S-Legal Index Difference Sequences

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Zeckendorf's theorem

Let $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

Theorem (Zeckendorf)

Every non-negative integer has a unique decomposition as a sum of distinct non-consecutive Fibonacci numbers.

Equivalent Definition (Fibonacci sequence)

Let F_n be the smallest positive integer which cannot be written as as a sum of non-consecutive terms in $\{F_1, \ldots, F_{n-1}\}$.

Using the new definition

- F_1 is the smallest positive integer which cannot be written as a sum of non-consecutive terms in $\{\}$. Thus, $F_1 = 1$.
- F_2 is the smallest positive integer which cannot be written as a sum of non-consecutive terms in $\{1\}$. Thus, $F_2 = 2$.
- F_3 is the smallest positive integer which cannot be written as a sum of non-consecutive terms in $\{1, 2\}$. Thus, $F_3 = 3$.

Using the new definition

- F_4 is the smallest positive integer which cannot be written as a sum of non-consecutive terms in $\{1, 2, 3\}$. The allowed sums are:

$$1,\ 2,\ 3,\ \text{and } 1+3.$$

Thus, $F_4 = 5$.

- F_5 is the smallest positive integer which cannot be written as a sum of non-consecutive terms in $\{1, 2, 3, 5\}$. The allowed sums are:

$$1,\ 2,\ 3,\ 1+3,\ 5,\ 1+5,\ \text{and } 2+5,$$

Thus, $F_5 = 8$.

Interpretation of consecutiveness

We put the Fibonacci sequence in a 1D array of boxes.

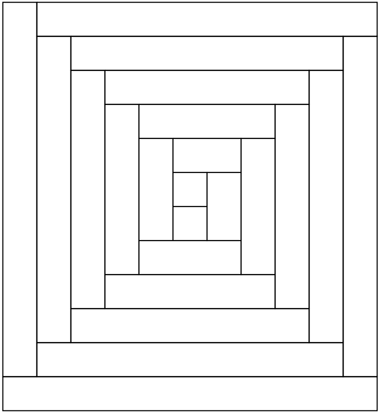
F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	$\cdots$

1	2	3	5	8	13	21	34	$\cdots$

A legal decomposition does not have summands that correspond to boxes sharing an edge.

A 1D array of boxes is pretty boring... What is a cool arrangement of boxes? The Fibonacci spiral!

The Fibonacci... spiral?

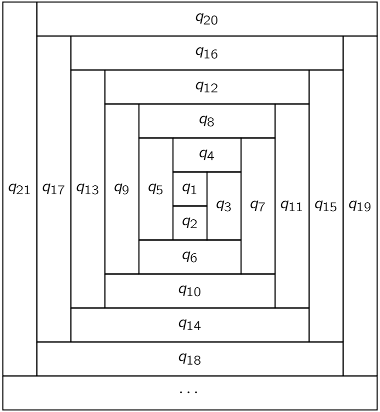


This is the Fibonacci quilt. In terms of adjacency, it is equivalent to the Fibonacci spiral. (But it is more compact to display.)

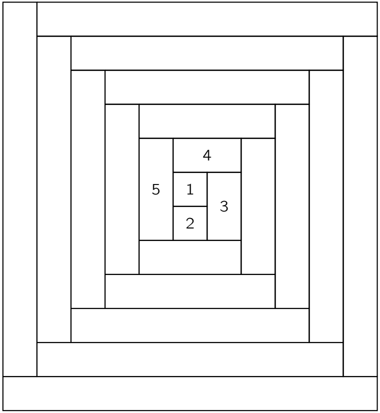
The Fibonacci quilt sequence

Definition (Fibonacci quilt sequence; Miller and Newlon, 2019)

Let q_n be the smallest positive integer which cannot be written as as a sum of non-adjacent terms in $\{q_1, \ldots, q_{n-1}\}$. Two terms are adjacent if their boxes share an edge in the quilt.



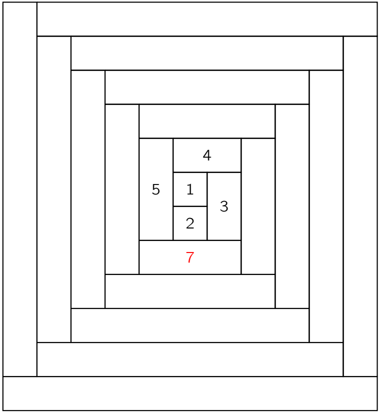
Computing the Fibonacci quilt sequence



All legal sums with at least two elements are at least 6, thus:

$q_1 = 1,$
$q_2 = 2,$
$q_3 = 3,$
$q_4 = 4,$
$q_5 = 5.$

Computing the Fibonacci quilt sequence

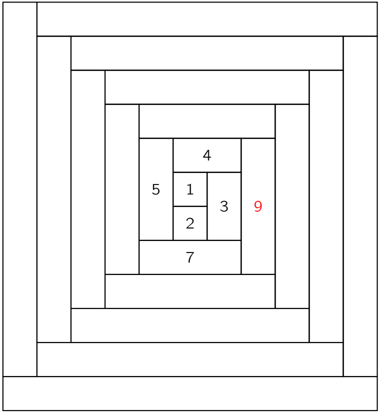


The possible legal sums using $\{1, 2, 3, 4, 5\}$ are:

$$1 = 1,$$
$$2 = 2,$$
$$3 = 3,$$
$$4 = 4,$$
$$5 = 5,$$
$$6 = 2 + 4,$$
$$8 = 3 + 5.$$

Thus, $q_6 = 7$.

