Finite conductor models for zeros near the central point of elliptic curve L-functions

Steven J Miller
Williams College

Steven.J.Miller@williams.edu
http://www.williams.edu/go/math/sjmiller

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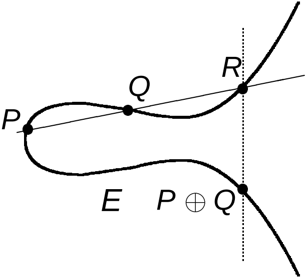
Outline

- Review elliptic curves.
- Introduce relevant RMT ensembles.
- Results for large conductors.
- Results towards Birch and Swinnerton-Dyer.
- Data for small conductors.
- Reconciling theory and data.

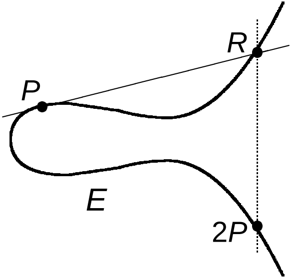
Elliptic Curves

Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Elliptic curve L-function

$E : y^2 = x^3 + ax + b$, associate L-function

$$L(s,E) \;=\; \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s,E)$ at $s = 1/2$.

One parameter family

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T), A(T), B(T) \in \mathbb{Z}[T].$$

Silverman's Specialization Theorem

Assume (geometric) rank of $\mathcal{E}/\mathbb{Q}(T)$ is r. Then for all $t \in \mathbb{Z}$ sufficiently large, each $E_t : y^2 = x^3 + A(t)x + B(t)$ has (geometric) rank at least r.

Average rank conjecture

For a generic one-parameter family of rank r over $\mathbb{Q}(T)$, expect in the limit half the specialized curves have rank r and half have rank $r + 1$.

Random Matrix Ensembles

1-Level Density

L-function $L(s,f)$: by RH non-trivial zeros $\frac{1}{2}+i\gamma_{f,j}$.
C_f: analytic conductor.
$\varphi(x)$: compactly supported even Schwartz function.

$$D_{1,f}(\varphi) \quad = \quad \sum_j \varphi\left(\frac{\log C_f}{2\pi}\gamma_{f,j}\right)$$

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Katz-Sarnak Conjecture:

$$D_{1,\mathcal{F}}(\varphi) = \lim_{N\to\infty}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{1,f}(\varphi) = \int \varphi(x)\rho_{G(\mathcal{F})}(x)dx$$
$$= \int \widehat{\varphi}(u)\widehat{\rho}_{G(\mathcal{F})}(u)du.$$

Orthogonal Random Matrix Models

RMT: $SO(2N)$: $2N$ eigenvalues in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$d\epsilon_0(\theta) \ \propto \ \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j.$$

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Independent Model:

$$\mathcal{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

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Interaction Model: Sub-ensemble of $SO(2N)$ with the last $2r$ of the $2N$ eigenvalues equal $+1$: $1 \;\leq\; j,k \;\leq\; N-r$:

$$d\varepsilon_{2r}(\theta) \;\propto\; \prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

Random Matrix Models and One-Level Densities

Fourier transform of 1-level density:

$$\hat{\rho}_0(u) = \delta(u) + \frac{1}{2}\eta(u).$$

Fourier transform of 1-level density (Rank 2, Indep):

$$\hat{\rho}_{2,\text{Independent}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right].$$

Fourier transform of 1-level density (Rank 2, Interaction):

$$\hat{\rho}_{2,\text{Interaction}}(u) = \left[\delta(u) + \frac{1}{2}\eta(u) + 2\right] + 2(|u| - 1)\eta(u).$$

Limiting Behavior
(joint with John Goes)

Comparing the RMT Models

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi}\gamma_{E_t,j}\right) = \int \varphi(x)\rho_{\mathcal{G}}(x)dx + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Supports Independent and not Interaction model in the limit.

Previous Results on low-lying zeros

Expect zeros near central point of size $\frac{1}{\log N_E}$.

Mestre: zero with imaginary part at most $\frac{B}{\log\log N_E}$.

Goal: bound (from above and below) number of zeros in a neighborhood of size $\frac{1}{\log N_E}$ near the central point in a family.

One-Level Density: Sketch of result

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\tilde{\gamma}_{j,f}}\phi(\tilde{\gamma}_{j,f}) = \left(r+\frac{1}{2}\right)\phi(0)+\hat{\phi}(0)+O\left(\frac{\log\log R}{\log R}\right)$$

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$$N_{ave}(\tau,R)\phi(0) \geq \left(r+\frac{1}{2}\right)\phi(0)+\hat{\phi}(0)+O\left(\frac{\log\log R}{\log R}\right)$$

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$$N_{ave}(\tau,R) \ge \left(r+\frac{1}{2}\right)+\frac{\widehat{\phi}(0)}{\phi(0)}+O\left(\frac{\log\log R}{\log R}\right)$$

Towards an average version of Birch and Swinnerton-Dyer

Theorem (Goes, M–)

Let $\tau = C(\phi)/\sigma$, where σ is the support of $\widehat{\phi}$, $C(\phi)$ is constant depending on the choice of test function, and $N_{\rm ave}(\tau, R)$ the average number of normalized zeros in $(-\tau, \tau)$ for $t \in [R, 2R]$. Then assuming GRH

$$N_{\rm ave}(\tau, R) \ \geq \ \left(r + \frac{1}{2}\right) + \frac{\widehat{\phi}(0)}{\phi(0)} + O\left(\frac{\log\log R}{\log R}\right).$$

Technical requirements for ϕ:

- ϕ even, positive in $(-\tau, \tau)$, negative elsewhere;
- ϕ monotonically decreasing on $(0, \tau)$;
- ϕ differentiable;
- $\widehat{\phi}$ compactly supported in $(-\sigma, \sigma)$.

Construction Preliminaries

- Convolution:

$$(A * B)(x) \ = \ \int_{-\infty}^{\infty} A(t)B(x-t)dt.$$

- Fourier Transform:

$$\begin{aligned} \widehat{A}(y) &= \int_{-\infty}^{\infty} A(x)e^{-2\pi ixy}dx \\ \widehat{A''}(y) &= -(2\pi y)^2\widehat{A}(y). \end{aligned}$$

- Lemma: $(\widehat{A * B})(y) = \widehat{A}(y) \cdot \widehat{B}(y)$;
  in particular, $(\widehat{A * A})(y) = \widehat{A}(y)^2 \geq 0$ if A is even.

Constructing good ϕ's

- Let h be supported in $(-1, 1)$.
- Let $f(x) = h(2x/\sigma)$, so f supported in $(-\sigma/2, \sigma/2)$.
- Let $g(x) = (f * f)(x)$, so g supported in $(-\sigma, \sigma)$.
  $\widehat{g}(y) = \widehat{f}(y)^2$.
- Let $\phi(y) := (\widehat{g + \beta^2 g''})(y) = \widehat{f}(y)^2 \left(1 - (2\pi\beta y)^2\right)$.
  For β sufficiently small above is non-negative.

