Stability of Matrix Recurrence Relations: k-nacci Counting and Asymptotics

Matrix Recurrence Relations Group and Predator-Prey Population Modeling Group (SMALL 2024 REU)

Speakers: Glenn Bruda, Pico Gilman, Beni Prapashtica, Daniel Son, Saad Waheed, and Janine Wang

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Recurrence Relations Primer

A well-known example:

Definition (Fibonacci Numbers)

$$F_n = F_{n-1} + F_{n-2}, \text{ with } F_0 = 0,\ F_1 = 1.$$

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...with different initial conditions:

Definition (Lucas Numbers)

$$L_n = L_{n-1} + L_{n-2}, \text{ with } L_0 = 2,\ L_1 = 1.$$

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...with different initial conditions:

Definition (Lucas Numbers)

$$L_n = L_{n-1} + L_{n-2}, \text{ with } L_0 = 2,\ L_1 = 1.$$

...and different recurrence depth:

Definition (j-nacci Numbers)

$$F_n^{(j)} = \sum_{i=0}^{j-1} F_{n-i-1}^{(j)}, \text{ with } F_0^{(j)}, F_1^{(j)}, \ldots, F_{j-2}^{(j)} = 0,\ \ F_{j-1}^{(j)} = 1.$$

Motivating Questions

- Can we apply recurrence relations to matrices?
- How should we do so?
- What applications could this have?

Matrix Recurrences in Population Modeling

Definition (Modeling Migrating Populations Using Leslie Matrices)

Let $x_1(t)$ and $x_2(t)$ represent the population of a species in two regions, R_1 and R_2 respectively. The following recurrence relation can be used to model the time evolution of these populations

$$\vec{x}_1(t) = [L - k_1\mathbf{I}] \cdot \vec{x}_1(t-1) + k_2 \cdot \vec{x}_2(t-1),$$

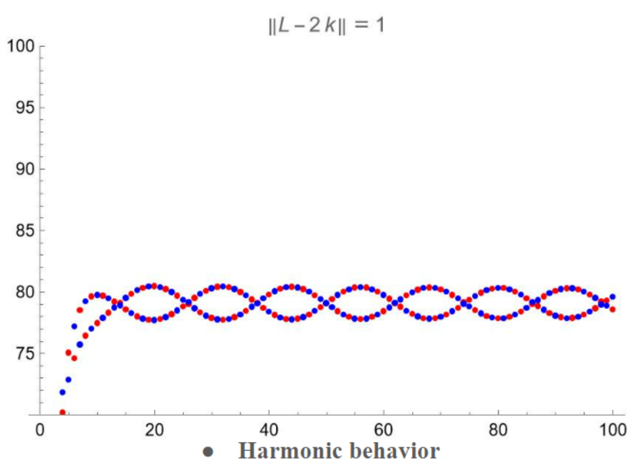
$$\vec{x}_2(t) = [L - k_2\mathbf{I}] \cdot \vec{x}_2(t-1) + k_1 \cdot \vec{x}_1(t-1),$$

where L is a Leslie matrix, defined as:

$$A = \begin{bmatrix} f_1 & f_2 & \dots & f_n \\ s_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & s_n & 0 \end{bmatrix}.$$

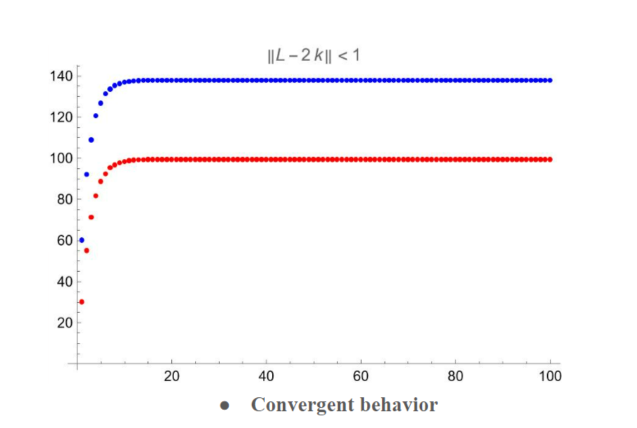
Behavior – I

Example (Base Case with $k_1 = k_2 = k$)