Computing the Fibonacci quilt sequence



The possible legal sums using $\{1, 2, 3, 4, 5, 7\}$ are:

$1 = 1,$
$2 = 2,$
$3 = 3,$
$4 = 4,$
$5 = 5,$
$6 = 2 + 4,$
$7 = 7,$
$8 = 3 + 5 = 1 + 7,$
$11 = 4 + 7.$

Thus, $q_7 = 9$.

Computing is hard

Using the definition of the Fibonacci quilt sequence to calculate terms is computationally expensive.

The following theorem makes it easier:

Proposition (Miller and Newlon, 2019)

Let q_n be the Fibonacci quilt sequence. For $n \geq 6$,

$$q_n = q_{n-1} + q_{n-5}.$$

The triangular quilt sequence

Definition (Triangular quilt sequence; SMALL, 2022)

Let t_n be the smallest positive integer which cannot be written as as a sum of non-adjacent terms in $\{t_1, \ldots, t_{n-1}\}$. Two terms are adjacent if their boxes share an edge in the Padovan spiral.

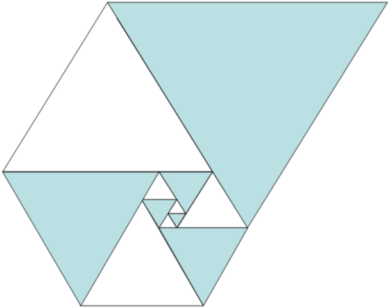


Figure: The Padovan Spiral.

Computing the triangular quilt sequence

Computing the triangular quilt sequence using its definition is computationally expensive. We were able to compute the first 50 terms of the triangular quilt sequence.

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	t_{13}	$\cdots$
1	2	3	5	6	11	12	20	23	40	46	80	92	$\cdots$

Question

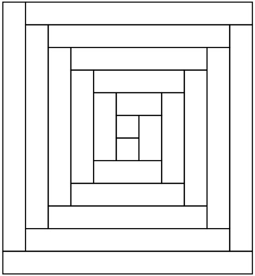
Like the Fibonacci quilt sequence, does the triangular quilt sequence follow a recurrence?

Surprising answer: Based on the first terms, apparently no.

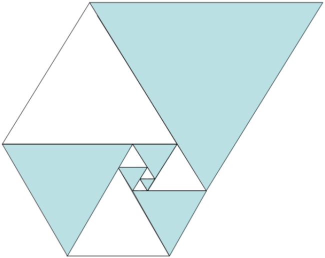
More surprising fact: This sequence is not listed on the Online Encyclopedia of Integer Sequences.

Simplifying the quilt sequences

Studying 2D constructions is hard.
Let's simplify the notion of adjacency.



Eventually, boxes q_i and q_j are adjacent iff $|i-j| \in \{1,3,4\}$.



Eventually, triangles t_j and t_i are adjacent iff $|i-j| \in \{1,5\}$.

Simplifying the quilt sequences

Fibonacci quilt sequence:

Eventually, boxes q_i and q_j are adjacent iff $|i-j| \in \{1, 3, 4\}$.

Triangular quilt sequence:

Eventually, triangles t_j and t_i are adjacent iff $|i-j| \in \{1, 5\}$.

Fibonacci quilt-like sequence:

a_i and a_j are adjacent iff $|i-j| \in \{1, 3, 4\}$

Triangular quilt-like sequence:

a_j and a_i are adjacent iff $|i-j| \in \{1, 5\}$.

S-LID decompositions

Fix a set S of positive integers. (For example, $\{1, 3, 4\}$ or $\{1, 5\}$.)

Definition (S-LID decomposition)

An S**-Legal Index Difference (**S**-LID) decomposition** using $\{a_i\}_{i \in I \subseteq \mathbf{Z}_{>0}}$ is a sum of the form

$$N = \sum_{\ell \in L} a_\ell$$

for finite $L \subseteq I$ such that $|\ell_1 - \ell_2| \notin S$ for all $\ell_1, \ell_2 \in L$.

Example

Let $S = \{2\}$ and $\{a_i\}_{i \in I} = \{a_1, a_2, a_3, a_4\}$. Then

- $a_1 + a_2$ is an S-LID decomposition using $\{a_i\}_{i \in I}$,
- $a_1 + a_2 + a_3$ is not an S-LID decomposition using $\{a_i\}_{i \in I}$ because $|3 - 1| = 2 \in S$.

