On the density of low-lying zeros of a large family of automorphic L-functions

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1. Introduction
2. Prior Work
3. Main Results

Montgomery Pair Correlation Conjecture

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Essentially, predicts that for f a Schwartz test function whose Fourier tranform has arbitrary compact support

$$\frac{1}{N(T)} \sum_{\substack{0 \le \gamma, \gamma' \le T \\ \gamma \neq \gamma'}} f\left((\gamma - \gamma') \frac{\log T}{2\pi}\right) \longrightarrow \int_{-\infty}^{\infty} f(x) W(x)\, dx, \quad T \to \infty.$$

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- Also agree with classical compact groups $O(N)$, $SO(\text{even})$, $SO(\text{odd})$, $U(N)$, $Sp(2N)$.

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- Katz-Sarnak (1999): introduced n-level density, distinguishes the classical compact groups, depends on behavior of eigenvalues near 1.
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- Katz-Sarnak density conjecture: behavior of low-lying zeros of a family of L-functions governed by behavior of eigenvalues of a classical compact group.
- Low-lying zeros related to infinitude of primes, Chebyshev's bias, Birch and Swinnerton-Dyer conjecture, class number bounds.

Density of low-lying zeros

Definition (1-level density)

Let Φ be a Schwartz function with $\mathrm{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$. Assume GRH and write $\rho_f = 1/2 + i\gamma_f$ for the non-trivial zeros of $L(s, f)$ counted with multiplicity. Then

$$\mathscr{OD}(f; \Phi) := \sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi} \log c_f\right),$$

is the *1-level density*, where c_f is the analytic conductor of f.

- 1-level density captures density of the zeros within height $O(1/\log c_f)$ of $s = 1/2$.
- Cannot asymptotically evaluate $\mathscr{OD}(f; \Phi)$ for a single f, must perform averaging over the family ordered by analytic conductor.

Katz-Sarnak Density Conjecture

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Let $\mathscr{F}(Q) := \{f \in \mathscr{F} : c_f = Q\}$ or $\mathscr{F}(Q) := \{f \in \mathscr{F} : c_f \leq Q\}$. Then for a Schwartz test function Φ whose Fourier transform has arbitrary compact support, we have that

$$\frac{1}{|\mathscr{F}(Q)|} \sum_{f \in \mathscr{F}(Q)} \mathscr{O}\mathscr{D}(f;\Phi) \longrightarrow \int_{-\infty}^{\infty} \Phi(x) W(G_{\mathscr{F}})(x)\, dx \quad \text{as} \quad Q \to \infty,$$

where $W(G_{\mathscr{F}})(x)$ is a distribution depending on the underlying symmetry group $G_{\mathscr{F}}$ associated to the family.

n-level density

Definition

In the setting as before, define the n*-level density* as

$$\mathscr{D}_n(f;\Phi) \;:=\; \sum_{\substack{j_1,\ldots,j_n \\ j_i\neq\pm j_k}} \prod_{i=1}^{n} \Phi_i\left(\frac{\gamma_f(j_i)}{2\pi}\log c_f\right).$$

- Computing n-level density for $n > 2$ requires knowledge of distribution of signs of the functional equation of each $L(s,f)$, which is beyond current theory.
- Hughes-Rudnick (2003): introduced n-th centered moments.

L-functions Attached to Cuspidal Newforms

Fix $f \in \mathcal{S}_k^{\text{new}}(q)$. Then for $\Re(s) > 1$, we define

$$\begin{aligned} L(s,f) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1} \\ &= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \end{aligned}$$

where χ_0 is the principal character mod q. Note, $L(s,f)$ can be analytically continued to an entire function on $\mathbb{C}$. Moreover, $L(s,f) = L(s,\overline{f})$.

Katz-Sarnak Density Conjecture for Orthogonal Symmetry

The symmetry type of the family of automorphic L-functions attached to holomorphic cuspidal newforms is orthogonal. Thus, the Katz-Sarnak density conjecture predicts that for test functions Φ whose Fourier transform has arbitrary compact support,

$$\frac{1}{|\mathscr{H}_k(Q)|} \sum_{f \in \mathscr{H}_k(Q)} \mathscr{O}\mathscr{D}(f;\Phi) \longrightarrow \int_{-\infty}^{\infty} \Phi(x) W(O)(x)\, dx \quad \text{as } Q \to \infty,$$

where O is the scaling limit of the group of square orthogonal matrices with density

$$W(O)(x) = 1 + \frac{1}{2}\delta_0(x),$$

where $\delta_0(x)$ denotes the Dirac delta function at $x = 0$.

