Modeling the Vanishing of L-functions at the Central Point

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SMALL REU at Williams College

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Outline

- Motivation
- Background
- Main question
- The model
- One-level density
- Pair-correlation

Conjecture (Montgomery-Dyson, 1970s)

High on the critical line, spacings between

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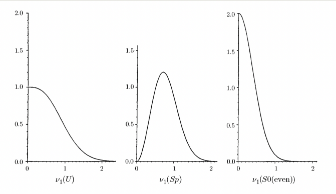
Conjecture (Katz-Sarnak, 1990s)

Katz-Sarnak conjectured that the following distributions match in the correct asymptotic limit:

- lowest-lying zeros at the critical point of families of L-functions,
- eigenvalues of random matrices from classical compact groups.



Source: N. M. Katz and P. Sarnak, *Zeros of zeta functions and symmetry*, Bulletin of the American Mathematical Society (1) **36** (1999), pages 1-26.

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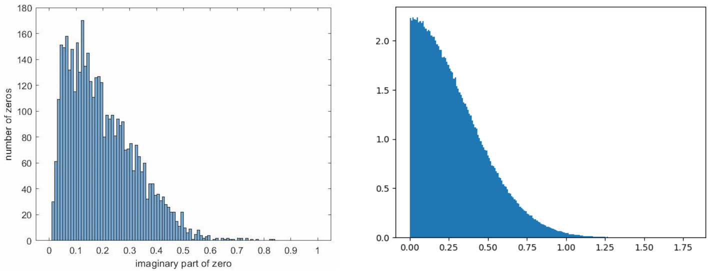

Comparison of distribution of lowest zeros for twists of an elliptic curve L-function and the corresponding eigenvalues from SO(even)

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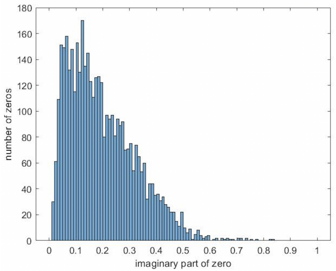

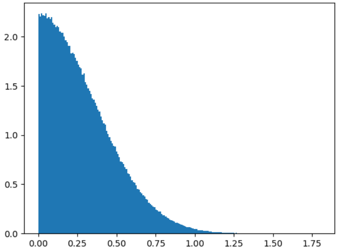

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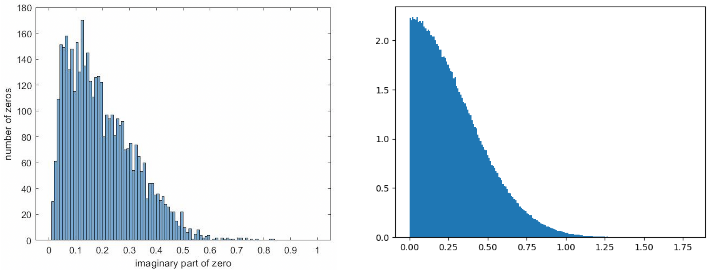

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Motivating Question

How accurately do eigenvalues of random matrices from classical compact groups model the lowest-lying zeros of families of L-functions associated to a cuspidal newform?

Let $f \in S_k^{\text{new}}(M, \chi_f)$ be a cuspidal newform of level odd prime M, weight k, and nebentypus χ_f. f has Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}$$

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Put $\lambda_f(n) = a_f(n)/n^{(k-1)/2}$. Then, for $\text{Re}(s) > 1$, the L-function attached to f is given by the Dirichlet series

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The functional equation of the completed L-function is given by

$$\Lambda(f, s) = \epsilon_f \Lambda(\overline{f}, 1 - s),$$

where ϵ_f is the root number.