Constructing good ϕ's (cont)

$N_{\rm ave}(\tau, R)$ is average number of zeros in $(-\tau, \tau)$, and

$$N_{\rm ave}(\tau, R) \ \geq \ \left(r + \frac{1}{2}\right) + \frac{\widehat{\phi}(0)}{\phi(0)} + O\left(\frac{\log\log R}{\log R}\right).$$

Want to maximize $\widehat{\phi}(0)/\phi(0)$, which is

$$\mathcal{P}_\beta \ := \ \frac{(\int_0^1 h(u)^2 du) + (\frac{2\beta}{\sigma})^2 (\int_0^1 h(u)h''(u)du)}{\sigma(\int_0^1 h(u)du)^2}.$$

Birch and Swinnerton-Dyer on "average"

Setting $\mathcal{P}_\beta = 0$ gives $\beta = \mathrm{C}(h)\sigma$ gives

Theorem (Goes, M–)

Assume GRH and let $\beta = \mathrm{C}(h)\sigma$ so that $\mathcal{P}_\beta = 0$. Then there are on average at least $r + \frac{1}{2}$ normalized zeros within the band $\left(-\frac{1}{2\pi \mathrm{C}(h)\sigma}, \frac{1}{2\pi \mathrm{C}(h)\sigma}\right)$ for $t \in [R, 2R]$.

Using $h(x) = (1 - x^2)^2$ gives at least $r + \frac{1}{2}$ normalized zeros on average within the band $\approx \left(-\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma}\right)$

Results for certain test functions

$h(x) = 0$ for $|x| > 1$, and

- Class: $h(x) = (1 - x^{2k})^{2j}$,$(j, k \in \mathbb{Z})$
  Optimum: $h(x) = (1 - x^2)^2$
  gives interval approximately $\left(-\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma}\right)$.

- Class: $h(x) = \exp\left(-1/(1 - x^{2k})\right)$,$(k \in \mathbb{Z})$
  Optimum: $h(x) = \exp\left(-1/(1 - x^2)\right)$
  gives approximately $\left(-\frac{0.558415}{\sigma}, \frac{0.558415}{\sigma}\right)$.

- Class: $h(x) = \exp\left(-k/(1 - x^2)\right)$
  Optimum: $h(x) = \exp\left(-.754212/(1 - x^2)\right)$
  gives approximately $\left(-\frac{0.552978}{\sigma}, \frac{0.552978}{\sigma}\right)$.

Upper bounds

Theorem (Goes, M–)

For an elliptic curve with explicit formulas as above, the number of normalized zeros within $(-\tau, \tau)$ is bounded above by $(r+\frac{1}{2}) + \frac{(r+\frac{1}{2})(\psi(0)-\psi(\tau))+\hat{\psi}(0)}{\psi(\tau)}$, for all strictly positive, even test functions monotonically decreasing over $(0, \infty)$.

Questions

Interesting Families

Let $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank r over $\mathbb{Q}(T)$.
Natural sub-families:

- Curves of rank r.
- Curves of rank $r + 2$.

Interesting Families

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Natural sub-families:

- Curves of rank r.
- Curves of rank $r+2$.

Question: Does the sub-family of rank $r+2$ curves in a rank r family behave like the sub-family of rank $r+2$ curves in a rank $r+2$ family?

Equivalently, does it matter how one conditions on a curve being rank $r+2$?

Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r+2$.

2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(T)$.
Density Conjecture (Generic Family) $\Longrightarrow$ 50% rank r, r+1.

For many families, observe
Percent with rank r $\approx$ 32%
Percent with rank r+1 $\approx$ 48%
Percent with rank r+2 $\approx$ 18%
Percent with rank r+3 $\approx$ 2%

Problem: small data sets, sub-families, convergence rate log(conductor).

Data on Excess Rank

$$y^2 + y = x^3 + Tx$$

Each data set 2000 curves from start. Last has conductors of size 10^{17}, but on logarithmic scale still small.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Results and Data

RMT: Theoretical Results ($N \to \infty$, Mean $\to$ 0.321)

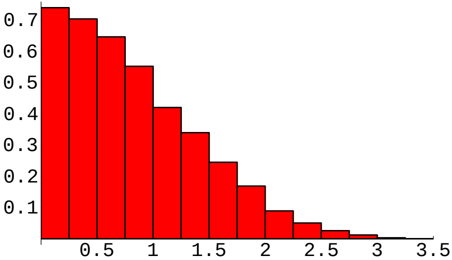


Figure 1a: 1st norm. evalue above 1: 23,040 SO(4) matrices
Mean $=$ .709, Std Dev of the Mean $=$ .601, Median $=$ .709

RMT: Theoretical Results ($N \to \infty$, Mean $\to 0.321$)

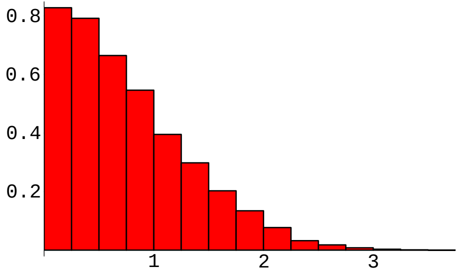


Figure 1b: 1st norm. evalue above 1: 23,040 SO(6) matrices
Mean $=$ .635, Std Dev of the Mean $=$ .574, Median $=$ .635

RMT: Theoretical Results ($N \to \infty$)

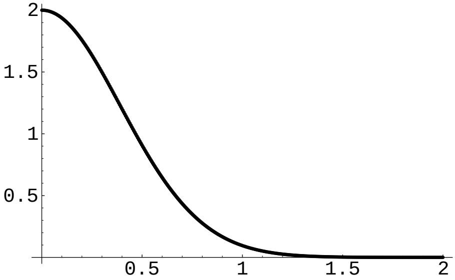


Figure 1c: 1st norm. evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)

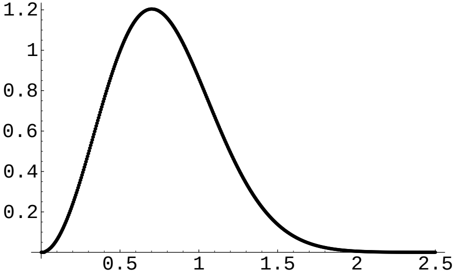


Figure 1d: 1st norm. evalue above 1: SO(odd)

Rank 0 Curves: 1st Normalized Zero above Central Point

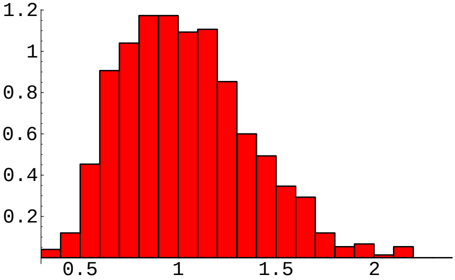


Figure 2a: 750 rank 0 curves from

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

$\log(\text{cond}) \in [3.2, 12.6]$, median $= 1.00$ mean $= 1.04$, $\sigma_\mu = .32$

