# Sensor sensitivity in several settings

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# Compression and Reconstruction

Underlying arbitrary image

- Suppose we are given an arbitrary image generated by an arbitrary function.

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Incomplete discretized data

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Complete discretized image

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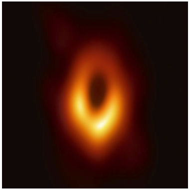
- A standard reconstruction algorithm uses the observed measurements to estimate the missing values.
- The reconstruction does not look good because the initial image is not spectrally compressible.

# Compression and Reconstruction



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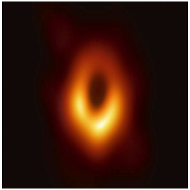

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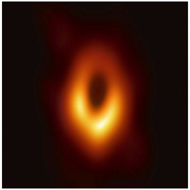

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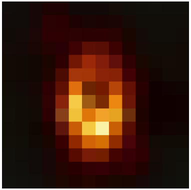

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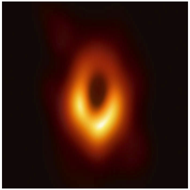

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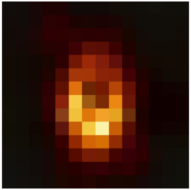

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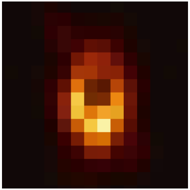

Complete discretized image

- Now what if we assume that the image is generated by an underlying smooth continuous signal.
- We observe only a discretized version of the signal, and some sampled values are missing. The white squares represent unknown measurements.
- A standard reconstruction algorithm uses the known measurements to estimate the missing values.
- The reconstruction closely resembles the complete discretized image that would have been observed if no measurements were missing because this intial signal is indeed spectrally compressible.

# Definitions

- For a function $g : \mathbb{Z}_N^2 \to \mathbb{C}$, define its Fourier ratio by

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- The corresponding norms are

$$\|\widehat{g}\|_{\ell^1} = \sum_{m \in \mathbb{Z}_N^2} |\widehat{g}(m)| \quad \text{and} \quad \|\widehat{g}\|_{\ell^2} = \left( \sum_{m \in \mathbb{Z}_N^2} |\widehat{g}(m)|^2 \right)^{1/2}.$$

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- We find bounds for certain families of functions across several geometries.

# Three geometric settings

| **Unit Square** | **Unit Sphere** | **Manifold** |
|---|---|---|
| Fourier modes | spherical harmonics | Laplace–Beltrami eigenfunctions |
| uniform grid cells | equal-area cells | equal-volume cells |
| Euclidean perturbations | geodesic perturbations | geodesic perturbations |

# Bounding the distribution of data

## Theorem

*Let $X$ be the unit square, the unit sphere, or a compact Riemannian manifold then,*

$$\mathrm{FR}_R(f) \lesssim_X \frac{\left|\int_X f\right|}{\|f\|_{L^2(X)}} + \sqrt{1+\log R}\,\frac{\|\nabla_X f\|_{L^2(X)}}{\|f\|_{L^2(X)}}.$$

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5. Count the frequencies in each setting so as to estimate the remaining spectral sum.

# Perturbations in sensor placement

## Theorem

*Let $X$ be the unit square, the unit sphere, or a compact Riemannian manifold. If $d_X(x_i, y_i) \leq \delta$, then*

$$\left( \frac{|X|}{q} \sum_{i=1}^{q} |f(y_i) - f(x_i)|^2 \right)^{1/2} \lesssim_X \delta \, \|\nabla_X f\|_{L^\infty(X)}.$$

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5. Square and average over the sensor locations to obtain the total placement error.

# Random sensor sampling

## Theorem

*Let $X$ be the unit square, the unit sphere, or a compact Riemannian manifold partitioned into $K$ equal-measure cells of diameter at most $h_K$. If sensors are placed uniformly at random, then*

$$\mathbb{E}[T_q] \;=\; K\,(H_K - H_{K-q})\,, \qquad \mathcal{E}_{\text{place}} \;\lesssim_X\; h_K\|\nabla_X f\|_{L^\infty(X)}.$$

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5. Use the cell diameter to bound the sensor-placement error.

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1. Smoothness controls the distribution of Fourier mass across several geometries.
2. Small errors in sensor placement produce proportionally small errors in the measured data.
3. Randomly placed sensors can be converted into approximately uniform samples by partitioning the space into equal-measure cells.
4. Together, these results give a geometry-independent framework for stable signal recovery.

# Future Work

## Definition

*In a Sobolev space $W^{s,p}$, a function $f(x)$ is said to have derivative $f'(x)$ if for any smooth function $\varphi(x)$ with compact support*

$$\int_{\Omega} f(x)\varphi'(x) = -\int_{\Omega} f'(x)\varphi(x)$$

*holds.*

1. We can weaken the restrictions on our functions $f$ but still get meaningful results about the fourier ratio of such functions.

# Future Work

## Conjecture

*Using new bounds for Sobolev Spaces, we conjecture that for any function $f \in W^{s,p}$ the bound on the Fourier Ratio is*

$$FR(f) \leq \sqrt{2}\frac{\left|\int_{[0,1]^2} f(u)\,du\right|}{\|f\|_{L^2([0,1]^2)}} + CN^{-s-1}\sqrt{1+\log N}\frac{\|f\|_{W^{s,p}([0,1]^2)}}{\|f\|_{L^2([0,1]^2)}} + 2CN^{-s}\frac{\|f\|_{W^{s,p}([0,1]^2)}}{\|f\|_{L^2([0,1]^2)}}.$$

*Where $C$ is some constant that depends on $s$ and $p$.*

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4. Function space $\mathcal{F} = \{f : X \to \mathbb{R}\}$

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## Conjecture

*Let $A^{s/2}$ be a non-negative, self-adjoint, generator with compact resolvent and L is the spectral cutoff of the orthonormal basis*

$$FR(f) \leq \frac{\left|\int_X f \,\mathrm{d}\mu\right|}{\|f\|_2} + C\frac{\|A^{s/2}f\|}{\|f\|_2} \begin{cases} 1, & s > \alpha \\ \sqrt{1+\log L}, & s = \alpha \\ L^{(\alpha-s)/2}, & 0 < s < \alpha \end{cases}$$

And we can apply this to fractals.

# References


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# Thanks ^–^