Behavior – II

Example (Base Case with $k_1 = k_2 = k$)



Behavior – III

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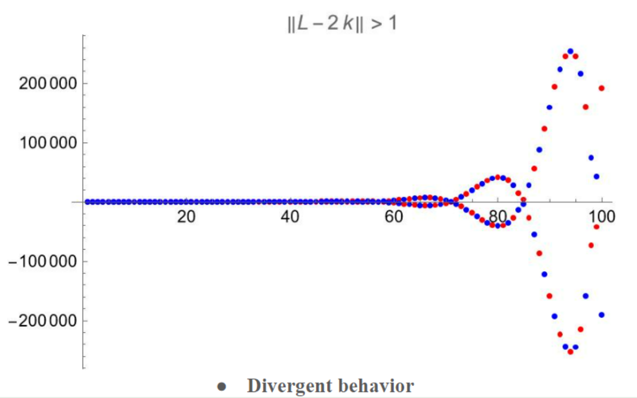


Stable Population of a Single-Species Model

Theorem (Bounded Fibonacci)

The recurrence relation

$$x_n = x_{n-1} + x_{n-2} - x_{n-1}x_{n-2}/M$$

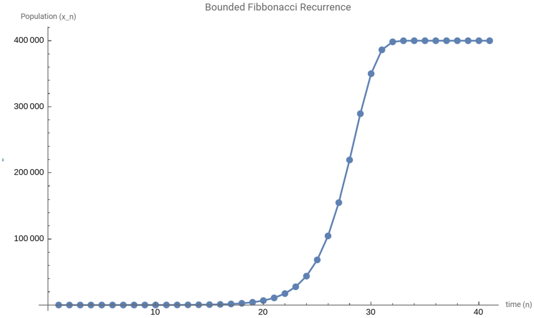
has a following closed form solution

$$x_n = M - M\left(1 - \frac{x_1}{M}\right)^{F_{n-1}}\left(1 - \frac{x_0}{M}\right)^{F_{n-2}}.$$

Moreover, x_n can be approximated by the following formula

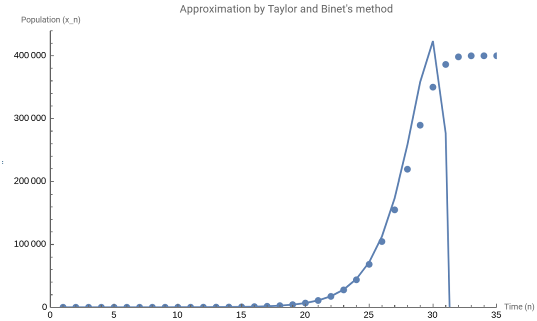
$$x_n \approx x_0 \frac{\varphi^n}{\sqrt{5}}\left(1 - \frac{\varphi^{n-3}}{\sqrt{5}M}x_0\right).$$

Plot of the Bounded Fibonacci



(* The line plot is the theoretical value computed by the closed - form solution.
The dots show the values computed by computing the recurrence via brute - force
*)

Approximation VS Reality



(* The curve shows the approximation and the points show the real value *)

Matrix Recurrence: The Simplest Case

Definition (Fibonacci Matrix Sequence)

Let A_0, A_1 be square matrices of same order. For $n \geq 2$,

$$A_n = A_{n-1}A_{n-2}.$$

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$A_0 = \mathbf{I}$, $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} =: Q$, known as the Fibonacci matrix.

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$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix}, \ldots$$

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$$A_n = \begin{bmatrix} F_{F_n+1} & F_{F_n} \\ F_{F_n} & F_{F_n-1} \end{bmatrix}$$

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$$A_n = \begin{bmatrix} F_{F_n+1} & F_{F_n} \\ F_{F_n} & F_{F_n-1} \end{bmatrix}$$

$$Q^m = \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} \implies A_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{F_n} = A_1^{F_n}$$

Matrix Recurrence: The Simplest Case

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Example:

$A_0 = \mathbf{I}$, $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} =: Q$, known as the Fibonacci matrix.