Rank 0 Curves: 1st Normalized Zero above Central Point

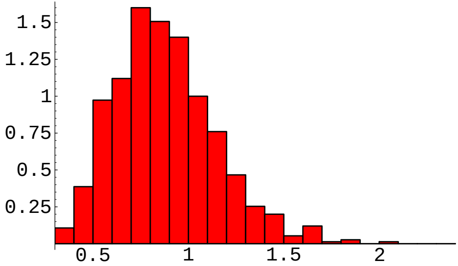


Figure 2b: 750 rank 0 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, $\sigma_\mu = .27$

Rank 2 Curves: 1st Norm. Zero above the Central Point

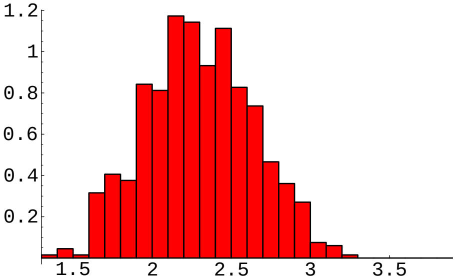


Figure 3a: 665 rank 2 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\mathrm{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

Rank 2 Curves: 1st Norm. Zero above the Central Point

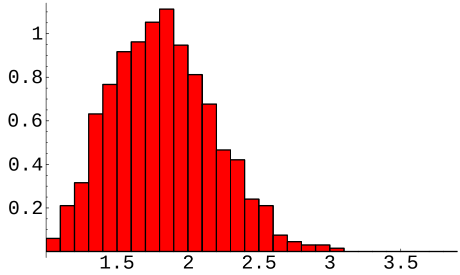


Figure 3b: 665 rank 2 curves from
$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

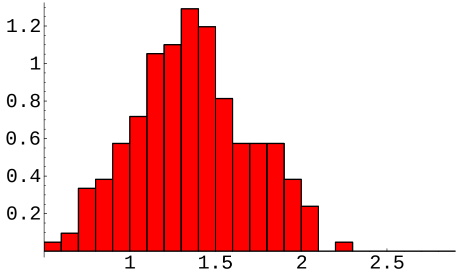


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

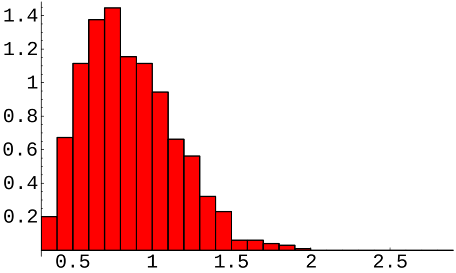


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\mathrm{cond}) \in [15.00, 16.00]$, median $=$ .81, mean $=$ .86.

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	Median $\widetilde{\mu}$	Mean μ	StDev σ_μ	log(cond)	Number
1: [0,1,1,1,T]	1.28	1.33	0.26	[3.93, 9.66]	7
2: [1,0,0,1,T]	1.39	1.40	0.29	[4.66, 9.94]	11
3: [1,0,0,2,T]	1.40	1.41	0.33	[5.37, 9.97]	11
4: [1,0,0,-1,T]	1.50	1.42	0.37	[4.70, 9.98]	20
5: [1,0,0,-2,T]	1.40	1.48	0.32	[4.95, 9.85]	11
6: [1,0,0,T,0]	1.35	1.37	0.30	[4.74, 9.97]	44
7: [1,0,1,-2,T]	1.25	1.34	0.42	[4.04, 9.46]	10
8: [1,0,2,1,T]	1.40	1.41	0.33	[5.37, 9.97]	11
9: [1,0,-1,1,T]	1.39	1.32	0.25	[7.45, 9.96]	9
10: [1,0,-2,1,T]	1.34	1.34	0.42	[3.26, 9.56]	9
11: [1,1,-2,1,T]	1.21	1.19	0.41	[5.73, 9.92]	6
12: [1,1,-3,1,T]	1.32	1.32	0.32	[5.04, 9.98]	11
13: [1,-2,0,T,0]	1.31	1.29	0.37	[4.73, 9.91]	39
14: [-1,1,-3,1,T]	1.45	1.45	0.31	[5.76, 9.92]	10
All Curves	1.35	1.36	0.33	[3.26, 9.98]	209
Distinct Curves	1.35	1.36	0.33	[3.26, 9.98]	196

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0 over $\mathbb{Q}(T)$

Family	Median $\widetilde{\mu}$	Mean μ	StDev σ_μ	log(cond)	Number
1: [0,1,1,1,T]	0.80	0.86	0.23	[15.02, 15.97]	49
2: [1,0,0,1,T]	0.91	0.93	0.29	[15.00, 15.99]	58
3: [1,0,0,2,T]	0.90	0.94	0.30	[15.00, 16.00]	55
4: [1,0,0,-1,T]	0.80	0.90	0.29	[15.02, 16.00]	59
5: [1,0,0,-2,T]	0.75	0.77	0.25	[15.04, 15.98]	53
6: [1,0,0,T,0]	0.75	0.82	0.27	[15.00, 16.00]	130
7: [1,0,1,-2,T]	0.84	0.84	0.25	[15.04, 15.99]	63
8: [1,0,2,1,T]	0.90	0.94	0.30	[15.00, 16.00]	55
9: [1,0,-1,1,T]	0.86	0.89	0.27	[15.02, 15.98]	57
10: [1,0,-2,1,T]	0.86	0.91	0.30	[15.03, 15.97]	59
11: [1,1,-2,1,T]	0.73	0.79	0.27	[15.00, 16.00]	124
12: [1,1,-3,1,T]	0.98	0.99	0.36	[15.01, 16.00]	66
13: [1,-2,0,T,0]	0.72	0.76	0.27	[15.00, 16.00]	120
14: [-1,1,-3,1,T]	0.90	0.91	0.24	[15.00, 15.99]	48
All Curves	0.81	0.86	0.29	[15.00,16.00]	996
Distinct Curves	0.81	0.86	0.28	[15.00,16.00]	863

Rank 2 Curves: 1st Norm Zero: one-param of rank 0 over $\mathbb{Q}(T)$

first set $\log(\text{cond}) \in [15, 15.5)$; second set $\log(\text{cond}) \in [15.5, 16]$.
Median $\widetilde{\mu}$, Mean μ, Std Dev (of Mean) σ_μ.