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Extending the Support

Theorem (Iwaniec-Luo-Sarnak '00)

Assume GRH. Then for Φ any even Schwartz function with $\mathrm{supp}(\widehat{\Phi}) \subset (-2, 2)$, we have that

$$\lim_{\substack{q\to\infty \\ \square\text{free}}} \frac{1}{|\mathscr{H}_k(q)|} \sum_{f\in\mathscr{H}_k(q)} \mathscr{O}\mathscr{D}(f;\Phi) \;=\; \int_{-\infty}^{\infty} \Phi(x) W(O)(x)\, dx,$$

where O denotes the orthogonal type, showing agreement with the Katz-Sarnak philosophy predictions.

Recent Breakthrough

Theorem (Baluyot-Chandee-Li '23)

Assume GRH. Let Φ *be an even Schwartz function such that* $\mathrm{supp}(\widehat{\Phi}) \subset (-4, 4)$, *and let* Ψ *be any smooth function compactly supported on* $\mathbb{R}^+$ *with* $\widehat{\Psi}(0) \neq 0$. *Then we have that*

$$\langle \mathscr{O}\mathscr{D}(f;\Phi)\rangle_* := \lim_{Q\to\infty} \frac{1}{N(Q)} \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{f\in\mathscr{H}_k(q)}^h \mathscr{O}\mathscr{D}(f;\Phi) = \int_{-\infty}^{\infty} \Phi(x) W(O)(x) dx,$$

where $N(Q)$ *is a normalizing factor, showing agreement with the Katz-Sarnak philosophy predictions.*

The n-th Centered Moments of the 1-level Density

We study the n-th centered moments of the 1-level density averaged over levels $q \asymp Q$.

Definition (n-th centered moments of the 1-level density)

In the setting as above, define the n*-th centered moment of the 1-level density* to be

$$\left\langle \prod_{i=1}^{n} [\mathscr{O}\mathscr{D}(f;\Phi_i) - \langle \mathscr{O}\mathscr{D}(f;\Phi_i)\rangle_*] \right\rangle_* .$$

Main Results

Theorem (Cheek-Gilman-Jaber-Miller-Tomé '24)

Assume GRH. For Ψ *non-negative and* Φ_i *even Schwartz functions with* $\operatorname{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$ *and* $\sigma \leq \min\left\{\frac{3}{2(n-1)}, \frac{4}{2n-\mathbf{1}_{2\nmid n}}\right\}$ *we have that*

$$\left\langle \prod_{i=1}^{n} \left(\mathscr{O}\mathscr{D}(f;\Phi_i) - \langle \mathscr{O}\mathscr{D}(f;\Phi_i)\rangle_*\right) \right\rangle_* = \frac{\mathbf{1}_{2|n}}{(n/2)!} \sum_{\tau \in S_n} \prod_{i=1}^{n/2} \int_{-\infty}^{\infty} |u| \widehat{\Phi}_{\tau(2i-1)}(u) \widehat{\Phi}_{\tau(2i)}(u)\, du.$$

As such, our work is a generalization of the BCL '23 $n = 1, \sigma = 4$ result.

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Remark

Notably, for $n = 3$, *we achieve* $\sigma = \sigma_i = 3/4$, *greater than currently best known* $\sigma = \sigma_i = 2/3$.

Main results

Corollary (Cheek-Gilman-Jaber-Miller-Tomé '24)

Let $\sigma_1 = 3/2$ and $\sigma_2 = 5/6$. Then the two-level density

$$\left\langle \sum\sum_{j_1 \neq \pm j_2} \Phi_1\left(\gamma_f(j_1)\right)\Phi_2\left(\gamma_f(j_2)\right) \right\rangle_* = 2\int_{-\infty}^{\infty} |u|\widehat{\Phi}_1(u)\widehat{\Phi}_2(u)\,du + \prod_{i=1}^{2}\left(\frac{1}{2}\Phi_i(0) + \widehat{\Phi}_i(0)\right)$$
$$- \Phi_1\Phi_2(0) - 2\widehat{\Phi_1\Phi_2}(0) + \mathscr{ODD}\ \Phi_1\Phi_2(0),$$

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where $\mathscr{ODD} := \langle (1-\epsilon_f)/2 \rangle_*$ denotes the proportion of forms with odd functional equation.

Remark

This is the first evidence of an interesting new phenomenon: only by taking different test functions are we able to extend the range in which the Katz-Sarnak density predictions hold. In particular, $\sigma_1 + \sigma_2 = 7/3 > 2$, where $\sigma_1 + \sigma_2 = 2$ was the previously best known.

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