More generally, for $A_0 = \mathbf{I}$ and arbitrary A_1, we have that

$$A_n = A_1^{F_n}.$$

Matrix Recurrence: Generalization

Definition (j-nacci Matrix Sequence)

Let $A_0, A_1, \ldots, A_{j-1}$ be square matrices of the same order. For $n \geq j$,

$$A_n = \prod_{i=1}^{j} A_{n-i}.$$

Matrix Recurrence: Generalization

Definition (j-nacci Matrix Sequence)

Let $A_0, A_1, \ldots, A_{j-1}$ be square matrices of the same order. For $n \geq j$,

$$A_n = \prod_{i=1}^{j} A_{n-i}.$$

- Can we get a better grasp of the A_n's beyond the initial conditions?

Multiplicity of Initial Matrices Theorem

Theorem (Multiplicity)

Let $j > 1$ be an integer and $A_0, A_1, \ldots, A_{j-1}$ be square matrices of the same order. Consider the j-nacci matrix sequence defined by these initial conditions and the recurrence relation

$$A_n = \prod_{i=1}^{j} A_{n-i}.$$

Then for all $n \geq j$, A_n is a product of $A_0, A_1, \ldots, A_{j-1}$'s where each A_k, with $0 \leq k \leq j-1$, has multiplicity

$$\#A_{k,n} := \sum_{i=0}^{k} F_{n-i-1}^{(j)}.$$

Proof of the Multiplicity Theorem

Proof:

Analogous to the 2×2 Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, we can construct the following $j \times j$ j-nacci matrix:

$$Q^{(j)} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

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This matrix encodes the recurrence relation for the j-nacci sequence such that

$$\begin{bmatrix} F_{n+1}^{(j)} \\ F_{n}^{(j)} \\ \vdots \\ F_{n-j+2}^{(j)} \end{bmatrix} = Q^{(j)} \begin{bmatrix} F_{n}^{(j)} \\ F_{n-1}^{(j)} \\ \vdots \\ F_{n-j+1}^{(j)} \end{bmatrix}.$$

Proof of the Multiplicity Theorem – cont.

$\#A_{k,n}$ denotes the multiplicity of A_k in A_n (where $0 \leq k \leq j-1$).

$$A_n = \prod_{i=1}^{j} A_{n-i}$$

$$\implies \#A_{k,n} = \sum_{i=1}^{j} \#A_{k,n-i}$$

Proof of the Multiplicity Theorem – cont.

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$$A_n = \prod_{i=1}^{j} A_{n-i}$$
$$\implies \#A_{k,n} = \sum_{i=1}^{j} \#A_{k,n-i}$$

This is just the j-nacci recurrence relation which means we can write

$$\begin{bmatrix} \#A_{k,n+1} \\ \#A_{k,n} \\ \vdots \\ \#A_{k,n-j+2} \end{bmatrix} = Q^{(j)} \begin{bmatrix} \#A_{k,n} \\ \#A_{k,n-1} \\ \vdots \\ \#A_{k,n-j+1} \end{bmatrix}.$$

Proof of the Multiplicity Theorem – cont.

Starting from initial conditions, we can apply the $Q^{(j)}$ matrix n times to recover the nth vector iteration. Specifically,

$$\begin{bmatrix} \#A_{k,n+j-1} \\ \#A_{k,n+j-2} \\ \vdots \\ \#A_{k,n} \end{bmatrix} = \left(Q^{(j)}\right)^n \begin{bmatrix} \#A_{k,j-1} \\ \#A_{k,j-2} \\ \vdots \\ \#A_{k,0} \end{bmatrix}.$$

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Different A_k's have different initial conditions since $\#A_{k,l} = \begin{cases} 1 & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$

$$A_{j-1} : \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad A_1 : \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \qquad A_0 : \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \Longrightarrow \qquad A_k : \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}$$

Proof of the Multiplicity Theorem – cont.

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$$\begin{bmatrix} \#A_{k,n+j-1} \\ \#A_{k,n+j-2} \\ \vdots \\ \#A_{k,n} \end{bmatrix} = \begin{bmatrix} q_{j-1,j-1} & \cdots & q_{j-1,k} & \cdots & q_{j-1,0} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ q_{0,j-1} & \cdots & q_{0,k} & \cdots & q_{0,0} \end{bmatrix} \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}.$$

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Proof of the Multiplicity Theorem – cont.