Family	$\widetilde{\mu}$	μ	σ_μ	Number	$\widetilde{\mu}$	μ	σ_μ	Number
1: [0,1,3,1,T]	1.59	1.83	0.49	8	1.71	1.81	0.40	19
2: [1,0,0,1,T]	1.84	1.99	0.44	11	1.81	1.83	0.43	14
3: [1,0,0,2,T]	2.05	2.03	0.26	16	2.08	1.94	0.48	19
4: [1,0,0,-1,T]	2.02	1.98	0.47	13	1.87	1.94	0.32	10
5: [1,0,0,T,0]	2.05	2.02	0.31	23	1.85	1.99	0.46	23
6: [1,0,1,1,T]	1.74	1.85	0.37	15	1.69	1.77	0.38	23
7: [1,0,1,2,T]	1.92	1.95	0.37	16	1.82	1.81	0.33	14
8: [1,0,1,-1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
9: [1,0,1,-2,T]	1.74	1.74	0.43	14	1.82	1.90	0.40	14
10: [1,0,-1,1,T]	2.00	2.00	0.32	22	1.81	1.94	0.42	18
11: [1,0,-2,1,T]	1.97	1.99	0.39	14	2.17	2.14	0.40	18
12: [1,0,-3,1,T]	1.86	1.88	0.34	15	1.79	1.87	0.39	22
13: [1,1,0,T,0]	1.89	1.88	0.31	20	1.82	1.88	0.39	26
14: [1,1,1,1,T]	2.31	2.21	0.41	16	1.75	1.86	0.44	15
15: [1,1,-1,1,T]	2.02	2.01	0.30	11	1.87	1.91	0.32	19
16: [1,1,-2,1,T]	1.95	1.91	0.33	26	1.98	1.97	0.26	18
17: [1,1,-3,1,T]	1.79	1.78	0.25	13	2.00	2.06	0.44	16
18: [1,-2,0,T,0]	1.97	2.05	0.33	24	1.91	1.92	0.44	24
19: [-1,1,0,1,T]	2.11	2.12	0.40	21	1.71	1.88	0.43	17
20: [-1,1,-2,1,T]	1.86	1.92	0.28	23	1.95	1.90	0.36	18
21: [-1,1,-3,1,T]	2.07	2.12	0.57	14	1.81	1.81	0.41	19
All Curves	1.95	1.97	0.37	350	1.85	1.90	0.40	388
Distinct Curves	1.95	1.97	0.37	335	1.85	1.91	0.40	366

Rank 2 Curves: 1st Norm Zero: 21 One-Param of Rank 0 over $\mathbb{Q}(T)$

- Observe the medians and means of the small conductor set to be larger than those from the large conductor set.
- For all curves the Pooled and Unpooled Two-Sample t-Procedure give t-statistics of 2.5 with over 600 degrees of freedom.
- For distinct curves the t-statistics is 2.16 (respectively 2.17) with over 600 degrees of freedom (about a 3% chance).
- Provides evidence against the null hypothesis (that the means are equal) at the .05 confidence level (though not at the .01 confidence level).

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

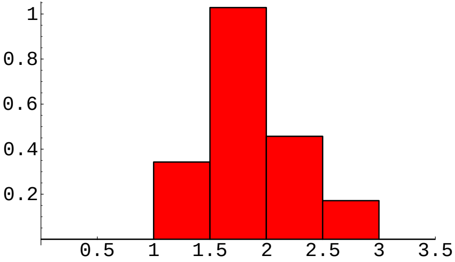


Figure 5*a*: 35 curves, $\log(\mathrm{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point

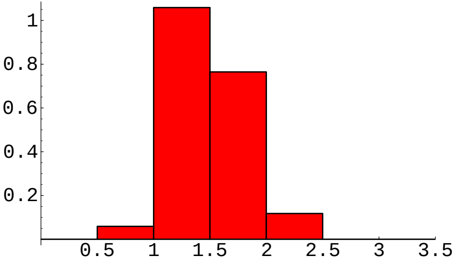


Figure 5b: 34 curves, $\log(\mathrm{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Rank 2 Curves: 1st Norm Zero: rank 2 one-param over $\mathbb{Q}(T)$

$\log(\text{cond}) \in [15, 16]$, $t \in [0, 120]$, median is 1.64.

Family	Mean	Standard Deviation	log(conductor)	Number
1: [1,T,0,-3-2T,1]	1.91	0.25	[15.74,16.00]	2
2: [1,T,-19,-T-1,0]	1.57	0.36	[15.17,15.63]	4
3: [1,T,2,-T-1,0]	1.29		[15.47, 15.47]	1
4: [1,T,-16,-T-1,0]	1.75	0.19	[15.07,15.86]	4
5: [1,T,13,-T-1,0]	1.53	0.25	[15.08,15.91]	3
6: [1,T,-14,-T-1,0]	1.69	0.32	[15.06,15.22]	3
7: [1,T,10,-T-1,0]	1.62	0.28	[15.70,15.89]	3
8: [0,T,11,-T-1,0]	1.98		[15.87,15.87]	1
9: [1,T,-11,-T-1,0]				
10: [0,T,7,-T-1,0]	1.54	0.17	[15.08,15.90]	7
11: [1,T,-8,-T-1,0]	1.58	0.18	[15.23,25.95]	6
12: [1,T,19,-T-1,0]				
13: [0,T,3,-T-1,0]	1.96	0.25	[15.23, 15.66]	3
14: [0,T,19,-T-1,0]				
15: [1,T,17,-T-1,0]	1.64	0.23	[15.09, 15.98]	4
16: [0,T,9,-T-1,0]	1.59	0.29	[15.01, 15.85]	5
17: [0,T,1,-T-1,0]	1.51		[15.99, 15.99]	1
18: [1,T,-7,-T-1,0]	1.45	0.23	[15.14, 15.43]	4
19: [1,T,8,-T-1,0]	1.53	0.24	[15.02, 15.89]	10
20: [1,T,-2,-T-1,0]	1.60		[15.98, 15.98]	1
21: [0,T,13,-T-1,0]	1.67	0.01	[15.01, 15.92]	2
All Curves	1.61	0.25	[15.01, 16.00]	64

Function Field Example (with Sal Butt, Chris Hall)

$y^2 = x^3 + (t^5 + a_1 t^4 + a_0)x + (t^3 + b_2 t^2 + b_1 t + b_0)$, $a_i, b_i \in \mathbb{F}_5$

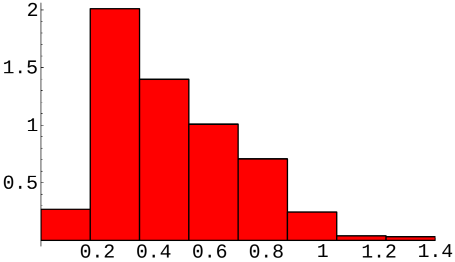


Figure 6a: Normalized first eigenangle: 719 rank 0 curves.

Function Field Example (with Sal Butt, Chris Hall)

$y^2 = x^3 + (t^5 + a_1 t^4 + a_0)x + (t^3 + b_2 t^2 + b_1 t + b_0)$, $a_i, b_i \in \mathbb{F}_5$

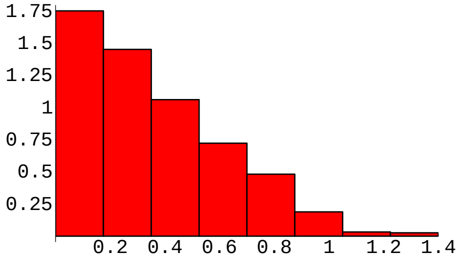


Figure 6*b*: Normalized first eigenangle: 978 curves (719 rank 0 curve, 254 rank 2 curves, 5 rank 4 curves).