Fix a cuspidal newform f, and consider its L-function $L(f,s)$. Given a quadratic character ψ_d, we create a twist as such

$$L(f\otimes\psi_d,s) \;=\; \sum_{n=1}^{\infty}\frac{\psi_d(n)\lambda_f(n)}{n^s} \;=\; \prod_p\left(1-\psi_d(p)\lambda_f(p)p^{-s}+\psi_d(p)\chi_f(p)p^{-2s}\right)^{-1}.$$

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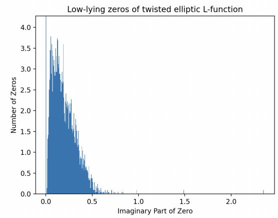
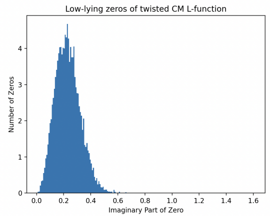
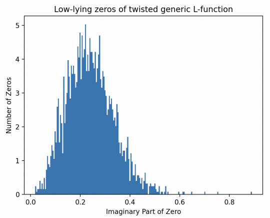
We create a family of L-functions by taking twists of $L(f, s)$ with fundamental discriminant $d \in \mathcal{D}$ ranging over

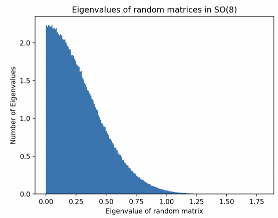
$$\mathcal{D}_f(X) := \begin{cases} \{d \in \mathcal{D} \,|\, 0 < d \leq X,\ \psi_d(M)\epsilon_f = +1\} & \chi_f \text{ principal, even twists,} \\ \{d \in \mathcal{D} \,|\, 0 < d \leq X,\ \psi_d(N)\epsilon_f = -1\} & \chi_f \text{ principal, odd twists,} \\ \{d \in \mathcal{D} \,|\, 0 < d \leq X,\ \psi_d(M) = \{\pm 1\}\} & \chi_f \text{ non-principal, } f = \overline{f}, \\ \{d \in \mathcal{D} \,|\, 0 < d \leq X\} & \chi_f \text{ non-principal, } f \neq \overline{f}. \end{cases}$$

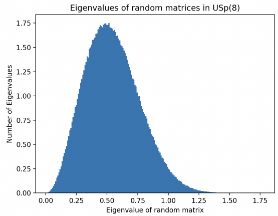
Is there any unexpected behavior that appears when we try to model the lowest-lying zeros of our family?

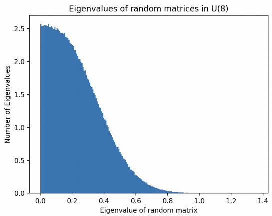












Before analyzing any behavior, we must ask:

Which classical compact groups model the lowest lying zeros of our family?

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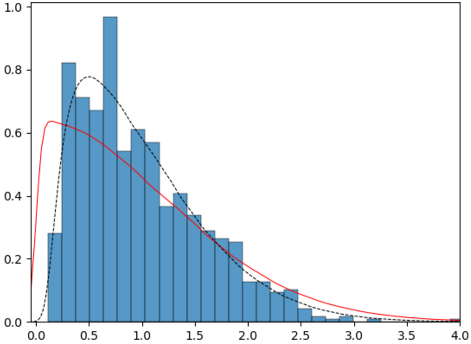
We computed the one-level density of our family and compared it to that of the groups U, Sp, and SO to determine the model:

$$\begin{aligned}\text{Principal nebentype, even twists} &\longleftrightarrow \text{SO(even)}\\ \text{Principal nebentype, odd twists} &\longleftrightarrow \text{SO(odd)}\\ \text{Non-principal nebentype and self-dual} &\longleftrightarrow \text{Sp}\\ \text{Generic} &\longleftrightarrow \text{U}\end{aligned}$$

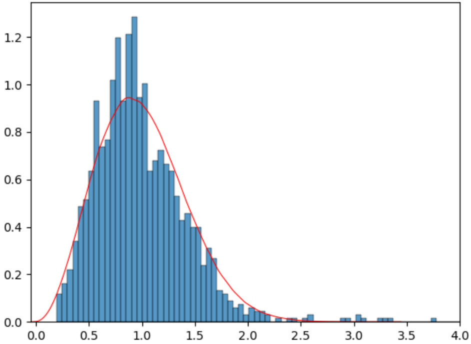
Lowest zeros (even twists) of 11.2.a.a

Eigenvalues from random matrices of SO(18)