Theorem (nth Power of the $Q^{(j)}$ Matrix, [1, page 159])

$$\left(Q^{(j)}\right)^n = \begin{bmatrix} F^{(j)}_{n+j-1} & \sum_{i=0}^{j-2} F^{(j)}_{n+j-2-i} & \cdots & \sum_{i=0}^{1} F^{(j)}_{n+j-2-i} & F^{(j)}_{n+j-2} \\ F^{(j)}_{n+j-2} & \sum_{i=0}^{j-2} F^{(j)}_{n+j-3-i} & \cdots & \sum_{i=0}^{1} F^{(j)}_{n+j-3-i} & F^{(j)}_{n+j-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F^{(j)}_{n+1} & \sum_{i=0}^{j-2} F^{(j)}_{n-i} & \cdots & \sum_{i=0}^{1} F^{(j)}_{n-i} & F^{(j)}_{n} \\ F^{(j)}_{n} & \sum_{i=0}^{j-2} F^{(j)}_{n-1-i} & \cdots & \sum_{i=0}^{1} F^{(j)}_{n-1-i} & F^{(j)}_{n-1} \end{bmatrix}$$

Proof of the Multiplicity Theorem – cont.

$$\begin{bmatrix} \#A_{k,n+j-1} \\ \#A_{k,n+j-2} \\ \vdots \\ \#A_{k,n} \end{bmatrix} = \left(Q^{(j)}\right)^n \begin{bmatrix} \#A_{k,j-1} \\ \#A_{k,j-2} \\ \vdots \\ \#A_{k,0} \end{bmatrix}$$

Proof of the Multiplicity Theorem – cont.

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Proof of the Multiplicity Theorem – cont.

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$$\#A_{k,n} = \sum_{i=0}^{k} F_{n-i-1}^{(j)}.$$

Stability Prerequisites

Definition (j-nacci Constant)

The j-nacci constant, denoted φ_j, is defined as the unique positive real number satisfying

$$x^j - \sum_{i=0}^{j-1} x^i \ = \ 0.$$

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Proposition (Asymptotic Form of the j-nacci Sequence)

Let $j > 1$ be an integer. Then

$$F_n^{(j)} \;\sim\; c_j \varphi_j^n,$$

where c_j is a positive constant.

- This result follows from one proved by E.P. Miles, Jr. in [2], namely that all roots in $\mathbb{C}\backslash\mathbb{R}_{\geq 0}$ of the j-nacci polynomial lie inside the unit circle.

The Stability Theorem

Theorem (Stability)

Let $j > 1$ be an integer and $A_0, A_1, \ldots, A_{j-1}$ be square matrices of the same order. Define the j-nacci matrix sequence $\{A_n\}$ using these initial conditions and the following recurrence relation (where the product can be taken in ***any*** *order)*

$$A_n = \prod_{0 \le i \le j-1} A_{n-j+i}.$$

Suppose that

$$\prod_{k=0}^{j-1} ||A_k||^{1-\varphi_j^{-k-1}} < 1,$$

where $||\cdot||$ is any submultiplicative matrix norm. Then $\{A_n\}$ converges to the zero matrix.

Proof of the Stability Theorem

Proof:

For any $n \geq j$,

$$||A_n|| \ \leq \ \prod_{k=0}^{j-1} ||A_k||^{\sum_{i=0}^{k} F_{n-i-1}^{(j)}}$$

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$$= \Theta\left(\prod_{k=0}^{j-1} ||A_k||^{\sum_{i=0}^{k} c_j \varphi_j^{n-i-1}}\right)$$

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$$\begin{aligned} ||A_n|| &\leq \prod_{k=0}^{j-1} ||A_k||^{\sum_{i=0}^{k} F_{n-i-1}^{(j)}} \\ &= \Theta\left(\prod_{k=0}^{j-1} ||A_k||^{\sum_{i=0}^{k} c_j \varphi_j^{n-i-1}}\right) \\ &= \Theta\left(\left(\prod_{k=0}^{j-1} ||A_k||^{1-\varphi_j^{-k-1}}\right)^{\frac{c_j \varphi_j^n}{\varphi_j - 1}}\right) \end{aligned}$$