Jacobi Ensembles

Repulsion or Attraction?

Conductors in $[15, 16]$; first set is rank 0 curves from 14 one-parameter families of rank 0 over $\mathbb{Q}$; second set rank 2 curves from 21 one-parameter families of rank 0 over $\mathbb{Q}$. The t-statistics exceed 6.

Family	2nd vs 1st Zero	3rd vs 2nd Zero	Number
Rank 0 Curves	2.16	3.41	863
Rank 2 Curves	1.93	3.27	701

The repulsion from extra zeros at the central point cannot be entirely explained by *only* collapsing the first zero to the central point while leaving the other zeros alone.

Can also interpret as attraction.

Comparison b/w One-Param Families of Different Rank, first normalized zero above the central point.

- First is the 701 rank 2 curves from the 21 one-parameter families of rank 0 over $\mathbb{Q}(T)$ with $\log(\mathrm{cond}) \in [15, 16]$;
- second is the 64 rank 2 curves from the 21 one-parameter families of rank 2 over $\mathbb{Q}(T)$ with $\log(\mathrm{cond}) \in [15, 16]$.

Family	Median	Mean	Std Dev	#
Rank 2 Curves (Rank 0 Families)	1.926	1.936	0.388	701
Rank 2 Curves (Rank 2 Families)	1.642	1.610	0.247	64

- t-statistic is 6.60, indicating the means differ.
- The mean of the first normalized zero of rank 2 curves in a family above the central point (for conductors in this range) depends on *how* we choose the curves.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	**23 Rank 4 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	64 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

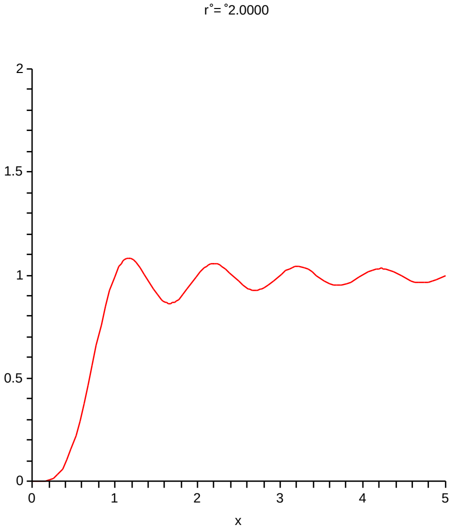
New model

The joint PDF of N pairs of eigenvalues $\{e^{i\theta_j}\}_{1\le j\le N}$, taken from random orthogonal matrices having other r fixed eigenvalues at $+1$ is

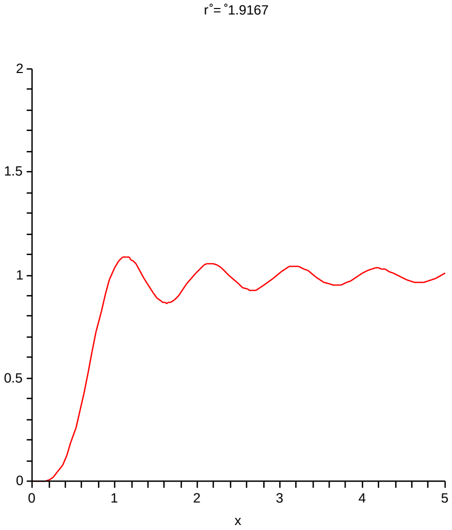
$$d\varepsilon_r(\theta_1,\dots,\theta_N) = C_{N,r}\prod_{j<k}(\cos\theta_k - \cos\theta_j)^2\prod_j(1-\cos\theta_j)^r\,d\theta_j.$$

- This probability measure is well defined for $r \in (-1/2,\infty)$.

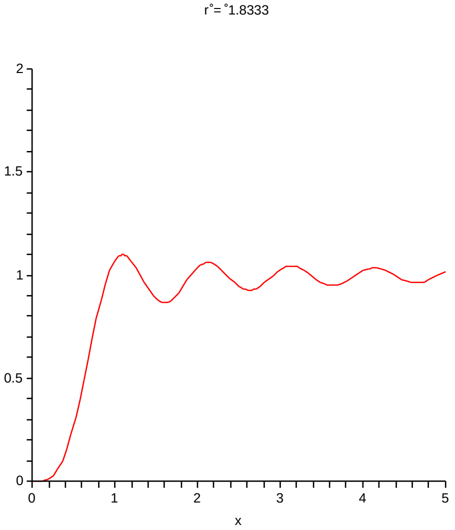
Example of Decreasing repulsion: $2 \geq r \geq 0$



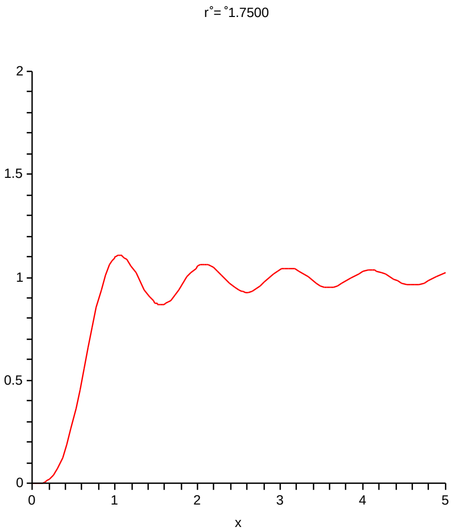
Example of Decreasing repulsion: $2 \geq r \geq 0$



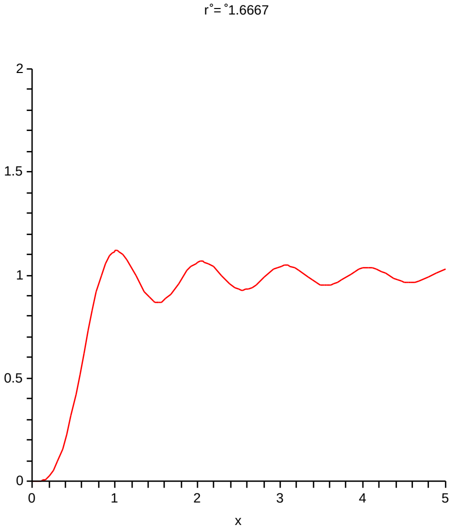
Example of Decreasing repulsion: $2 \geq r \geq 0$