Second lowest zeros (odd twists) of 11.2.a.a

Eigenvalues from random matrices of SO(19)

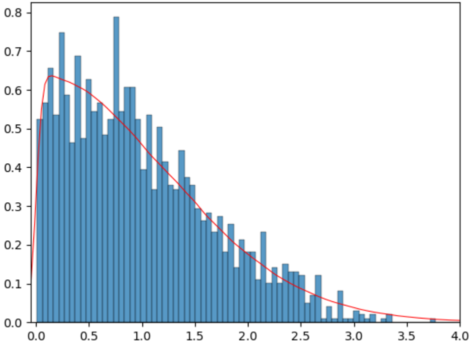


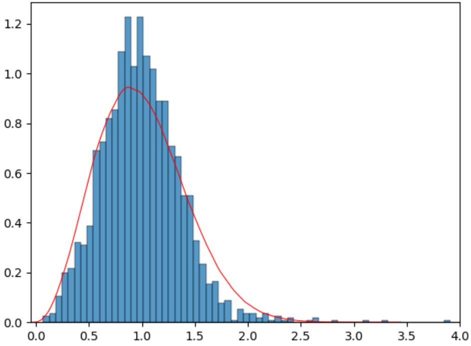
Lowest zeros (even twists) of 5.4.a.a

Eigenvalues from random matrices of SO(18)

Second lowest zeros (odd twists) of 5.4.a.a

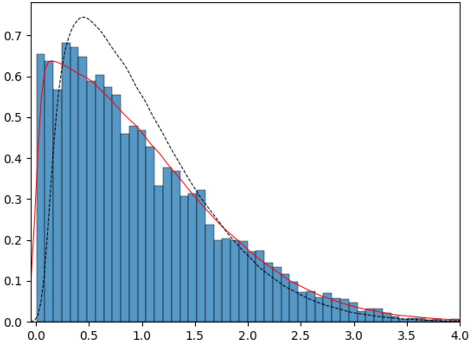
Eigenvalues from random matrices of SO(19)





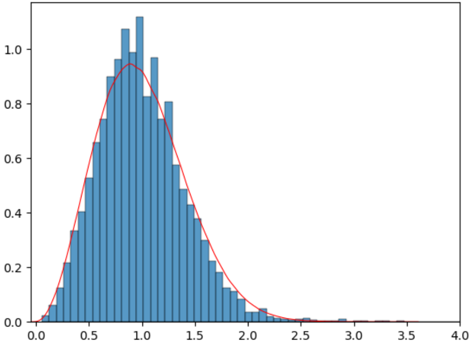
Lowest zeros (even twists) of 7.4.a.a

Eigenvalues of random matrices of SO(20)



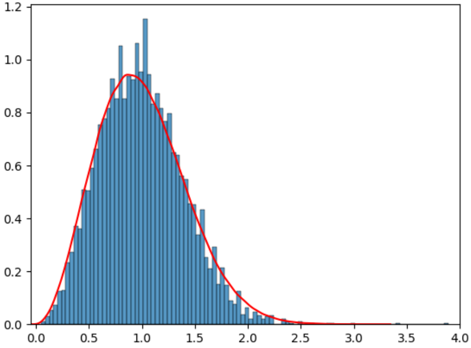
Lowest zeros (odd twists) of 7.4.a.a

Eigenvalues of random matrices of SO(21)



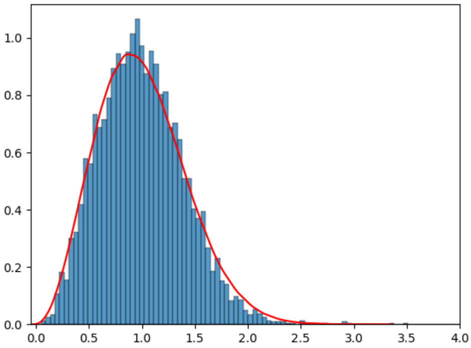
Lowest zeros ($\Delta = +1$) of 3.7.b.a

Eigenvalues of random matrices of Sp(20)