Extending the Stability Theorem to Sums of Products

Corollary (Stability for Sums of Products)

Let K be a finite set of integers each greater than 1. Let $A_0, A_1, \ldots, A_{\max(K)-1}$ be square matrices of the same order. Define the sequence of matrices $\{A_n\}$ using these initial conditions and the following recurrence relation (where each product can be taken in ***any*** *order)*

$$A_n = \sum_{j \in K} a_j \prod_{0 \leq i \leq j-1} A_{n-\max(K)+i}.$$

Suppose that for all $j \in K$,

$$\prod_{k=0}^{j-1} ||A_k||^{1-\varphi_j^{-k-1}} < 1,$$

where $||\cdot||$ is any submultiplicative matrix norm. Then $\{A_n\}$ converges to the zero matrix.

Example of the Stability Theorem

Example:

Let $M_0 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $M_1 = \begin{bmatrix} (2\sqrt{2})^{-1} & (2\sqrt{2})^{-1} \\ 0 & 0 \end{bmatrix}$ and for $n \geq 2$, define

$$M_n = M_{n-1} M_{n-2}.$$

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$$M_n \; = \; M_{n-1} M_{n-2}.$$

We have that $||M_0||_{\text{op}} = 3$ and $||M_1||_{\text{op}} = 1/2$. Thus

$$||M_0||_{\text{op}}^{1-\varphi^{-1}} ||M_1||_{\text{op}}^{1-\varphi^{-2}} \approx \; 0.9913 < 1.$$

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Using the Stability Theorem, we know that M_n converges to the zero matrix.

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We have that $||M_0||_{\text{op}} = 3$ and $||M_1||_{\text{op}} = 1/2$.

Conjecture

For any $n \geq 1$,

$$||M_n||_{\text{op}} \;=\; \sqrt{2}\frac{3^{F_{n-1}}}{(2\sqrt{2})^{F_n}}.$$

- Considering how this formula contains the values of the norms and entries of M_0 and M_1, we naturally ask if this is generalizable.

The Stability Theorem: Further Generalization

Definition (The (S, j)-nacci Matrix Sequences)

Let $A_0, A_1, \ldots, A_{j-1}$ be square matrices of the same order. Let $S \subseteq \{0, 1, \ldots, j-1\}$. For $n \geq j$,

$$A_{n+j} \;=\; \prod_{k \in S} A_{n+k}.$$

Definition (S-nacci constant)

For a finite subset $S \subseteq \mathbb{N}$, the S-nacci constant, denoted φ_S, is defined as the positive real satisfying

$$\sum_{\ell \in S} \frac{1}{\varphi_S^{\ell+1}} \;=\; 1.$$

Note that for $S = \{0, 1, 2, \ldots, j-1\}$, this definition coincides with φ_j.

Stability Conjecture for the (S, j)-nacci matrix sequences

Conjecture (Stability for the (S, j)-nacci sequences)

*Let $A_0, A_1, \ldots, A_{j-1}$ be square matrices of the same order. For the matrix sequence $\{A_n\}$ defined by these initial conditions and the following recurrence relation (where the product can be taken in **any** order)*

$$A_{n+j} = \prod_{k \in S} A_{n+k},$$

with $S \subseteq \{0, 1, \ldots, j-1\}$ such that $S + 1 \not\subseteq m\mathbb{Z}$ for any integer $m > 1$. Suppose

$$\prod_{k=0}^{j-1} ||A_k||^{\sum_{i \in S}^{k} \varphi_S^{-i-1}} < 1.$$

Then $\{A_n\}$ converges to the zero matrix.

Further Directions for Investigation

- Investigate further types of matrix products (e.g. Kronecker product, Hadamard product)
- Consider p-adic versions
- Investigate recurrences defined by combining matrix multiplication and addition, for example of the form

$$M_n \ = \ (M_{n-1} + M_{n-2})^d$$

References and Thanks

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[1] G. Y. Lee et al. "The Binet Formula and Representations of k-Generalized Fibonacci Numbers". In: *Fibonacci Quarterly* 39.2 (2001), pp. 158–164.

[2] E.P. Miles Jr. "Generalized Fibonacci numbers and associated matrices". In: *American Mathematical Monthly* 67 (1960), pp. 745–752.