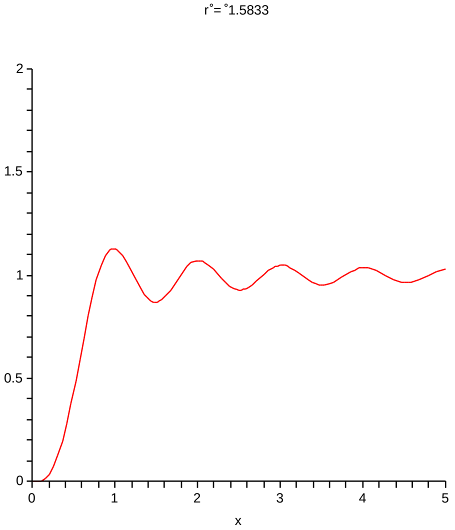
Example of Decreasing repulsion: $2 \geq r \geq 0$



Example of Decreasing repulsion: $2 \geq r \geq 0$



Example of Decreasing repulsion: $2 \geq r \geq 0$



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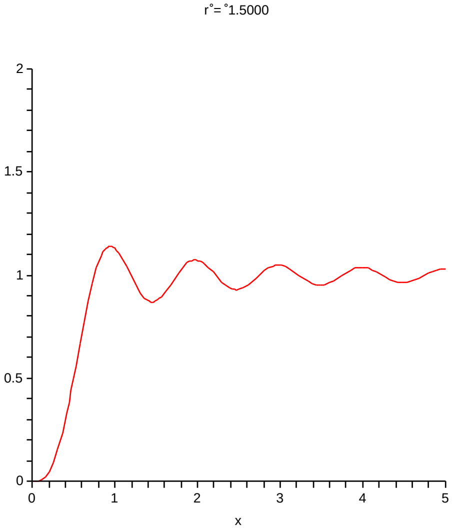

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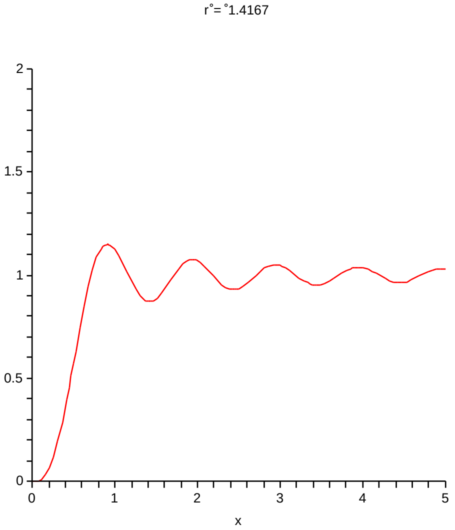
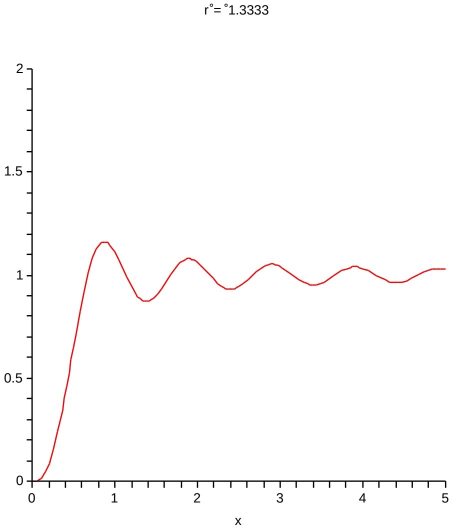
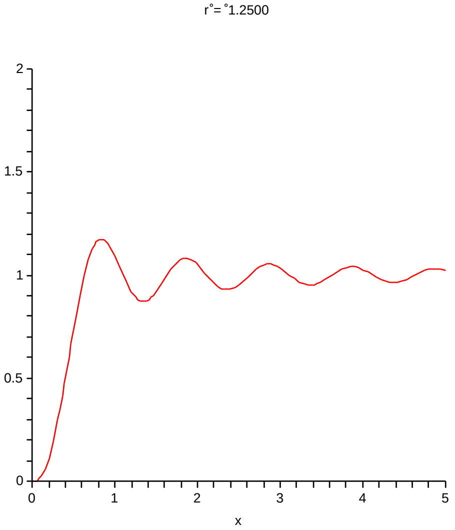


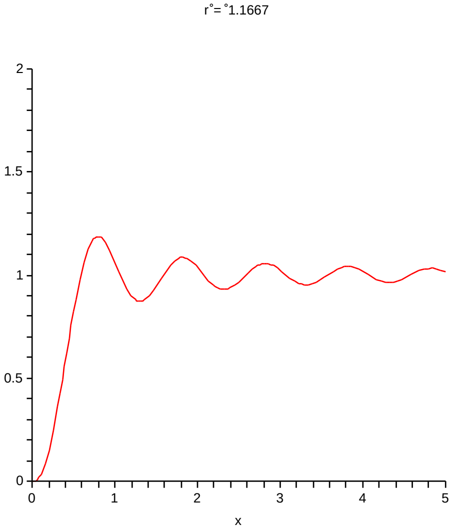
Example of Decreasing repulsion: $2 \geq r \geq 0$



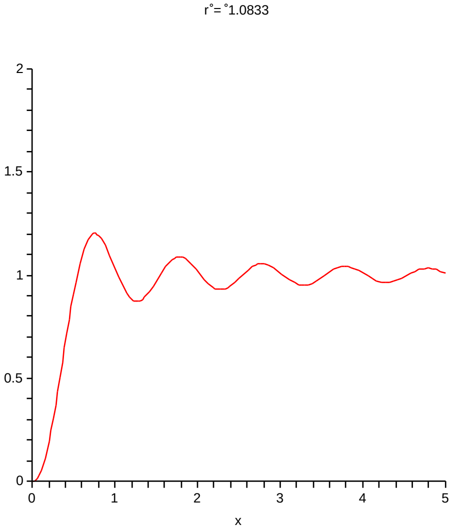
Example of Decreasing repulsion: $2 \geq r \geq 0$



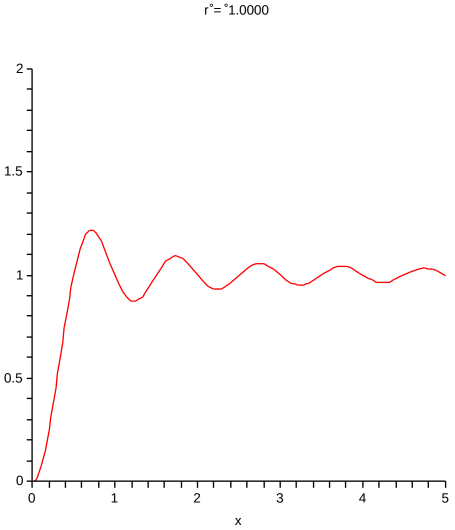
Example of Decreasing repulsion: $2 \geq r \geq 0$