S-LID sequences

We use S-LID decompositions to construct a sequence.

Definition (S-LID sequence)

The S-**LID sequence** $\{a_i\}_{i=1}^{\infty}$ is defined by:

- a_n is the smallest positive integer that does not have an S-LID decomposition using $\{a_i\}_{i=1}^{n-1}$.

Example

- The $\{\}$-LID sequence is $\{2^{i-1}\}_{i=1}^{\infty}$,
- The $\{1\}$-LID sequence is the Fibonacci sequence,
- Understanding the $\{1, 3, 4\}$ and $\{1, 5\}$-LID sequences will help us understand the Fibonacci and triangular quilt sequences.

Lower bound

Lemma

Let $\{a_i\}_{i=1}^{\infty}$ *be the S-LID sequence, and let* $k = \max S$*. Then, for all* $i \geq k+2$*,*

$$a_i \geq a_{i-1} + a_{i-(k+1)}.$$

Proof of lower bound

- 0 has S-LID decomposition using $a_1, \dots, a_{i-(k+2)}$.
- 1 has S-LID decomposition using $a_1, \dots, a_{i-(k+2)}$.

$\vdots$

- $a_{i-(k+1)} - 1$ has S-LID decomposition using $a_1, \dots, a_{i-(k+2)}$.

We can add a_{i-1} to each decomposition, and obtain S-LID decompositions.

Proof of lower bound

- a_{i-1} has S-LID dec. using $a_1, \ldots, a_{i-1}$.
- $a_{i-1} + 1$ has S-LID dec. using $a_1, \ldots, a_{i-1}$.

$\vdots$

- $a_{i-1} + a_{i-(k+1)} - 1$ has S-LID dec. using $a_1, \ldots, a_{i-1}$.

Therefore,

$$a_i \geq a_{i-1} + a_{i-(k+1)}.$$

Sharpness of lower bound

We have

$$a_i \geq a_{i-1} + a_{i-(k+1)}.$$

Thus,

$$a_i = a_{i-1} + a_{i-(k+1)}$$

iff $a_{i-1} + a_{i-(k+1)}$ does not have a S-LID decomposition using $a_1, \ldots, a_{i-1}$.

Sharpness of lower bound

For some sequences, this lower bound is sharp:

Example

The first terms of the $\{1, 2, 4\}$-LID sequence are

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	$\cdots$
1	2	3	4	6	7	9	12	16	22	29	38	50	$\cdots$

Sharpness of lower bound

In other cases, the behavior is much less predictable:

Example

The first terms of the $\{1, 5\}$-LID sequence are

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	$\cdots$
1	2	3	5	8	13	14	21	24	43	51	67	105	$\cdots$

For almost all of the S we consider, the S-LID sequence is mysterious and does not appear on OIES.

Bounding possible sums

To prove that $a_i = a_{i-1} + a_{i-(k+1)}$ in certain cases, we bound the largest possible S-LID decomposition using $\{a_j\}_{j=1}^{i-1}$.

Lemma

If k is the maximal element of S, then for all $\ell \geq 1$ we have

$$a_\ell > a_{\ell-(k-c)} + a_{\ell-2(k-c)} + \cdots$$

whenever $k \geq 2(c+1)$.

We prove this by iteratively applying the lower bound lemma.

Our results

Theorem (SMALL '22)

For all $k \geq 6$, the $\{1, 2, \ldots, k-2, k\}$-LID sequence satisfies

$$a_i = a_{i-1} + a_{i-(k+1)}$$

for all $i \geq k+2$.

Theorem (SMALL '22)

For all $k \geq 12$, the $\{1, 2, \ldots, k-3, k-1, k\}$-LID sequence satisfies

$$a_i = a_{i-1} + a_{i-(k+1)}$$

for all $i \geq k+3$.

Our results

Conjecture (SMALL '22)

Fix any $c > 0$. Then for all $k \gg 0$, the $\{1, 2, \ldots, k-c-1, k-c+1, \ldots, k\}$-LID sequence satisfies

$$a_i = a_{i-1} + a_{i-(k+1)}$$

for all $i \gg 0$.

Conjecture (SMALL '22)

Fix any finite set T of positive integers. Let $k \gg 0$, and $S = \{1, \ldots, k\} \setminus (k - T)$. Then, the S-LID sequence satisfies

$$a_i = a_{i-1} + a_{i-(k+1)}$$

for all $i \gg 0$.

Future work

- An extension of this work would be to analyze sparser sets S where we do not expect the same recurrence relation to hold.
- We would like to understand the probabilistic behavior of those S-LID sequences that exhibit more "chaotic" behavior.
- We are also interested in finding more efficient algorithms to generate S-LID sequences.

Acknowledgments and References

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References:

- Steven J. Miller and Alexandra Newlon. "The Fibonacci quilt game". In: *Fibonacci Quart.* 58.2 (2020), pp. 157–168. issn: 0015-0517. arXiv: 1909.01938 [math.NT].