorf Game starting from this triple beginning with the player who did not place down the final number.

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Theorem (SMALL 2023)

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For $n = 4$ or n odd, Player 1 wins the Build-Up 1-2-3 Game. Otherwise (i.e., when $n \neq 4$ is even), Player 2 wins.

The proof gives explicit winning strategies and uses the results of the previous table.

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Create code that calculates the winning player more efficiently.

Solve the reversed game for other starting positions.

Acknowledgements

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P. Baird-Smith, A. Epstein, K. Flint and S. J. Miller, *The Zeckendorf Game*, Combinatorial and Additive Number Theory III, CANT, New York, USA, 2017 and 2018, Springer Proceedings in Mathematics & Statistics **297** (2020), 25–38.

P. Baird-Smith, A. Epstein, K. Flint and S. J. Miller, *The Generalized Zeckendorf Game*, Proceedings of the 18th International Conference on Fibonacci Numbers and Their Applications, Fibonacci Quarterly **57** (2019), no. 5, 1–14.

Z. X. Batterman, A. Jambhale, S. J. Miller, A. L. Narayanan, K. Sharma, A. K. Yang, C. Yao. *The Reversed Zeckendorf Game*, to appear in the Proceedings of the 21st International Conference on Fibonacci Numbers and Their Applications, Fibonacci Quarterly.

A. Cusenza, A. Dunkelberg, K. Huffman, D. Ke, D. Kleber, M. McClatchey, S. J. Miller, C. Mizgerd, V. Tiwari, J. Ye, X. Zheng, *Bounds on Zeckendorf Games*, Fibonacci Quarterly **60** (2022), no. 1, 57–71.