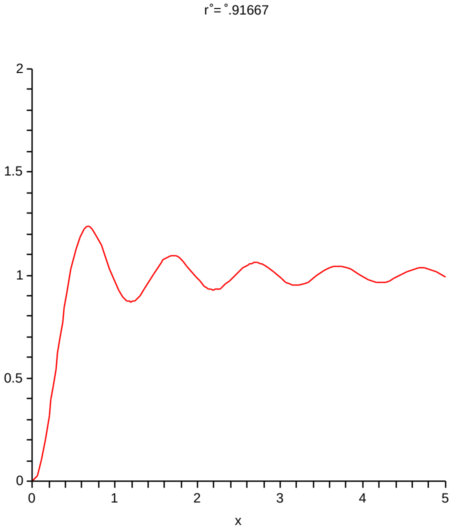
Example of Decreasing repulsion: $2 \geq r \geq 0$



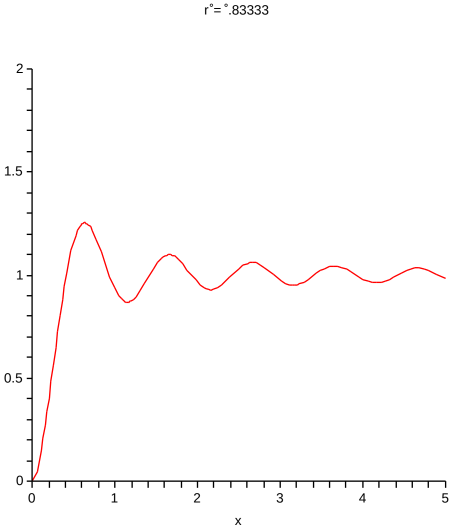
Example of Decreasing repulsion: $2 \geq r \geq 0$



Example of Decreasing repulsion: $2 \geq r \geq 0$



Example of Decreasing repulsion: $2 \geq r \geq 0$



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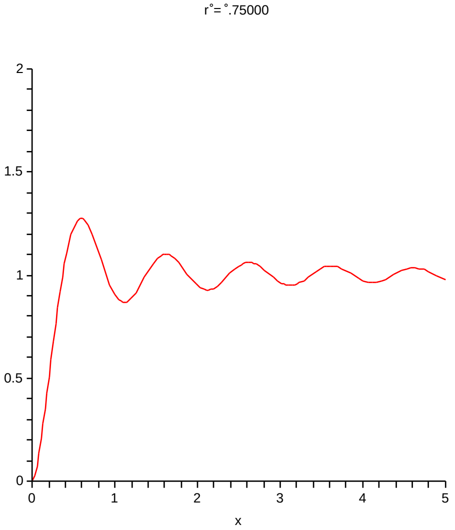
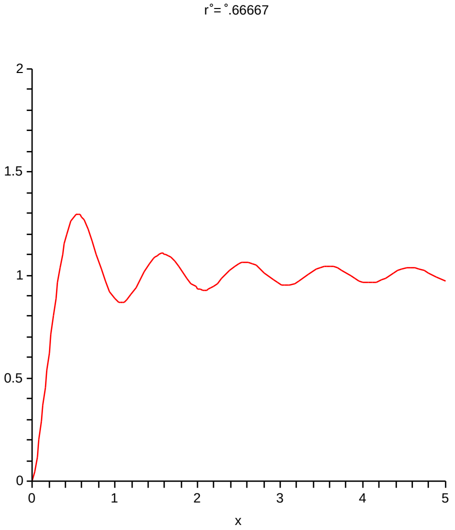
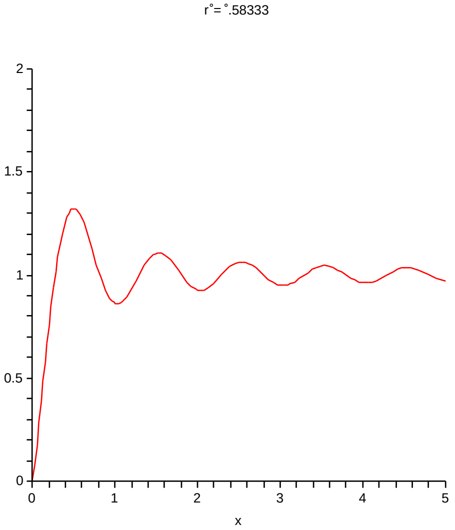


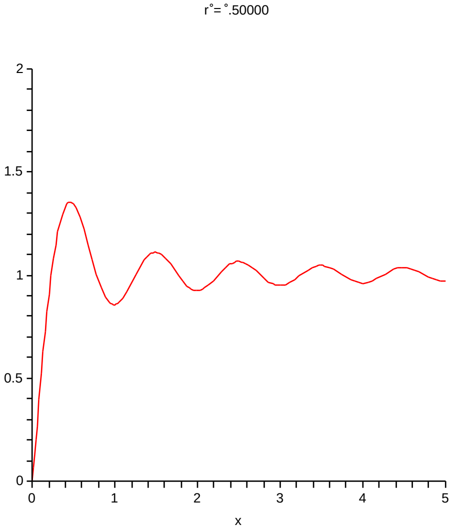
Example of Decreasing repulsion: $2 \geq r \geq 0$



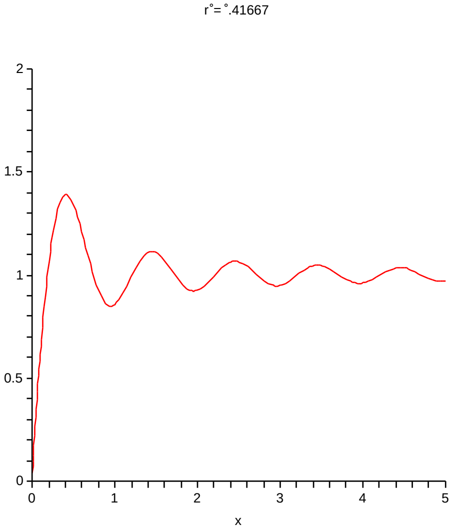
Example of Decreasing repulsion: $2 \geq r \geq 0$



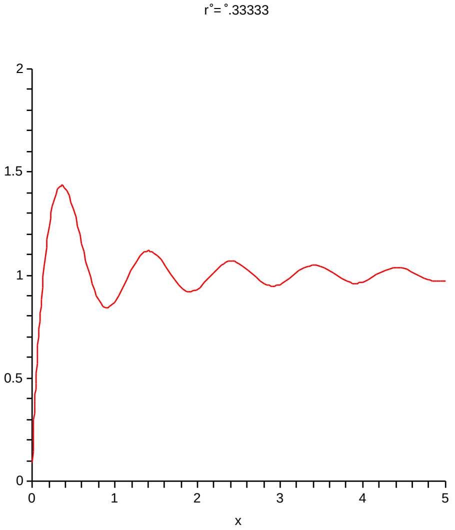
Example of Decreasing repulsion: $2 \geq r \geq 0$