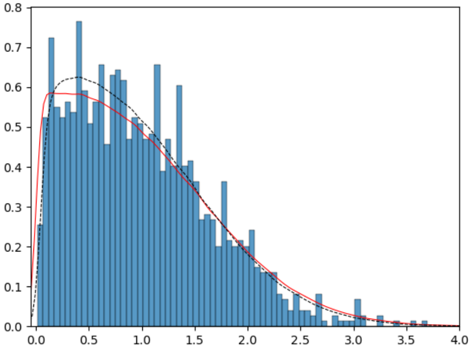
Lowest zeros ($\Delta = -1$) of 7.3.b.a

Eigenvalues of random matrices of Sp(20)



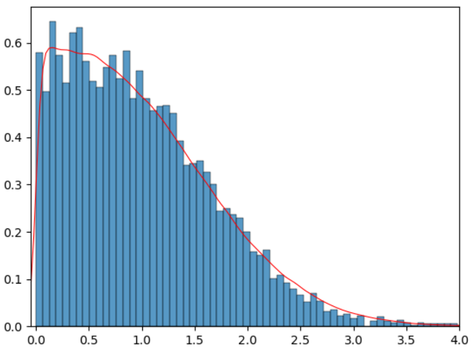
Lowest zeros (twists) of 13.2.e.a

Eigenvalues of random matrices of Sp(20)



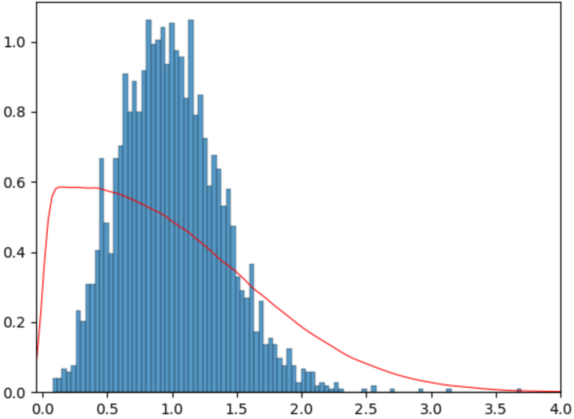
Lowest zeros (twists) of 17.2.d.a

Eigenvalues of random matrices of Sp(20)



Lowest zeros (twists) of 11.7.b.b

Eigenvalues of random matrices of U(9)



For large T, we denote the pair-correlation of a family of twists of a given form f by

$$P(f \otimes \psi_d; \varphi) = \sum_{0<\gamma,\gamma'<T} \varphi(\gamma - \gamma'),$$

where the γ's are the imaginary part of the zeros and φ a (holomorphic) test function.

Pair-correlation function of a family of twists

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where the γ's are the imaginary part of the zeros and φ a (holomorphic) test function. Using the ratios conjecture and analyticity, we expand the above using series expansions

$$P(f \otimes \psi_d; \varphi) := \frac{T}{2\pi} R \Bigg[h(0) + \int_{\mathbb{R}} h(y) \Bigg(1 - \left(\frac{\sin \pi y}{\pi y} \right)^2 + \frac{e_1 - e_2 \sin^2 \pi y}{R^2} - \frac{e_3 \pi y \sin 2\pi y}{R^3} + O(R^{-4}) \Bigg) dy \Bigg] + O(T^{\varepsilon + 1/2}),$$

where

$$R = \log \left(\frac{\sqrt{M} |d| T}{2\pi e} \right).$$

Compare the U(N) pair-correlation

$$1 - \left(\frac{\sin \pi y}{\pi y}\right)^2 - \frac{\sin^2 \pi y}{3N^2},$$

to the pair-correlation for our form f, we compare the term

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Conjecture (Montgomery, 1973)

High on the critical line, the spacing between pairs of the Riemann zeta function is asymptotically

$$1 - \left(\frac{\sin \pi u}{\pi u}\right)^2.$$

We would like to thank our mentor, Professor Steven J. Miller, and previous years of SMALL for their contributions.

Thanks to our SMALL 2023 faculty, research assistants, and peers for their support.

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