Schreier-Type Sets and Linear Recurrences

Hùng V. Chu

Washington and Lee University
hchu@wlu.edu

in joint work with Kevin Beanland, Carrie E. Finch-Smith, Yubo Geng*, Nurettin Irmak, Julian King*, Steven J. Miller, László Szalay, Garrett Tresch, Zachary Louis Vasseur*, Sindy Xin Zhang*

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Schreier sets

A finite set $F \subset \mathbb{N}$ is said to be Schreier if $F = \emptyset$ or $\min F \geq |F|$.

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Not Schreier: $\{3, 7, 10, 13\}$

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It began with the observation of A. Bird:

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$$\mathcal{A}_5 = \{\{2,5\},\{3,5\},\{4,5\},\{3,4,5\},\{5\}\} \Longrightarrow |\mathcal{A}_5| = 5$$

Schreier sets and the Fibonacci sequence

A. Bird (2012)

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$\min F \geq |F| \longrightarrow q \min F \geq p|F|, p, q \in \mathbb{N}$

The s-step Fibonacci numbers

For $s \geq 2$, define $F_{2-s}^{(s)} = \cdots = F_0^{(s)} = 0, F_1^{(s)} = 1$, and

$$F_n^{(s)} = F_{n-1}^{(s)} + \cdots + F_{n-s}^{(s)}, \text{ for } n \geq 2.$$

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$$\mathcal{A}_n^{(s-1)} := \left\{ F \subseteq \{\underbrace{1, \ldots 1}_{s-1}, \ldots, \underbrace{n-1, \ldots, n-1}_{s-1}, n\} : \begin{array}{c} n \in F, \\ F \text{ is Schreier} \end{array} \right\}$$

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C., Irmak, Miller, Szalay, and Zhang (2024)

$$|\mathcal{A}_n^{(s-1)}| = F_n^{(s)}$$

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$$|\mathcal{S}_n^{p/q}| = \sum_{k=1}^{q} (-1)^{k+1} \binom{q}{k} |\mathcal{S}_{n-k}^{p/q}| + |\mathcal{S}_{n-(p+q)}^{p/q}|.$$

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Proof 2: the subsequence method. Rely on $(|\mathcal{S}_n^{p/q}|)_{n=1}^{\infty}$ is a periodic subsequence of a sequence satisfying a relatively simple recurrence.

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$33, 38, 48, 55, 66, 81, 93, 114, 136, 159, 195, 229, 273, 331,$
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$$\begin{cases} (|\mathcal{S}_n^{2/3}|)_{n=1}^{\infty} & \text{subseq of } (a_n)_{n=0}^{\infty} \\ (a_n)_{n=1}^{\infty} & \text{satisfies } a_n = a_{n-3} + a_{n-5} \end{cases} \Rightarrow \text{Recurrence for } (|\mathcal{S}_n^{2/3}|)_{n=1}^{\infty}$$

No integers from a fixed arithmetic progression

For $b \geq 2$ and $0 \leq a < b$, let $g^{(a,b)}(n)$ be the n^{th} positive integer that is not $a \mod b$.

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C., Geng, King, Miller, Tresch, Vasseur (2025), C. (2026)

$$|\mathcal{D}_n^{(a,b)}| = \sum_{k=1}^{b} (-1)^{k-1} \binom{b}{k} |\mathcal{D}_{n-k}^{(a,b)}| + |\mathcal{S}_{n-2b+1}^{(a,b)}|, \quad n \geq 2b$$

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C. (2026)

$$|\mathcal{M}_{2n-1}| = F_n, \quad n \geq 1$$

Problem 1

$$\mathcal{A}^{(s)}_{p,q,n} := \left\{ F \subset \{\underbrace{1,\ldots,1}_{s},\ldots,\underbrace{n-1,\ldots,n-1}_{s},n\} : \begin{array}{c} n \in F, \\ q \min F \geq p|F| \end{array} \right\}$$

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Nonlinear Schreier-type condition $(\min F)^q \geq |F|^p$, $p, q \in \mathbb{N}$

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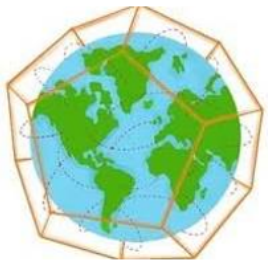
C., Irmak, Miller, Szalay, and Zhang (2024)

$$|\mathcal{A}_n^p| = K_{n+1,p-1}$$

Polymath Junior Program



- Provide research opportunities to undergraduates.
- Online, runs in the spirit of the Polymath Project.
- Projects run by researcher with experience in undergraduate mentoring.
- Most 15-25 students, a main mentor, grad students / postdocs assisting.

https://geometrynyc.wixsite.com/polymathreu