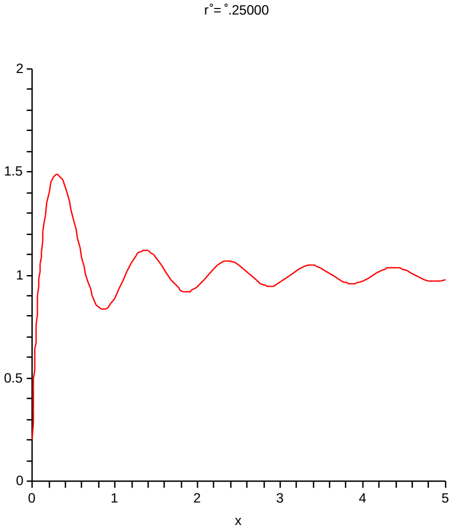
Example of Decreasing repulsion: $2 \geq r \geq 0$



Example of Decreasing repulsion: $2 \geq r \geq 0$



Example of Decreasing repulsion: $2 \geq r \geq 0$



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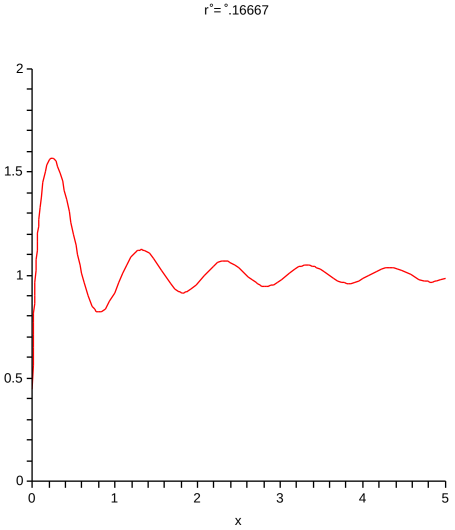

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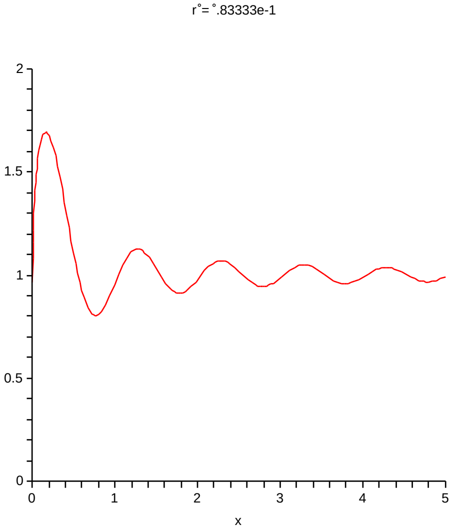
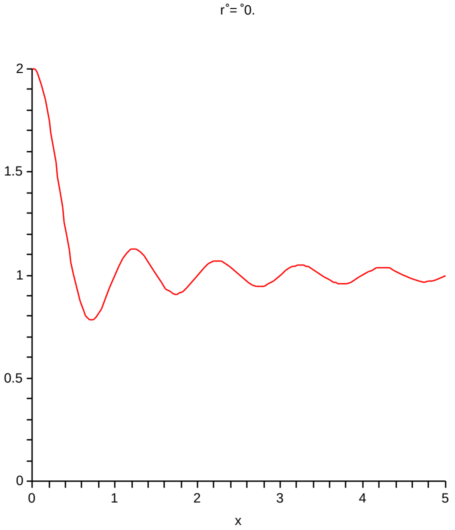


Example of Decreasing repulsion: $2 \geq r \geq 0$



The Condensation Parameter r

For simplicity, assume that $\mathcal{E}$ is an even orthogonal family depending on a parameter $T \to \infty$.

The Condensation Parameter r

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- By suitably decreasing r as T increases, the statistics of eigenvalues in this model match many of the theoretical and experimental features observed in the critical zeros of $\mathcal{E}$:
  - "Repulsion" of eigenvalues away from central point when $r > 0$. (The larger r, the more repulsion.)
  - "Independent" model statistics when $r = 0$.

The Effect of the Parameter r

- As r varies from r_0 to 0 the “central repulsion” decreases and, at $r = 0$, it disappears completely.

- Increasing r merely tends to shift all the eigenvalues to the right: they are pushed away, but the relative spacings between them are basically unchanged.

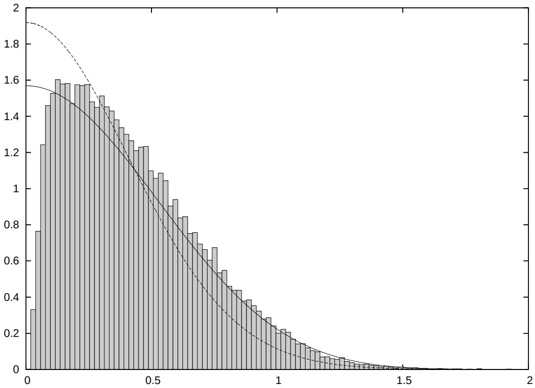
Additional inputs

- Discretization (values at central point).
- $N_{\text{effective}}$ (from lower order terms).

Goal is to model one-parameter families with finite conductor.

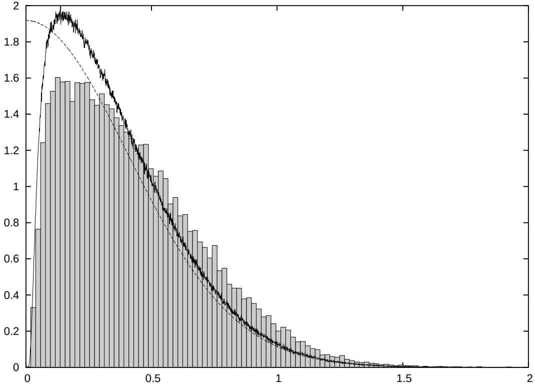
Will study simpler family of quadratic twists of a fixed E.

Modeling lowest zero (data & calculations from Duc Khiem Huynh)



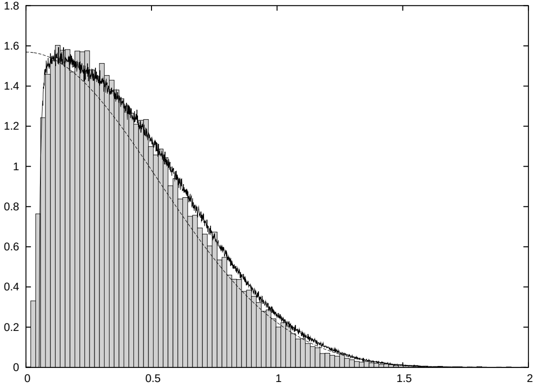
Lowest zero for $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero (data & calculations from Duc Khiem Huynh)



Lowest zero for $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$ (bar chart), lowest eigenvalue of SO(2N) with $N_0 = 12$ (solid) with discretisation and with standard $N_0 = 12.26$ (dashed) without discretisation.

Modeling lowest zero (data & calculations from Duc Khiem Huynh)



Lowest zero for $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$ (bar chart), lowest eigenvalue of SO(2N) effective N of $N_{\text{eff}} = 2$ (solid) with discretisation and with effective N of $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

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Caveat: this bibliography hasn't been updated much from a previous talk, and could be a little out of date. It is meant to serve as a first reference.

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